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OUTER AUTOMORPHISM GROUPS OF CERTAIN POLYGONAL PRODUCTS OF GROUPS

GOANSU KIM

ABSTRACT. We show that certain polygonal products of any four groups, amalgamating central subgroups with trivial intersections, have Property E. Using this result, we derive that outer automorphism groups of polygonal products of four polycyclic-by-finite groups, amalgamating central subgroups with trivial intersections, are residually finite.

1. Introduction

Polygonal products of groups were introduced by A. Karrass, A. Pietrowski and D. Solitar [6] in the study of the subgroup structure of the Picard group $PSL(2, Z[i])$, which is a polygonal product of four finite groups amalgamating cyclic subgroups, with trivial intersections. In [4], Allenby and Tang proved that polygonal products of four finitely generated (briefly, f.g.) free abelian groups, amalgamating cyclic subgroups with trivial intersections, are residually finite (briefly, $\mathcal{RF}$). And they gave an example of a polygonal product of four f.g. nilpotent groups which is not $\mathcal{RF}$. However, certain polygonal products of f.g. nilpotent groups are $\mathcal{RF}$ or π_c [1, 8, 10]. In particular, polygonal products of polycyclic-by-finite groups, amalgamating central subgroups with trivial intersections, are known to be conjugacy separable [9, 11]. Unlike the case for residual finiteness or for conjugacy separability, most polygonal products of four finitely generated abelian groups amalgamating cyclic subgroups with trivial intersections are not subgroup separable [7].

In this paper we prove that polygonal products of any four groups, amalgamating central subgroups with trivial intersections, have Property E. Using this result, we derive that outer automorphism groups of polygonal products of four polycyclic-by-finite groups, amalgamating central subgroups with trivial intersections, are residually finite (Theorem 4.5).

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2. Preliminaries

Briefly, polygonal products of groups can be considered as follows [4]: Let P be a polygon. Assign a group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product* G is defined to be the group presented by the generators and relations of the vertex groups together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_0, G_1, \ldots, G_n$, amalgamating the (edge) subgroups $H_1, \ldots, H_n, H_0$ with *trivial intersections*, if $G_{i-1} \cap G_i = H_i$ and $H_{i-1} \cap H_i = 1$, where $0 \leq i \leq n$ and the subscripts i are taken modulo $n+1$. We only consider the case $n \geq 3$ (see [4]).

We introduce some definitions and results that we shall use in this paper.

If A, B are groups, $G = A *_H B$ denotes the generalized free product of A and B amalgamating the subgroup H. If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G.

If $g \in G$, Inn g denotes the inner automorphism of G induced by g.

$Out(G)$ denotes the outer automorphism group of G.

$x \sim_G y$ means that x and y are conjugate in G.

$\mathcal{RF}$ is an abbreviation for "residually finite".

$Z(G)$ is the center of G and $Z_A(x) = \{g \in A \mid gx = xg\}$.

Definition 2.1. By a *conjugating endomorphism/automorphism* of a group G we mean an endomorphism/automorphism α which is such that, for each $g \in G$, there exists $k_g \in G$, depending on g, so that $\alpha(g) = k_g^{-1} g k_g$.

Definition 2.2 (Grossman [5]). A group G has *Property A* if, for each conjugating automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$, i.e., $\alpha = \text{Inn } k$.

We extend Grossman's Property A to include endomorphisms.

Definition 2.3 ([3]). A group G has *Property E* if, for each conjugating endomorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$, i.e., $\alpha = \text{Inn } k$.

Clearly, every abelian group has Property E and every group having Property E has Property A. We will make use of the following result of Grossman [5]:

Theorem 2.4 (Grossman [5]). *Let B be a finitely generated, conjugacy separable group with Property A. Then $Out(B)$ is $\mathcal{RF}$.*

Theorem 2.5 ([3]). *Non trivial free products of groups have Property E.*

Theorem 2.6 ([12, Theorem 4.6]). *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If* $||x|| = 0$, *then* $||y|| \leq 1$ *and, if* $y \in A$, *then there is a sequence* $h_1, h_2, \ldots, h_r$ *of elements in* H *such that* $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x$.
(2) *If* $||x|| = 1$, *then* $||y|| = 1$ *and, either* $x, y \in A$ *and* $x \sim_A y$, *or* $x, y \in B$ *and* $x \sim_B y$.
(3) *If* $||x|| \geq 2$, *then* $||x|| = ||y||$ *and* $y \sim_H x^*$, *where* x^* *is a cyclic permutation of* x.

3. A criterion

Theorem 3.1. *Let* $G = A *_H B$, *where* $A \neq H \neq B$ *and* $H \subset Z(B)$. *Suppose* A *has Property E and the following conditions hold:*

($C1$) *If* $u \in A$ *and* $u^{-1}hu = h$ *for all* $h \in H$ *then* $u \in H$.
($C2$) *There exists an element* $a \in A$ *such that* $\{a\}^A \cap H = \emptyset$ *and if* $u^{-1}au = h'ah$, *where* $u \in A$ *and* $h', h \in H$, *then* $h' = h^{-1}$.

Then G *has Property E.*

Proof. Let α be a conjugating endomorphism of G and $\alpha(g) = k_g^{-1}gk_g$ for $g \in G$. Without loss of generality, we can assume $\alpha(a) = a$, where a satisfies (C2). We shall show that α is an inner automorphism of G as follows:

(I) For each $y \in B$, we can choose $k_y \in A$.

Let $1 \neq y \in B$ and $k_y = u_1u_2 \cdots u_r$ be an alternating product of the shortest length in G such that $\alpha(y) = k_y^{-1}yk_y$. Then $k_{ya}^{-1}(ya)k_{ya} = \alpha(ya) = \alpha(y)\alpha(a) = k_y^{-1}yk_y \cdot a = u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1}yu_1 \cdot u_2 \cdots u_{r-1} \cdot u_r \cdot a$. Thus,

$$ya \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1}yu_1 \cdot u_2 \cdots u_{r-1} \cdot u_r a u_r^{-1}. \tag{3.1}$$

(a) $y \in B \backslash H$.

(i) Suppose $u_1 \in A \backslash H$. If $u_r \in B \backslash H$ and $r \geq 2$, then the R.H.S. of (3.1) is cyclically reduced of length $2(r+1) \geq 6$. Since the L.H.S. of (3.1) is of length 2, this case does not occur by Theorem 2.6.

If $u_r \in A \backslash H$ then, by (C2), $u_r a u_r^{-1} \notin H$. The R.H.S. of (3.1) is cyclically reduced of length $2r$. Since the L.H.S. of (3.1) is of length 2, we have $r = 1$. Hence $k_y = u_1 \in A$.

(ii) Suppose $u_1 \in B \backslash H$. Since $H \subset Z(B)$ and $y \in B \backslash H$, $u_1^{-1}yu_1 \notin H$.

If $u_r \in A \backslash H$ then, by (C2), $u_r a u_r^{-1} \notin H$. Hence the R.H.S. of (3.1) is cyclically reduced of length $2(r-1)$. Since the L.H.S. of (3.1) is cyclically reduced of length 2, we have $r \leq 2$ and $k_y = u_1u_2 \in BA$. Then, from (3.1), we have $ya \sim_G u_1^{-1}yu_1 \cdot u_2 a u_2^{-1}$, where both sides are cyclically reduced of length 2. Thus, by Theorem 2.6, $ya \sim_H u_1^{-1}yu_1 \cdot u_2 a u_2^{-1}$, which implies that $y = h^{-1}(u_1^{-1}yu_1)h_1$ and $a = h_1^{-1}(u_2 a u_2^{-1})h$ for some $h, h_1 \in H$. By (C2), $h = h_1$. This implies that $u_1^{-1}yu_1 = h_1 y h_1^{-1}$. Thus $\alpha(y) = k_y^{-1}yk_y = u_2^{-1}u_1^{-1}yu_1u_2 = u_2^{-1}(h_1 y h_1^{-1})u_2$. This means that we can choose $k_y = h_1^{-1}u_2 \in A$.

If $u_r \in B \backslash H$, then the R.H.S. of (3.1) is cyclically reduced of length $2r$. Since the L.H.S. of (3.1) is of length 2, we have $r = 1$. Hence $k_y = u_1 \in B$.

Then, from (3.1), we have $ya \sim_G u_1^{-1}yu_1a$, where both sides are cyclically reduced of length 2. Thus, by Theorem 2.6, $ya \sim_H u_1^{-1}yu_1 \cdot a$, which implies that $y = h^{-1}(u_1^{-1}yu_1)h_1$ and $a = h_1^{-1}ah$ for some $h, h_1 \in H$. By (C2), $h = h_1$. This implies that $u_1^{-1}yu_1 = h_1yh_1^{-1}$. Thus $\alpha(y) = k_y^{-1}yk_y = u_1^{-1}yu_1 = h_1yh_1^{-1}$. This means that we can choose $k_y = h_1^{-1} \in H \subset A$.

(b) $y = h \in H$.

Since $H \subset Z(B)$, we may assume that $u_1 \in A$. Suppose $r \geq 2$. Since $u_1 \in A\backslash H$ and $H \subset Z(B)$, if $u_1^{-1}yu_1 \in H$ then $u_2^{-1}u_1^{-1}yu_1u_2 = u_1^{-1}yu_1$. This reduces the length of $k_y = u_1u_2\cdots u_r$. Hence we may assume $u_1^{-1}yu_1 \notin H$. If $u_r \in B\backslash H$, then the R.H.S. of (3.1) is cyclically reduced of length $2r \geq 4$. If $u_r \in A\backslash H$ and $r \geq 2$, then the R.H.S. of (3.1) is cyclically reduced of length $2(r-1) \geq 2$. Since the L.H.S. of (3.1) is of length at most 1 ($y \in H$), neither case satisfies (3.1) by Theorem 2.6. Hence $r \leq 1$. This means $k_y = u_1 \in A$.

(II) There exists a fixed element $u \in Z_A(a)$ so that $k_y = u$ for all $y \in B$.

Fix $b \in B\backslash H$ and fix $k_b = w \in A$ (by (I)). Let $y \in B\backslash H$. Then $k_{by}^{-1}(by)k_{by} = \alpha(by) = \alpha(b)\alpha(y) = w^{-1}bw \cdot k_y^{-1}yk_y$, where $k_y \in A$ by (I). This implies

$$by \sim_G b \cdot wk_y^{-1} \cdot y \cdot k_yw^{-1}. \tag{3.2}$$

Since $b, y \in B\backslash H$, if $wk_y^{-1} \in A\backslash H$ then the R.H.S. of (3.2) is cyclically reduced of length 4, in which case (3.2) cannot hold by Theorem 2.6. Thus $wk_y^{-1} \in H$. Let $k_y = h_y^{-1}w$, where $h_y \in H$ depends on y.

Note that $k_{ba}^{-1}(ba)k_{ba} = \alpha(ba) = \alpha(b)\alpha(a) = k_b^{-1}bk_b \cdot a = w^{-1}h_bbh_b^{-1}w \cdot a = w^{-1}bw \cdot a$ ($H \subset Z(B)$). This means

$$ba \sim_G b \cdot waw^{-1}. \tag{3.3}$$

Since $\{a\}^A \cap H = \emptyset$, $waw^{-1} \in A\backslash H$. Hence both sides of (3.3) are cyclically reduced of length 2. This implies $ba \sim_H b \cdot waw^{-1}$ by Theorem 2.6. Thus $b = h^{-1}bh_1$ and $a = h_1^{-1}waw^{-1}h$ for some $h, h_1 \in H$. Since $H \subset Z(B)$ and $b \in B$, we have $h_1 = h$. Hence, $a = h_1^{-1}waw^{-1}h_1$. Let $u = h_1^{-1}w$. Then $u \in Z_A(a)$.

For each $y \in B\backslash H$, we have $\alpha(y) = k_y^{-1}yk_y = w^{-1}h_yyh_y^{-1}w = w^{-1}yw = w^{-1}h_1yh_1^{-1}w = u^{-1}yu$. Hence $\alpha(y) = u^{-1}yu$ for all $y \in B\backslash H$. Now, for $v \in H$, let $b \in B\backslash H$ and consider $\alpha(v) = \alpha(vb \cdot b^{-1}) = \alpha(vb)\alpha(b^{-1}) = u^{-1}(vb)u \cdot u^{-1}b^{-1}u = u^{-1}vu$. Hence $\alpha(y) = u^{-1}yu$ for all $y \in B$.

(III) G has Property E.

Let $\overline{\alpha} = \text{Inn } u^{-1} \circ \alpha$. Then, by (II), $\overline{\alpha}(y) = y$ for all $y \in B$. Moreover, since $u \in Z_A(a)$, $\overline{\alpha}(a) = u\alpha(a)u^{-1} = uau^{-1} = a$. We shall show that $\overline{\alpha}$ is an inner automorphism of G. For convenience, we again use $\overline{\alpha}(g) = k_g^{-1}gk_g$ for $g \in G$.

Let $x \in A\backslash H$ and $k_x = u_1u_2\cdots u_r$ be an alternating product of the shortest length from G such that $\overline{\alpha}(x) = k_x^{-1}xk_x$. As before, we have $k_{xa}^{-1}(xa)k_{xa} = \overline{\alpha}(xa) = k_x^{-1}xk_x \cdot a = u_r^{-1}\cdots u_1^{-1}xu_1\cdots u_r \cdot a$. Hence

$$xa \sim_G u_{r-1}^{-1}\cdots u_2^{-1} \cdot u_1^{-1}xu_1 \cdot u_2\cdots u_{r-1} \cdot u_rau_r^{-1}. \tag{3.4}$$

We now show: (a) $u_1 \in A$ and (b) $r \leq 1$.

(a) Suppose $u_1 \in B \backslash H$. If $u_r \in A \backslash H$, then $u_r a u_r^{-1} \notin H$ by (C2). Hence the R.H.S. of (3.4) is cyclically reduced of length $2r$. If $u_r \in B \backslash H$, then the R.H.S. of (3.4) is cyclically reduced of length $2(r+1)$. Since the L.H.S. of (3.4) is $xa \in A$, neither of these cases can occur by Theorem 2.6. Hence $u_1 \in A$.

(b) Suppose $u_1 \in A$ and $r \geq 2$. First, consider the case $u_1^{-1} x u_1 \notin H$. If $u_r \in B \backslash H$ then the R.H.S. of (3.4) is cyclically reduced of length $2r$. And if $u_r \in A \backslash H$ $(r \geq 3)$ then the R.H.S. of (3.4) is cyclically reduced of length $2(r-1)$, since $u_r a u_r^{-1} \in A \backslash H$ by (C2). As before, since the length of the L.H.S. of (3.4) is at most 1, neither of these cases can occur. Therefore, we can assume $u_1^{-1} x u_1 = h_1 \in H$. Since $u_2 \in B \backslash H$ and $H \subset Z(B)$, $u_2^{-1} u_1^{-1} x u_1 u_2 = u_1^{-1} x u_1$. This reduces the length of k_x. Hence, this case cannot occur.

Hence $u_1 \in A$ and $r \leq 1$, i.e., $k_x \in A$ for each $x \in A \backslash H$. Therefore, the restriction of $\overline{\alpha}$ to A is a conjugating endomorphism of A. Since A has property E, there exists $c \in A$ such that $\overline{\alpha}(x) = c^{-1} x c$ for all $x \in A$. Then $h = \overline{\alpha}(h) = c^{-1} h c$ for all $h \in H$. By (C1), we must have $c \in H$. Hence $\overline{\alpha}(b) = b = c^{-1} b c$ for all $b \in B$. Thus $\overline{\alpha} = \mathrm{Inn}\ c$ on G and hence α is an inner automorphism of G. This shows that G has Property E. □

4. Polygonal products of abelian groups

In this section we consider a polygonal product G of four groups A_0, A_1, A_2 and A_3, amalgamating central subgroups H_1, H_2, H_3 and H_0, with trivial intersections, that is, $H_i, H_{i+1} \subset Z(A_i)$ and $H_i \cap H_{i+1} = 1$ (the subscripts i are taken modulo 4). Then the reduced polygonal product P of G is the polygonal product of H_0H_1, H_1H_2, H_2H_3 and H_3H_0, amalgamating subgroups H_1, H_2, H_3 and H_0. Then we have

$$P = (H_0 * H_2) \times (H_1 * H_3) \tag{4.1}$$

and

$$G = (((P *_{H_0H_1} A_0) *_{H_1H_2} A_1) *_{H_2H_3} A_2) *_{H_3H_0} A_3. \tag{4.2}$$

On the other hand, if we put $E = A_0 *_{H_1} A_1$, $F = A_3 *_{H_3} A_2$ and $S = H_0 * H_2$, then we have

$$G = E *_S F. \tag{4.3}$$

Lemma 4.1. *Let G be a polygonal product of groups A_0, A_1, A_2, and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections. If x, y are in the centers of vertex groups and $x \sim_G y$ then $x = y$.*

Proof. As in (4.3), let $G = E *_S F$, where $E = A_0 *_{H_1} A_1$, $F = A_3 *_{H_3} A_2$ and $S = H_0 * H_2$. Without loss of generality, we assume $1 \neq x \in Z(A_0)$ and $x \sim_G y$.

(1) Suppose $x \sim_G h$ for some $h \in S$. We may assume that h is cyclically reduced in $S = H_0 * H_2$. By Theorem 2.6 (1), there exist cyclically reduced elements $s_i \in S$ such that $x \sim_E s_1 \sim_F s_2 \sim_E \cdots \sim_E s_r \sim_F h$. Since the s_i are cyclically reduced in $S = H_0 * H_2$, $\|s_i\| = 1$ or $\|s_i\| = 2n$. Consider $x \sim_E s_1$, where $E = A_0 *_{H_1} A_1$. Since $x \in A_0$, by Theorem 2.6, we must have $\|s_1\| = 1$, that is, either $1 \neq s_1 \in H_0$ or $1 \neq s_1 \in H_2$. In both cases s_1 has the minimal length 1 in its conjugacy class in $E = A_0 *_{H_1} A_1$, since $H_1 \in Z(E)$ and $H_0 \cap H_1 = 1 = H_1 \cap H_2$. It follows from Theorem 2.6 that $s_1 \in A_0$ and $x \sim_{A_0} s_1$. Since $x \in Z(A_0)$, we have $x = s_1$. Now, consider $x = s_1 \sim_F s_2$. Then, as before, we have $x = s_2$. Inductively, we have $x = h$. Since $x \sim_G y$, $y \sim_G h$. Then, as in above, $y = h$. Therefore $x = y$.

(2) Suppose $x \not\sim_G h$ for any $h \in S$. Then x has the minimal length 1 in its conjugacy class in $G = E *_S F$. By Theorem 2.6, $x \sim_G y$ implies that $x, y \in E$ and $x \sim_E y$. If $x \in H_1 \subset Z(E)$, then $x = y$. If $x \notin H_1$, then x has the minimal length 1 in its conjugacy class in $E = A_0 *_{H_1} A_1$. Thus, by Theorem 2.6, $x, y \in A_0$ and $x \sim_{A_0} y$. Since $x \in Z(A_0)$, we have $x = y$. $\square$

Lemma 4.2. *Let G be a polygonal product of groups A_0, A_1, A_2, and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections.*
(1) *If $u^{-1}hu = h$ for all $h \in H_0H_1$, where $u \in G$, then $u \in A_0$.*
(2) *If $u^{-1}hu = h$ for all $h \in H_1H_2$, where $u \in G$, then $u \in A_1$.*
(3) *If $u^{-1}hu = h$ for all $h \in H_2H_3$, where $u \in G$, then $u \in A_2$.*
(4) *If $u^{-1}hu = h$ for all $h \in H_3H_0$, where $u \in G$, then $u \in A_3$.*

Proof. We only prove (1), since the others are similar. Let $E = A_0 *_{H_1} A_1$ and $F = A_3 *_{H_3} A_2$. Then $G = E *_S F$, where $S = H_0 * H_2$.

Suppose $\|u\| \geq 1$ and $u = f_1e_1 \cdots$, where $f_1 \in F \backslash S$ and $e_1 \in E \backslash S$. Choose $1 \neq h \in H_1$. Then $\|u^{-1}hu\| = 2\|u\| + 1$. Hence $u^{-1}hu \neq h$. Thus we must have $u = e_1f_1 \cdots$.

Suppose $u = e_1f_1 \cdots$ and $\|u\| \geq 2$, where $e_1 \in E \backslash S$ and $f_1 \in F \backslash S$. Since $H_1 \subset Z(E)$, for $1 \neq h \in H_1$, we have $u^{-1}hu = \cdots f_1^{-1}e_1^{-1}he_1f_1 \cdots = \cdots f_1^{-1}hf_1 \cdots$. Thus $\|u^{-1}hu\| = 2\|u\| - 1 \geq 3$. Hence $u^{-1}hu \neq h$. Therefore $u = e_1 \in E$ if $u^{-1}hu = h$ for all $h \in H_1$.

We shall show that if $u \in E = A_0 *_{H_1} A_1$ and $u^{-1}hu = h$ for all $h \in H_0$, then $u \in A_0$.

Suppose $u = c_1a_1 \cdots \in E = A_0 *_{H_1} A_1$ where $c_i \in A_1 \backslash H_1$ and $a_i \in A_0 \backslash H_1$. Let $1 \neq h \in H_0$. Then $\|u^{-1}hu\| = 2\|u\| + 1$. Hence $u^{-1}hu \neq h$. Thus we must have $u = a_1c_1 \cdots$.

Suppose $\|u\| \geq 2$ and $u = a_1c_1 \cdots \in E = A_0 *_{H_1} A_1$, where $a_i \in A_0 \backslash H_1$ and $c_i \in A_1 \backslash H_1$. Since $H_0 \subset Z(A_0)$ for $1 \neq h \in H_0$, we have $u^{-1}hu = \cdots c_1^{-1}a_1^{-1}ha_1c_1 \cdots = \cdots c_1^{-1}hc_1 \cdots$. Thus $\|u^{-1}hu\| = 2\|u\| - 1 \geq 3$. Hence $u^{-1}hu \neq h$. This shows that if $u \in E$ and $u^{-1}hu = h$ for all $h \in H_0$ then $u = a_1 \in A_0$.

Therefore if, for $u \in G$, $u^{-1}hu = h$ for all $h \in H_0H_1$ then $u \in A_0$. $\square$

To prove that our polygonal product G in (4.2) has Property E, we begin the following simple observation.

Lemma 4.3. *If A and B have Property E, then $G = A \times B$ has Property E.*

Proof. Let $\alpha(g) = k_g^{-1} g k_g$ be a conjugating endomorphism of G. Since $G = A \times B$, we may assume $k_g = (x_a, y_b)$ for $g = (a, b)$, where $a, x_a \in A$ and $b, y_b \in B$. Hence $\alpha(a, b) = (x_a^{-1} a x_a, y_b^{-1} b y_b)$. Define $\alpha_A(a) = x_a^{-1} a x_a$ and $\alpha_B(b) = y_b^{-1} b y_b$. Then α_A, α_B are conjugating endomorphisms of A, B, respectively. Since A and B have Property E, there exist $x \in A$ and $y \in B$ such that $\alpha_A = \text{Inn } x$ and $\alpha_B = \text{Inn } y$. Then $\alpha = \text{Inn } (x, y)$. Hence $G = A \times B$ has Property E. $\square$

Theorem 4.4. *Let G be a polygonal product of groups A_0, A_1, A_2, and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections. Then G has Property E.*

Proof. Let P be the reduced polygonal product of H_0H_1, H_1H_2, H_2H_3, and H_3H_0, amalgamating H_1, H_2, H_3, and H_0. Then $P = (H_0 * H_2) \times (H_1 * H_3)$. Since free products of nontrivial groups have Property E by Theorem 2.5, $H_0 * H_2$ and $H_1 * H_3$ have Property E. Then, by Lemma 4.3, $P = (H_0 * H_2) \times (H_1 * H_3)$ has Property E.

Let $P_i = (\cdots(P *_{H_0H_1} A_0) *_{H_1H_2} \cdots) *_{H_iH_{i+1}} A_i$ for $i = 0, 1, 2, 3$. Then $G = P_3$. Inductively, by assuming that P_0, P_1, and P_2 have Property E, we shall show that $G = P_3 = P_2 *_{H_3H_0} A_3$ has Property E. We prove that conditions in Theorem 3.1 are satisfied.

(C1) Suppose $u \in P_2$ and $u^{-1}hu = h$ for all $h \in H_3H_0$. Since P_2 is a polygonal product of A_0, A_1, A_2, and H_3H_0, amalgamating subgroups H_1, H_2, H_3, and H_0, by Lemma 4.2 $u \in H_3H_0$.

(C2) Choose $1 \neq a \in H_1$. Then, by Lemma 4.1, we have $\{a\}^{P_2} \cap H_3H_0 = \emptyset$. Suppose $u \in P_2$ and $u^{-1}au = h'ah$ for $h', h \in H_3H_0$. Hence $a \sim_{P_2} ahh'$. Consider $P_2 = E *_S F$, where $E = A_0 *_{H_1} A_1$, $F = H_0H_3 *_{H_3} A_2$ and $S = H_0 * H_2$. Since $a \in H_1 \subset E$, a has the minimal length 1 in its conjugacy class in P_2. Hence, by Theorem 2.6, $ahh' \in E$ and $a \sim_E ahh'$. Since $a \in H_1 \subset Z(E)$, we have $a = ahh'$. Hence $h' = h^{-1}$ as required.

Thus, by Theorem 3.1, G has Property E. $\square$

By Theorem 2.4 we have the following:

Theorem 4.5. *Let G be a polygonal product of groups A_0, A_1, A_2, and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections. If G is finitely generated and conjugacy separable then $Out(G)$ is residually finite.*

Since polygonal products of polycyclic-by-finite groups A_0, A_1, A_2, and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections, are conjugacy separable [9, 11], by Theorem 2.4 we have the followings:

Corollary 4.6. *Let G be a polygonal product of polycyclic-by-finite groups A_0, A_1, A_2 and A_3, amalgamating central subgroups H_1, H_2, H_3, and H_0, with trivial intersections. Then $Out(G)$ is residually finite.*

Corollary 4.7. *Let G be a polygonal product of finitely generated abelian groups A_0, A_1, A_2, and A_3, amalgamating subgroups H_1, H_2, H_3, and H_0, with trivial intersections. Then $Out(G)$ is residually finite.*

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Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea
E-mail address: gskim@yu.ac.kr