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SOME RESULTS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of preconvexity neighborhoods, c-concave functions. We study some properties for c-convex functions, and characterize c-convex functions and c-concave functions by using the preconvexity neighborhoods.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set X and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [4] yields a topological space.

In this paper, we study some basic properties on preconvexity spaces and c-convex functions. We define the notion of preconvex neighborhood in the same way as a proximity neighborhood is defined by a proximity, and characterize c-convex functions and c-concave functions by using the preconvex neighborhoods.

Definition 1.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

(1) If $A \subseteq B$, then $A\sigma B$.
(2) If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
(3) If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
(4) If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space. A convexity is a reflexive and transitive relation. In a preconvexity space (X, σ), $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset A. A is called convex[1] if $G(A) = A$.

Theorem 1.2 ([1]). *For a preconvexity space* (X, σ),

(1) $G(\emptyset) = \emptyset$.

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(2) $A \subseteq G(A)$ *for all* $A \subseteq X$.
(3) *If* $A \subseteq B$, *then* $G(A) \subseteq G(B)$.
(4) $G(G(A)) = G(A)$ *for* $A \subseteq X$.

Theorem 1.3 ([1]). *If σ is a preconvexity on X and $A \subseteq X$, then $G(A) = \cap\{C : G(C) = C$ and $A \subseteq C\}$.*

Theorem 1.4 ([1]). *Let σ be a preconvexity on X and $A, B \subseteq X$. Then*
(1) $A\sigma B$ *if and only if* $A \subseteq G(B)$.
(2) $A\sigma B$ *if and only if* $G(A)\sigma G(B)$.

In [3] Kay and Womble introduced the following definition:

A family $\mathcal{C}$ of subsets of a set X is termed a convexity structure for X, and the pair $(X, \mathcal{C})$ is called a convexity space, whenever the following two conditions hold:
(a) $\emptyset$ and X belong to $\mathcal{C}$.
(b) $\cap_{F \in \mathcal{F}} F \in \mathcal{C}$ for each subfamily $\mathcal{F} \subseteq \mathcal{C}$.

If in a preconvexity space (X, σ), we take $\mathcal{C} = \{C : C \subseteq X$ and $G(C) = C\}$, then $\mathcal{C}$ is a convexity structure for X called the convexity structure determined [3] by σ and $(X, \mathcal{C})$ is a convexity space.

A function $h : P(X) \to P(X)$ is called a hull operator [3] on X if the following conditions are satisfied:
(1) $h(A) \supseteq A$.
(2) $B \supseteq A$ implies $h(B) \supseteq h(A)$.
(3) $h(h(A)) = h(A)$.
(4) $h(\emptyset) = \emptyset$.

Let h be a hull operator on X. A preconvexity σ is said to be associated with the convexity space (X, h) if $x \in h(A)$ implies $x\sigma A$. If in addition, $x\sigma A$ implies $x \in h(A)$, then σ is said to be compatible [1] with (X, h).

Definition 1.5 ([1]). Let $(X, h_1), (Y, h_2)$ be convexity spaces. A function $f : X \to Y$ is said to be a convex function on X if for each $A \subseteq X$, $f(h_1(A)) \subseteq h_2(f(A))$. The function f is said to be concave if for each $A \subseteq X$, $f(h_1(A)) \supseteq h_2(f(A))$.

Definition 1.6 ([1]). Let σ_1, σ_2 be two preconvexities on the convexity spaces (X, h_1) and (Y, h_2), respectively. A function $f : X \to Y$ is said to be *c*-convex if $A\sigma_1 B$ implies $f(A)\sigma_2 f(B)$. The function f is a c-isomorphism or convexeomorphism if and only if f is one-to-one, onto and both f and f^{-1} are *c*-convex.

2. Main results

Lemma 2.1. *Let (X, σ) be a preconvexity space and $A, B \subseteq X$. Then $A \not\sigma B$ if and only if there is an element $x \in A$ such that $x \not\sigma B$.*

Proof. It is obvious from Theorem 1.4(1). □

Lemma 2.2. *Let (X,σ) be a preconvexity space. Then for all $A \subseteq X$, $G(A)\sigma A$.*

Proof. From $G(A)\sigma G(A)$ and Definition 1.1(3), it follows the result. □

Definition 2.3. Let (X,σ) be a preconvexity space and $A \subseteq X$. A is called a preconvexity neighborhood of x, denoted by $x \lhd A$, if $x \not\sigma (X-A)$. A is called a preconvexity neighborhood of B, denoted by $B \lhd A$, if $B \not\sigma (X-A)$.

In a convexity space $\mathcal{C}$, a subset A in X is called a convexity neighborhood of x if there exists a $F \in \mathcal{C}$ such that $x \in X - F \subseteq A$.

Theorem 2.4. *Let (X,σ) be a preconvexity space and $A \subseteq X$. For $x \in X$, A is a preconvexity neighborhood of x if and only if it is a convexity neighborhood of x in the convexity space determined by σ.*

Proof. Let $x \lhd A$; then by Definition 2.3, $x \lhd A$ if and only if $x \not\sigma(X-A)$ if and only if $x \notin G(X-A)$ if and only if $x \in X - G(X-A) \subseteq A$. Thus we get the result because $G(X-A)$ is a convex set. □

Theorem 2.5. *Let (X,σ) be a preconvexity space and $A \subseteq X$. If A is a convexity neighborhood of $B \subseteq X$, then it is a preconvexity neighborhood of B in the convexity space determined by σ.*

Proof. Let A be a convexity neighborhood of $B \subseteq X$; then there is a convex set $F \subseteq X$ such that $B \subseteq X - F \subseteq A$. If $x\sigma(X-A)$ for all $x \in B$, then $B\sigma(X-A)$. Thus $B \subseteq G(X-A) \subseteq G(F) = F$. This is a contradiction. Hence $x \not\sigma(X-A)$ for some $x \in B$, and so $B \not\sigma(X-A)$. □

In the following example we can show that the converse in Theorem 2.5 is not always true.

Example 2.6. Let $X = \{a,b,c,d\}$ and define a relation σ on $P(X)$ given by $\{d\}\sigma\{a\}$, $\{d\}\sigma\{c\}$, $X\sigma\{b\}$ and $X\sigma\{a,c,d\}$, and the general condition that $A\sigma B$ if $A \subseteq B$. Then the relation σ is a preconvexity on X and $G(\emptyset) = \emptyset$, $G(X) = X$, $G(\{d\}) = \{d\}$, $G(\{a,d\}) = \{a,d\}$, $G(\{c,d\}) = \{c,d\}$. The convexity structure determined by σ is $\mathcal{C} = \{X, \emptyset, \{d\}, \{a,d\}, \{c,d\}\}$. Let $A = \{a,d\}$ and $B = \{a,b,d\}$; since $A = \{a,d\} \not\sigma \{c\} = X - B$, B is a preconvexity neighborhood of A but it is not a neighborhood of A in the convexity space $\mathcal{C}$.

Theorem 2.7. *For a preconvexity space (X,σ) if $A \lhd B$ and $B \subseteq D$, then $A \lhd D$.*

Proof. If $A \not\lhd D$, then $A\sigma(X-D)$. Say $B \subseteq D$; then it is $A \not\lhd B$ from the transitive property of preconvexity. □

Theorem 2.8. *Let $f : X \to Y$ be a bijective function on two preconvexities (X,σ) and (Y,μ). Then f is c-convex if and only if for $C, D \subseteq Y$ whenever $C \lhd D$, $f^{-1}(C) \lhd f^{-1}(D)$.*

Proof. ($\Rightarrow$) Assume $f^{-1}(C) \not\lhd f^{-1}(D)$ for $C, D \subseteq Y$. Then $f^{-1}(C)\sigma(X - f^{-1}(D))$. Since f is surjective and c-convex, it follows $C\mu(Y - D)$. Hence $C \not\lhd D$.

($\Leftarrow$) Assume $f(A) \not\mu f(B)$ for $A, B \subseteq X$. Then by Lemma 2.1, there exists an element $y \in f(A)$ such that $y \not\mu f(B)$. Let $f(x) = y$ for some $x \in A$. We can say $y \lhd (Y - f(B))$ because of $y \not\mu f(B) = Y - (Y - f(B))$, and so $f^{-1}(y) \lhd (X - B)$. Now we get $x = f^{-1}(y) \not\sigma B$ from definition of the preconvexity neighborhood. Hence $A \not\sigma B$. □

Theorem 2.9. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then f is c-convex if and only if $G(f(G(A))) = G(f(A))$ for all $A \subseteq X$.*

Proof. ($\Rightarrow$) Let f be c-convex and $y \in G(f(G(A)))$; then $y\mu f(G(A))$. Since f is c-convex, from Lemma 2.2, we get $f(G(A))\mu f(A)$. The transitive property gives $y\mu f(A)$. Thus $G(f(G(A))) \subseteq G(f(A))$. The other inclusion is obvious.

($\Leftarrow$) Suppose that $G(f(G(A))) = G(f(A))$ for all $A \subseteq X$. Let $A\sigma B$ for $A, B \subseteq X$; then by Theorem 1.4(1) and hypothesis, we get the following relationship:

$f(A) \subseteq G(f(A)) \subseteq G(f(G(B))) = G(f(B))$.

By Theorem 1.4(1), $f(A)\mu f(B)$. □

From Theorem 2.9, we get the following corollary:

Corollary 2.10 (Theorem 12 [1]). *Let σ_1 and σ_2 be compatible preconvexities on the convexity spaces (X, h_1) and (Y, h_2), respectively. Then $f : X \to Y$ is convex if and only if f is c-convex.*

Theorem 2.11. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following are equivalent:*

(1) *f is c-convex.*
(2) *For $A, B \subseteq X$ if $A\sigma B$, then $f(G(A))\mu G(f(B))$.*
(3) *$f(G(A))\mu f(A)$ for $A \subseteq X$.*

Proof. (1) $\Rightarrow$ (2) Let $A\sigma B$ for $A, B \subseteq X$; then by Theorem 1.4(2) and Lemma 2.2, $G(A)\sigma G(B)\sigma B$. Since f is c-convex, $f(G(A))\mu f(B)\mu G(f(B))$.

(2) $\Rightarrow$ (3) It is obvious from Lemma 2.2.

(3) $\Rightarrow$ (1) Let $A\sigma B$ for $A, B \subseteq X$; then $A \subseteq G(B)$, and so $f(A) \subseteq f(G(B))$. Hence f is c-convex by Definition 1.1(1) and condition (3). □

Definition 2.12. Let σ_1, σ_2 be two preconvexities on the preconvexity spaces X, Y, respectively. A function $f : X \to Y$ is said to be c-concave if for $C, D \subseteq Y$ whenever $C\sigma_2 D$, $f^{-1}(C)\sigma_1 f^{-1}(D)$.

Theorem 2.13. *Let $f : X \to Y$ be a bijective function on two preconvexity spaces (X, σ) and (Y, μ). Then f is c-concave if and only if for $A, B \subseteq X$ whenever $A \lhd B$, $f(A) \lhd f(B)$.*

Proof. ($\Rightarrow$) Let $f(A) \not\lhd f(B)$ for $A, B \subseteq X$; then $f(A)\mu(Y - f(B))$. Since f is c-concave, we have $A\sigma(X - B)$, and so $A \not\lhd B$.

($\Leftarrow$) Let $f^{-1}(C) \not\sigma f^{-1}(D)$; then there exists an element $x \in f^{-1}(C)$ such that $x \not\sigma f^{-1}(D)$, and let $f(x) = y$. It follows $x \lhd (X - f^{-1}(D))$, and so $y \lhd (Y - D)$. By Lemma 2.1, we get $C \not\mu D$. □

Theorem 2.14. *Let σ_1 and σ_2 be compatible preconvexities on the convexity spaces (X, h_1) and (Y, h_2), respectively and let $f : X \to Y$ be a bijective function. Then f is c-concave if and only if it is concave.*

Proof. ($\Rightarrow$) Let $y \in h_2(f(A))$ for $A \subseteq X$; then $y\sigma_2 f(A)$. Since f is c-concave and injective, it follows $f^{-1}(y)\sigma_1 A$. Since σ_1 is compatible with h_1, we have $f^{-1}(y) \in h_1(A)$ and so $y \in f(h_1(A))$.

($\Leftarrow$) Let $f^{-1}(C) \not\sigma_1 f^{-1}(D)$ for each $C, D \subseteq Y$; then there exists $x \in f^{-1}(C)$ such that $x \not\sigma_1 f^{-1}(D)$, and so $x \notin h_1(f^{-1}(D))$. Since f is concave and surjective, $f(x) \notin h_2(f(f^{-1}(D))) = h_2(D)$. Thus we get $f(x) \not\sigma_2 D$, and so $C \not\sigma_2 D$. □

Corollary 2.15. *Let $f : X \to Y$ be a bijective function on two preconvexity spaces (X, σ) and (Y, μ). Then f is a convexeomorphism if and only if f is c-convex and c-concave.*

Proof. From Definition 1.6, we get the result. □

Theorem 2.16. *Let $f : X \to Y$ be a function on two convexity spaces (X, h_1) and (Y, h_2). Then f is concave if and only if $f(h_1(A)) = h_2(f(h_1(A)))$ for $A \subseteq X$.*

Proof. ($\Rightarrow$) Let f be a concave function; then $h_2(f(h_1(A))) \supseteq f(h_1(A)) \supseteq h_2(f(h_1(A)))$ for $A \subseteq X$. Hence we get $f(h_1(A)) = h_2(f(h_1(A)))$.

($\Leftarrow$) It is obvious. □

Theorem 2.17. *Let $f : X \to Y$ be a function on two convexity spaces (X, C_1) and (Y, C_2), then f is concave if and only if for each $A \in C_1$, $f(A) \in C_2$.*

Proof. It is obvious from Theorem 2.16. □

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