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Implicative Closure Operators

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Abstract

In this paper, we investigate the properties of implicative closure operators on the stsc-quantale L. We find implicative closure operators induced by a function.

Key words : stsc-quantales, implicative closure spaces, C-maps

1. Introduction and preliminaries

Closure operators play an important role in topological spaces, lattices, Boolean algebras, convex sets, deductive systems [1-4, 8,12]. Recently, Gerla *et al.* [3,4] studied fuzzy closure operators as extensions of closure operators. Bělohlávek [2] and Rodriguez *et al.* [12] introduced fuzzy closure spaces and implicative closure spaces, respectively, as a sense that if λ_1 is almost a subset λ_2, then the closure of λ_1 is almost a subset of the closure of λ_2. They have been developed in many view points.

On the other hand, quantales were introduced by Mulvey [10,11] as the non-commutative generalization of the lattice of open sets in topological spaces. Recently, quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [5], which supports part of foundation of theoretic computer science. Recently, Höhle [7] developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices.

In this paper, we investigate the properties of implicative closure operators on the stsc-quantale L. We find implicative closure operators induced by a function. We give their examples.

Definition 1.1. [7, 10,11] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(Q2) $(L, \odot)$ is a commutative semigroup;

(Q3) $a = a \odot 1$, for each $a \in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Remark 1.2. [7](1) A completely distributive lattice is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is, $x \odot y \leq z$ iff $x \leq (y \to z)$.

In this paper, we always assume that $(L, \leq, \odot, ^*)$ is a stsc-quantale with an order-reversing involution* defined by $x^* = x \to 0$.

Lemma 1.3. [7,13] Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(5) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$

(6) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$.

(7) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(8) $x \odot (x \to y) \leq y$ and

$$x \to y \leq (y \to z) \to (x \to z).$$

(9) $(x \to y) \odot z \leq x \to (y \odot z)$.

(10) $x \to y \leq (x \odot z) \to (y \odot z)$.

(11) $x \to y = 1$ iff $x \leq y$.

(12) $x \to y = y^* \to x^*$.

Definition 1.4. [1,2,8] A function $E : X \times X \to L$ is called an $\odot$-equivalence relation if it satisfies the following conditions:

(E1) $E(x, x) = 1$,

(E2) $E(x, y) = E(y, x)$,

(E3) $E(x, y) \odot E(y, z) \leq E(x, z)$.

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All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x \in X$, $\lambda, \mu \in L^X$ and $\alpha \in L$,

(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$;

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$;

(3) $\overline{1}(x) = 1$, $\overline{\alpha}(x) = \alpha$ and $\overline{0}(x) = 0$;

(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$ and $(\lambda \to \alpha)(x) = \lambda(x) \to \alpha$;

(5) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$.

We denote $[\lambda, \mu] = \bigwedge_{x \in X}(\lambda(x) \to \mu(x))$, for $\lambda, \mu \in L^X$.

2. Implicative closure operators

Definition 2.1. [12] An operator $C : L^X \to L^X$ is called an implicative closure operator on X if it satisfies the following conditions: for each $\alpha \in L$ and $\lambda, \mu \in L^X$,

(C1) $\mu \leq C(\mu)$,

(C2) if $\lambda \leq \mu$, then $C(\lambda) \leq C(\mu)$,

(C3) $C(C(\mu)) \leq C(\mu)$,

(C4) $C(\alpha \odot \mu) \geq \alpha \odot C(\mu)$.

The pair (X, C) is an implicative closure space. Let C_1 and C_2 be implicative closure operators. We say C_1 is coarser than C_2 iff $C_1(\lambda) \leq C_2(\lambda)$ for each $\lambda \in L^X$.

Let (X, C_1) and (Y, C_2) be implicative closure spaces. A function $f : X \to Y$ is called a C-map if $f(C_1(\mu)) \leq C_2(f(\mu))$ for each $\mu \in L^X$.

Remark 2.2. We define implicative closure operators $C_1, C_2 : L^X \to L^X$ as follows:

$$C_1(\lambda) = \lambda, \quad C_2(\lambda) = \overline{1}, \ \forall \lambda \in L^X.$$

If C is an implicative closure operator, then $C_0 \leq C \leq C_1$.

Theorem 2.3. (1) If $C : L^X \to L^X$ satisfies the conditions (C2), then the condition (C4) holds iff C satisfies (C4)* $C(k \to \lambda) \leq k \to C(\lambda)$ for $k \in L$ and $\lambda \in L^X$.

(2) If $C : L^X \to L^X$ satisfies the conditions (C2) and (C4), then $[\lambda_1, \lambda_2] \leq [C(\lambda_1), C(\lambda_2)]$.

Proof. (1) ($\Rightarrow$) Since $k \odot (k \to \lambda) \leq \lambda$ from Lemma 1.3(8), by (C2) and (C4), we have

$$\begin{aligned} k \odot C(k \to \lambda) &\leq C\Big(k \odot (k \to \lambda)\Big) \\ &\leq C(\lambda). \end{aligned}$$

It implies $C(k \to \lambda) \leq k \to C(\lambda)$.

($\Leftarrow$) Since $\mu \leq k \to (k \odot \mu)$, we have

$$\begin{aligned} C(\mu) &\leq C(k \to (k \odot \mu)) \\ &\leq k \to C(k \odot \mu). \end{aligned}$$

It implies $k \odot C(\lambda) \leq k \odot C(\lambda)$.

(2) Since $\lambda_1 \odot [\lambda_1, \lambda_2] \leq \lambda_2$, by (C2) and (C4), we have

$$\begin{aligned} [\lambda_1, \lambda_2] \odot C(\lambda_1) &\leq C([\lambda_1, \lambda_2] \odot \lambda_1) \\ &\leq C(\lambda_2). \end{aligned}$$

Hence $[\lambda_1, \lambda_2] \leq [C(\lambda_1), C(\lambda_2)]$.

□

Remark 2.4. (1) If $C : L^X \to L^X$ satisfies the conditions (C1), (C2) and (C3), then C is a fuzzy closure operator in Biacino and Gerla's sense [3,4].

(2) If $C : L^X \to L^X$ satisfies the conditions (C1), (C3) and (C2)*,

(C2)* $[\lambda_1, \lambda_2] \leq [C(\lambda_1), C(\lambda_2)]$,

then C is a fuzzy closure operator in a Bělohlávek's sense [1,2]. In (C2)*, if $[\lambda_1, \lambda_2] = 1$ implies $[C(\lambda_1), C(\lambda_2)] = 1$, then the condition (C2) holds. The condition (C2)* is interpreted if λ_1 is almost a subset λ_2, then the closure of λ_1 is almost a subset of the closure of λ_2.

(3) If C is an implicative closure operator, by Theorem 2.3(2), then C is a fuzzy closure operator in a Bělohlávek's sense [1,2].

Example 2.5. Let $(L = [0, 1], \odot)$ be a stsc-quantale defined by

$$x \odot y = (x + y - 1) \vee 0,$$

$$x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{a, b\}$ and $\mu, \lambda_1, \lambda_2 \in L^X$ as follows:

$$\mu(a) = 0.4, \mu(b) = 0.7, \quad \lambda_1(a) = 0.5, \lambda_1(b) = 0.5,$$
$$\lambda_2(a) = 0.2, \lambda_2(b) = 0.5.$$

Define $C : L^X \to L^X$ as follows:

$$C(\lambda) = \begin{cases} \mu & \text{if } \lambda \leq \mu, \\ \overline{1} & \text{otherwise.} \end{cases}$$

(1) Since C satisfies the conditions (C1), (C2) and (C3), then C is a fuzzy closure operator in a sense Biacino and Gerla [3,4].

(2) Since $0.7 = [\lambda_1, \lambda_2] \not\leq [C(\lambda_1), C(\lambda_2)] = [\overline{1}, \mu] = 0.4$, C is not a fuzzy closure operator in a Bělohlávek's sense.

(3) Since $\mu = C(0.7 \odot \lambda_1) \not\geq 0.7 \odot C(\lambda_1) = \overline{0.7}$, C is not an implicative closure operator.

Theorem 2.6. Let $E \in L^{X \times X}$ be an $\odot$-equivalence relation. Define an operator $C : L^X \to L^X$ as follows:

$$C(\lambda)(x) = \bigvee_{z \in X} (\lambda(z) \odot E(z, x)).$$

Then C is an implicative closure operator on X.

Proof. (C1) It follows from $C(\lambda)(x) \leq \lambda(x) \odot E(x,x) = \lambda(x)$.

(C2) and (C4) are easily proved.

(C3) It follows from:

$$\begin{aligned} & C(C(\lambda))(x) \\ &= \bigvee_{z\in X}(C(\lambda)(z) \odot E(z,x)) \\ &= \bigvee_{z\in X}\Big(\bigvee_{y\in X}(\lambda(y) \odot (E(y,z) \odot E(z,x))\Big) \\ &= \bigvee_{y\in X}\Big(\lambda(y) \odot \bigvee_{z\in X}(E(y,z) \odot E(z,x)))\Big) \\ &\leq \bigvee_{y\in X}(\lambda(y) \odot E(y,x)) \\ &= C(\lambda)(x). \end{aligned}$$

Hence C is an implicative closure operator on X.

□

Theorem 2.7. For $\mu \in L^X$, we define an operator $C_\mu : L^X \to L^X$ as follows:

$$C_\mu(\lambda)(x) = [\lambda, \mu] \to \mu(x).$$

Then the following properties.

(1) C_μ is an implicative closure operator on X.

(2) $I_\mu(\lambda^*)^* = C_{\mu^*}(\lambda)$, where $I_\mu(\lambda)(x) = [\mu, \lambda] \odot \mu(y)$ for all $\lambda \in L^X$.

Proof. (1) (C1) Since $\lambda \odot [\lambda, \mu] \leq \mu$, then

$$\lambda \leq [\lambda, \mu] \to \mu = C_\mu(\lambda).$$

(C2) If $\lambda_1 \leq \lambda_2$, then $[\lambda_1, \mu] \geq [\lambda_2, \mu]$. So,

$$C_\mu(\lambda_1) = [\lambda_1, \mu] \to \mu \leq [\lambda_2, \mu] \to \mu = C_\mu(\lambda_2).$$

(C3) Since

$$\begin{aligned} & [\lambda, \mu] \to \mu \leq [\lambda, \mu] \to \mu \\ & \Leftrightarrow [\lambda, \mu] \odot ([\lambda, \mu] \to \mu) \leq \mu, \end{aligned}$$

then $[\lambda, \mu] \leq [[\lambda, \mu] \to \mu, \mu]$. It implies

$$\begin{aligned} C_\mu(C_\mu(\lambda)) &= [C_\mu(\lambda), \mu] \to \mu \\ &= [([\lambda, \mu] \to \mu), \mu] \to \mu \\ &\leq [\lambda, \mu] \to \mu \\ &= C_\mu(\lambda). \end{aligned}$$

(C4) Since $[k \odot \lambda, \mu] = k \to [\lambda, \mu]$ from Lemma 1.3(3)and (7), we have

$$\begin{aligned} & k \odot [k \odot \lambda, \mu] \odot ([\lambda, \mu] \to \mu) \\ &= k \odot (k \to [\lambda, \mu]) \odot ([\lambda, \mu] \to \mu) \\ &\leq [\lambda, \mu]) \odot ([\lambda, \mu] \to \mu) \leq \mu. \end{aligned}$$

It implies

$$\begin{aligned} k \odot C_\mu(\lambda) &= k \odot ([\lambda, \mu] \to \mu) \leq [k \odot \lambda, \mu] \to \mu \\ &= C_\mu(k \odot \lambda). \end{aligned}$$

(2) By Lemma 1.3(7)and (12), we have

$$\begin{aligned} I_\mu(\lambda^*)^* &= ([\mu, \lambda^*] \odot \mu)^* = [\mu, \lambda^*] \to \mu^* \\ &= [\lambda, \mu^*] \to \mu^* = C_{\mu^*}(\lambda). \end{aligned}$$

□

Example 2.8. Let $(L = [0,1], \odot)$ be a stsc-quantale defined by

$$x \odot y = (x + y - 1) \vee 0,$$

$$x \to y = (1 - x + y) \wedge 1.$$

Let $X = \{a, b\}$ and $\mu, \lambda, \rho \in L^X$ as follows:

$$\begin{aligned} & \mu(a) = 0.5, \mu(b) = 0.8, \ \ \lambda(a) = 0.9, \lambda(b) = 0.3, \\ & \rho(a) = 0.7, \rho(b) = 0.4. \end{aligned}$$

Since $[\mu, \lambda] = 0.5$ and $[\lambda^*, \mu^*] = 0.5$, we have

$$\begin{aligned} I_\mu(\lambda)^* &= ([\mu, \lambda] \odot \mu)^* = 0.5 \to \mu^* \\ &= [\lambda^*, \mu^*] \to \mu^* = C_{\mu^*}(\lambda^*). \end{aligned}$$

Lemma 2.9. Let $f : X \to Y$ be a function. Then we have the following properties.

(1) $f^{\leftarrow}(k \to \rho) = k \to f^{\leftarrow}(\rho)$ for each $\rho \in L^Y$.

(2) $f^{\to}(k \to \lambda) \leq k \to f^{\to}(\lambda)$ for each $\lambda \in L^X$.

(3) $[f^{\to}(\lambda), \rho] = [\lambda, f^{\leftarrow}(\rho)]$ for each $\lambda \in L^X$ and $\rho \in L^Y$.

(4) $[\nu, \rho] \leq [f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)]$ for each $\nu, \rho \in L^Y$. If f is surjective, the equality holds.

(5) $[\lambda, \mu] \leq [f^{\to}(\lambda), f^{\to}(\mu)]$ for each $\lambda, \mu \in L^Y$. If f is injective, the equality holds.

Proof. (1) It is obvious.

(2)

$$\begin{aligned} f^{\to}(k \to \lambda)(y) &= \bigvee_{x\in f^{-1}(\{y\})}(k \to \lambda)(x) \\ &\leq k \to \bigvee_{x\in f^{-1}(\{y\})} \lambda(x) \\ &= k \to f^{\to}(\lambda)(y). \end{aligned}$$

(3) We have $[f^{\to}(\lambda), \rho] = [\lambda, f^{\leftarrow}(\rho)]$ from

$$\begin{aligned} & [f^{\to}(\lambda), \rho] \\ &= \bigwedge_{y\in Y}(f^{\to}(\lambda)(y) \to \rho(y)) \\ &= \bigwedge_{y\in Y}\Big(\bigvee_{x\in f^{-1}(\{y\})} \lambda(x) \to \rho(f(x))\Big) \\ &= \bigwedge_{y\in Y} \bigwedge_{x\in f^{-1}(\{y\})}\Big(\lambda(x) \to \rho(f(x))\Big) \\ &= \bigwedge_{x\in X}\Big(\lambda(x) \to \rho(f(x))\Big) \\ &= [\lambda, f^{\leftarrow}(\rho)]. \end{aligned}$$

(4) and (5) are similarly proved.

□

Example 2.10. Let $(L = [0,1], \odot)$ be an operation $\odot$ defined as

$$x \odot y = \begin{cases} 0 & \text{if } x + y \leq 1, \\ x \wedge y & x + y > 1. \end{cases}$$

$$x \rightarrow y = \begin{cases} (1-x) \vee y & \text{if } x > y, \\ 1 & x \leq y. \end{cases}$$

Since $\odot$ is a left-continuous t-norm, by remark 1.2(2), $(L = [0,1], \odot)$ is a stsc-quantale. Let $f : N \rightarrow \{y\}$ be a function and $\lambda(n) = 0.3 - \frac{1}{n}$.

$$\begin{aligned} f^{\rightarrow}(0.3 \rightarrow \lambda)(y) &= \bigvee_{n \in N}(0.3 \rightarrow \lambda(n)) \\ &= \bigvee_{n \in N}(0.7 \vee (0.3 - \tfrac{1}{n})) = 0.7. \end{aligned}$$

$$0.3 \rightarrow f^{\rightarrow}(\lambda)(y) = 0.3 \rightarrow \bigvee_{n \in N} \lambda(n) = 1.$$

In general, $f^{\rightarrow}(k \rightarrow \lambda) \neq k \rightarrow f^{\rightarrow}(\lambda)$.

Example 2.11. Let $(L = [0,1], \odot)$ be a stsc-quantale defined as in Example 2.8. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ and $f : X \rightarrow Y$ be a function defined by

$$f(a) = f(b) = x, \ \ f(c) = y.$$

Put $\lambda, \mu \in L^X$ and $\nu, \rho \in L^Y$ as follows:

$$\lambda(a) = 0.4, \ \lambda(b) = 0.7, \ \lambda(c) = 0.3,$$

$$\mu(a) = 0.8, \ \mu(b) = 0.5, \ \mu(c) = 0.6,$$

$$\nu(x) = 0.5, \ \nu(y) = 0.4, \ \nu(z) = 0.6,$$

$$\rho(x) = 0.6, \rho(y) = 0.4, \ \rho(z) = 0.3.$$

(1) $[f^{\rightarrow}(\lambda), \nu] = [\lambda, f^{\leftarrow}(\nu)] = 0.8$.

(2) Since f is not injective, we have $0.8 = [\lambda, \mu] < [f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu)] = 1$.

(3) Since f is not surjective, we have $0.7 = [\nu, \rho] < [f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)] = 1$.

Theorem 2.12. Let C_i be implicative closure operators on X for each $i \in \Gamma$. Define $C = \bigwedge_{i \in \Gamma} C_i$ is an implicative closure operator on X

Proof. (C1), (C2) and (C4) are easily proved.
(C3)

$$\begin{aligned} C(C(\lambda)) &= C(\textstyle\bigwedge_{i \in \Gamma} C_i(\lambda)) \leq \bigwedge_{i \in \Gamma} C_i(C_i(\lambda)) \\ &\leq \textstyle\bigwedge_{i \in \Gamma} C_i(\lambda) = C(\lambda). \end{aligned}$$

□

Theorem 2.13. Let $f : X \rightarrow Y$ be a function. For $\rho \in L^Y$ and $C_\rho \in (L^Y)^{L^Y}$, we define an operator $f^{\Leftarrow} : (L^Y)^{L^Y} \rightarrow (L^X)^{L^X}$ as follows:

$$f^{\Leftarrow}(C_\rho)(\lambda)(x) = f^{\leftarrow}(C_\rho(f^{\rightarrow}(\lambda))(x).$$

Then $f^{\Leftarrow}(C_\rho) = C_{f^{\leftarrow}(\rho)}$ is an implicative closure operator on X.

Proof. It implies

$$\begin{aligned} & f^{\Leftarrow}(C_\rho)(\lambda)(x) \\ &= f^{\leftarrow}(C_\rho(f^{\rightarrow}(\lambda))(x) \\ &= f^{\leftarrow}([f^{\rightarrow}(\lambda), \rho] \rightarrow \rho)(x) \\ &= [f^{\rightarrow}(\lambda), \rho] \rightarrow \rho(f(x)) \text{ (by Lemma 2.9(2))} \\ &= [\lambda, f^{\leftarrow}(\rho)] \rightarrow f^{\leftarrow}(\rho)(x) \text{ (by Lemma 2.9(3))} \\ &= C_{f^{\leftarrow}(\rho)}(\lambda)(x). \end{aligned}$$

By Theorem 2.7(1), $f^{\Leftarrow}(C_\rho) = C_{f^{\leftarrow}(\rho)}$ is an implicative closure operator on X.

□

Theorem 2.14. (1) An operator $C : L^Y \rightarrow L^Y$ is an implicative closure operator iff there exists a family $\{\mu_i \mid i \in I\}$ such that

$$C(\lambda)(y) = \bigwedge_{i \in I}([\lambda, \mu_i] \rightarrow \mu_i(y)).$$

(2)In (1), let $f : X \rightarrow Y$ be a function. Define $f^{\Leftarrow}(C) : L^Y \rightarrow L^Y$ as follows

$$f^{\Leftarrow}(C)(\rho)(x) = \bigwedge_{i \in I}([\rho, f^{\leftarrow}(\mu_i)] \rightarrow f^{\leftarrow}(\mu_i)(x)).$$

Then $f^{\Leftarrow}(C)$ is the coarsest implicative closure on X which f is an C-map.

Proof. (1) ($\Rightarrow$) Let $F = \{C(\mu) \mid \mu \in L^Y\}$ be a family. Since

$$\begin{aligned} & \textstyle\bigwedge_{C(\mu) \in F}([\lambda, C(\mu)] \rightarrow C(\mu)(y)) \\ & \leq [\lambda, C(\lambda)] \rightarrow C(\lambda)(y) \\ & = 1 \rightarrow C(\lambda)(y) \\ & = C(\lambda)(y). \end{aligned}$$

we have $C(\lambda)(y) \geq \bigwedge_{C(\mu) \in F}([\lambda, C(\mu)] \rightarrow C(\mu)(y))$. Conversely, since $[\lambda, C(\mu)] \odot \lambda \leq C(\mu)$, we have

$$\begin{aligned} [\lambda, C(\mu)] \odot C(\lambda)(y) &\leq C([\lambda, C(\mu)] \odot \lambda)(y) \\ &\leq C(C(\mu))(y) \\ &\leq C(\mu)(y). \end{aligned}$$

It implies $C(\lambda)(y) \leq \bigwedge_{C(\mu) \in F}([\lambda, C(\mu)] \rightarrow C(\mu)(y))$.

($\Rightarrow$) Put $C_{\mu_i}(\lambda) = ([\lambda, \mu_i] \rightarrow \mu_i)$. By Theorem 2.12, $C(\lambda)(y) = \bigwedge_{i \in I} C_{\mu_i}(\lambda)(y)$ and C is is an implicative closure operator.

(2) Since $C(\lambda) = \bigwedge_{i \in \Gamma} C_{\mu_i}(\lambda)$, we have

$$\begin{aligned} f^{\Leftarrow}(\textstyle\bigwedge_{i \in \Gamma} C_{\mu_i})(\rho) &= f^{\leftarrow}(\textstyle\bigwedge_{i \in \Gamma} C_{\mu_i})(f^{\rightarrow}(\rho)) \\ &= \textstyle\bigwedge_{i \in \Gamma} f^{\leftarrow}(C_{\mu_i})(f^{\rightarrow}(\rho)) \\ &= \textstyle\bigwedge_{i \in \Gamma} \Big([f^{\rightarrow}(\rho), \mu_i] \rightarrow f^{\leftarrow}(\mu_i)\Big) \\ &= \textstyle\bigwedge_{i \in \Gamma} \Big([\rho, f^{\leftarrow}(\mu_i)] \rightarrow f^{\leftarrow}(\mu_i)\Big) \\ &= \textstyle\bigwedge_{i \in \Gamma} C_{f^{\leftarrow}(\mu_i)}(\rho). \end{aligned}$$

A function $f : (X, f^{\Leftarrow}(C)) \to (Y, C)$ is a C-map from:

$$\begin{aligned} f^{\to}(f^{\Leftarrow}(C)(\lambda)) &= f^{\to}(f^{\Leftarrow}(\bigwedge_{i\in I} C_{\mu_i})(\lambda)) \\ &= f^{\to}(\bigwedge_{i\in I} C_{f^{\leftarrow}(\mu_i)}(\lambda)) \\ &\leq \bigwedge_{i\in I} f^{\to}(C_{f^{\leftarrow}(\mu_i)}(\lambda)) \\ &= \bigwedge_{i\in I} f^{\to}([\lambda, f^{\leftarrow}(\mu_i)] \to f^{\leftarrow}(\mu_i)) \\ &= \bigwedge_{i\in I}[\lambda, f^{\leftarrow}(\mu_i)] \to f^{\to}(f^{\leftarrow}(\mu_i)) \\ &\leq \bigwedge_{i\in I}([f^{\to}(\lambda), \mu_i] \to \mu_i) \\ &= C(f^{\to}(\lambda)). \end{aligned}$$

If $f : (X, C^*) \to (Y, C)$ is a C-map, then

$$\begin{aligned} & f^{\to}(C^*(\lambda)) \leq C(f^{\to}(\lambda)) \\ \Leftrightarrow\ & C^*(\lambda) \leq f^{\leftarrow}(C(f^{\to}(\lambda))) \\ \Leftrightarrow\ & C^*(\lambda) \leq f^{\Leftarrow}(C)(\lambda). \end{aligned}$$

Hence $f^{\Leftarrow}(C)$ is the coarsest implicative closure on X which f is a C-map.

□

Example 2.15. Let $L, X, Y, f, \lambda, \mu \in L^X$ and $\nu, \rho \in L^Y$ defined as in Example 2.11. Then

$$C_\rho \wedge C_\nu(\omega) = ([\omega, \rho] \to \rho) \wedge ([\omega, \nu] \to \nu).$$

By Theorem 2.14, we obtain

$$\begin{aligned} & f^{\Leftarrow}(C_\rho \wedge C_\nu)(\lambda) \\ &= f^{\Leftarrow}(C_\rho)(\lambda) \wedge f^{\Leftarrow}(C_\nu)(\lambda) \\ &= C_{f^{\leftarrow}(\rho)}(\lambda) \wedge C_{f^{\leftarrow}(\nu)}(\lambda) \\ &= ([\lambda, f^{\leftarrow}(\rho)] \to f^{\leftarrow}(\rho)) \wedge ([\lambda, f^{\leftarrow}(\nu)] \to f^{\leftarrow}(\nu)). \end{aligned}$$

Since $[\lambda, f^{\leftarrow}(\rho)] = 0.9$ and $[\lambda, f^{\leftarrow}(\nu)] = 0.8$, we have

$$f^{\Leftarrow}(C_\rho \wedge C_\nu)(\lambda)(a) = (0.9 \to 0.6) \wedge (0.8 \to 0.5) = 0.7,$$

$$f^{\Leftarrow}(C_\rho \wedge C_\nu)(\lambda)(b) = (0.9 \to 0.6) \wedge (0.8 \to 0.5) = 0.7,$$

$$f^{\Leftarrow}(C_\rho \wedge C_\nu)(\lambda)(c) = (0.9 \to 0.4) \wedge (0.8 \to 0.4) = 0.5.$$

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