International Journal of Fuzzy Logic and Intelligent Systems, vol. 8, no. 1, March 2008 pp. 6−10

An (r,s)-derived sets and double fuzzy closure operators

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Abstract

In this paper, we introduce the concept of fuzzy (r, s)-adherence points, fuzzy (r, s)-accumulation points and fuzzy (r, s)-derived sets in double fuzzy topology. We investigate some of their properties. The relationship with double fuzzy closure operator was studied.

Key words : fuzzy (r, s)-adherence points, fuzzy (r, s)-accumulation points, (r, s)-derived sets

Introduction

Chang introduced fuzzy topological spaces in [4]. In a Chang fuzzy topological space, each fuzzy set is either open or not. Later Chang's idea was developed by Goguen [9] who replaced the unit interval $[0, 1]$ by a more general lattice L.

An essentially more general notion of fuzzy topology, in which each fuzzy set has a certain degree of openness, was introduced by Kubiak [12] and Šostak [16].

The notion of intuitionistic fuzzy sets was introduced by Atanassove [1, 2, 3]. Çoker and his colleagues [5, 7, 6] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [14, 15] introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [6] and fuzzy topological spaces.

Recently, Garcia and Rodabaugh [8] could prove that the term " intuitionistic " is unsuitable in mathematics and its applications. Their conclusion was to work under the name "double". The notions studied under the name " intuitionistic " were given new names. Double fuzzy topology, which is the range of our research replaced intuitionistic fuzzy topology in Kubiak-Šostak sense.

In this paper, we continuous our study for the properties of double fuzzy topological spaces. The concepts of fuzzy (r, s)-adherence points, fuzzy (r, s)-accumulation points and fuzzy (r, s)-derived sets are introduced. Their properties and the relationship with double fuzzy closure operators are characterized.

Manuscript received Oct. 16, 2007; revised Mar. 11, 2008.
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1. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_\circ = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X.

A fuzzy point x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = t$ if $y = x$ and $x_t(y) = 0$ otherwise, for each $y \in X$. The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. The negative of this statement is denoted by $\lambda \bar{q} \mu$.

A function $f^{\rightarrow} : I^X \rightarrow I^Y$ and its inverse $f^{\leftarrow} : I^Y \rightarrow I^X$ are defined by $f^{\rightarrow}(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{\leftarrow}(\nu)(x) = \nu(f(x))$, $\forall \lambda \in I^X$, $\nu \in I^Y$ and $x \in X$. Notions and notations not described in this paper are standard and usual.

1.1 Definition ([11]). *Let λ, $\mu \in I^X$. Define the fuzzy quasi-difference of λ and μ, denoted by $\lambda \backslash \mu$, as*

$$(\lambda \backslash \mu)(x) = \begin{cases} \lambda(x), & \text{if } \mu(x) = 0; \\ 0, & \text{if } \lambda(x) \geq \mu(x) > 0; \\ \lambda(x), & \text{if } \lambda(x) < \mu(x). \end{cases}$$

Lemma 1.1 ([11]). *For λ, $\mu \in I^X$ and $x_t \in Pt(X)$, the following properties hold:*

(1) $\lambda \backslash \mu \leq \lambda$.

(2) $\lambda \backslash \underline{0} = \lambda$.

(3) If $x_t \notin \lambda$, then $\lambda \backslash x_t = \lambda$. If $x_t \in \lambda$, then, for each $y \in X$,

$$(\lambda \backslash x_t)(y) = \begin{cases} \lambda(y), & \text{if } y \neq x; \\ 0, & \text{if } y = x. \end{cases}$$

(4) $(\lambda \vee \mu)\backslash x_t \leq (\lambda\backslash x_t) \vee (\mu\backslash x_t)$.

(5) If $f : X \to Y$ *is injective, then* $f^{\to}(\lambda\backslash\mu) = f^{\to}(\lambda) \vee f^{\to}(\mu)$.

1.2 Definition ([14]). *Let* X *be a non-empty set. A double fuzzy topology* $(\hat{\mathcal{T}}, \check{\mathcal{T}})$ *on* X *is functions* $\hat{\mathcal{T}}, \check{\mathcal{T}} : I^X \to I$ *which satisfy the following properties:*

(1) $\hat{\mathcal{T}}(\lambda) \leq \check{\mathcal{T}}(\lambda)'$,

(2) $\hat{\mathcal{T}}(\lambda_1 \wedge \lambda_2) \geq \hat{\mathcal{T}}(\lambda_1) \wedge \hat{\mathcal{T}}(\lambda_2)$, *and* $\check{\mathcal{T}}(\lambda_1 \wedge \lambda_2) \leq \check{\mathcal{T}}(\lambda_1) \vee \check{\mathcal{T}}(\lambda_2)$,

(3) $\hat{\mathcal{T}}(\bigvee \lambda_i) \geq \bigwedge \hat{\mathcal{T}}(\lambda_i)$, *and* $\check{\mathcal{T}}(\bigvee \lambda_i) \leq \bigvee \check{\mathcal{T}}(\lambda_i)$.

The triplet $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ is said to be a double fuzzy topological space. Also, we call $\hat{\mathcal{T}}(\lambda)$ a gradation of openness of λ and $\check{\mathcal{T}}(\lambda)$ a gradation of non openness of λ. Let $(X, \hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$ and $(Y, \hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ be two double fuzzy topological spaces. A function $f : X \to Y$ is said to be a double fuzzy continuous if

$$\hat{\tau}_1(f^{\leftarrow}(\nu)) \geq \hat{\tau}_2(\nu) \quad \text{and} \quad \check{\tau}_1(f^{\leftarrow}(\nu)) \leq \check{\tau}_2(\nu),$$

for each $\nu \in I^Y$.

1.3 Definition ([13]). *A double fuzzy closure space is an ordered pair* $(X, \mathcal{C})$, *where* $\mathcal{C} : I^X \times I_0 \times I_1 \to I^X$ *is a function satisfying the following axioms:*

(1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$,

(2) $\lambda \leq \mathcal{C}(\lambda, r, s)$,

(3) If $\lambda_1 \leq \lambda_2$, *then* $\mathcal{C}(\lambda_1, r, s) \leq \mathcal{C}(\lambda_2, r, s)$,

(4) $\mathcal{C}(\lambda, r_1, s_1) \leq \mathcal{C}(\lambda, r_2, s_2)$ *if* $r_1 \leq r_2$ *and* $s_1 \geq s_2$,

(5) $\mathcal{C}(\lambda \vee \mu, r, s) = \mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s)$.

A double fuzzy closure space is called topological provided that

$$\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s) \quad \text{for each} \quad \lambda \in I^X,\ r \in I_0,\ s \in I_1.$$

A continuous function between two double fuzzy closure spaces $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ (DC-map, for short) is a function $f : X \to Y$ such that for each $\lambda \in I^X$, , $r \in I_0$, $s \in I_1$, $f^{\to}(\mathcal{C}_1(\lambda, r, s)) \leq \mathcal{C}_2(f^{\to}(\lambda), r, s)$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ two double fuzzy closure operators on X. We say $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ ($\mathcal{C}_2$ is coarser than $\mathcal{C}_1$) iff $\mathcal{C}_2(\lambda, r, s) \leq \mathcal{C}_1(\lambda, r, s)$, for each $\lambda \in I^X$, $r \in I_0$, $s \in I_1$.

Theorem 1.1 ([13]). *Let* $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ *be a double fuzzy topology. Then for each* $r \in I_0$, $s \in I_1$, $\lambda \in I^X$ *we define an operator* $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}} : I^X \times I_0 \times I_1 \to I^X$ *as follows*

$$\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \hat{\mathcal{T}}(\mu') \geq r, \check{\mathcal{T}}(\mu') \leq s\}.$$

Then, $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}$, *is a double fuzzy closure operator.*

2. Connection with double fuzzy closure operators

Let $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ be a double fuzzy topological space. For each $x_t \in Pt(X)$, $r \in I_0$ and $s \in I_1$, we denote $\mathcal{N}(x_t, r, s) = \{\mu \in I^X \mid x_t q \mu$ and $\hat{\mathcal{T}}(\mu) \geq r, \check{\mathcal{T}}(\mu) \leq s\}$.

2.1 Definition. *Let* $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ *be a double fuzzy topological space,* $\lambda \in I^X$, $x_t \in Pt(X)$, $r \in I_0$ *and* $s \in I_1$. x_t *is called an fuzzy* (r, s)*-adherent point of* λ *if for every* $\mu \in \mathcal{N}(x_t, r, s)$ *we have* $\lambda q \mu$. x_t *is called a fuzzy* (r, s)*-accumulation point of* λ *if for every* $\mu \in \mathcal{N}(x_t, r, s)$, *we have* $\mu q(\lambda\, x_t)$. *Define the fuzzy* (r, s)*-derived set of* λ, *denote by* $\mathcal{D}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$, *as*

$$\mathcal{D}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a fuzzy } (r, s)\text{-accumulation point of } \lambda\}.$$

Theorem 2.1. *Let* $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ *be a double fuzzy topological space. For each* $\lambda \in I^X$, $r \in I_0$ *and* $s \in I_1$, *we have*

$$\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a fuzzy } (r, s)\text{-adherent point of } \lambda\}.$$

Proof. Put

$$\nu = \bigvee\{x_t \in Pt(X) \mid x_t \quad \text{is a fuzzy } (r, s)\text{-adherent point of } \lambda\}.$$

Suppose $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) \not\leq \nu$. Then there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)(x) \geq t > \nu(x).$$

Since $\nu(x) < t$, x_t is not a fuzzy (r, s)-adherent point of λ. Hence there exists $\mu \in N(x_t, r, s)$ such that $\lambda \bar{q} \mu$, that is, $\lambda \leq \mu'$. Then $\lambda \leq \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) \leq \mu'$. Since $x_t q \mu$ and $\mu \leq \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)'$, we have $x_t q \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$. It implies $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)(x) < t$. It is contradiction.

Suppose $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) \not\geq \nu$. Then there exists a fuzzy (r, s)-adherent point $x_t \in Pt(X)$ of λ such that

$$\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)(x) < t \leq \nu(x).$$

Since $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)(x) < t$, then $x_t q \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)'$ and $\hat{\mathcal{T}}(\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)) \geq r$, $\check{\mathcal{T}}(\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)) \leq s$ from the definition of $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}$ and (3) in Definition 1.2. Moreover, since $\lambda \leq \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$, $\lambda \bar{q} \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$. So x_t is not a fuzzy (r, s)-adherent point of λ. It is a contradiction. □

Theorem 2.2. *Let* $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ *be a double fuzzy topological space. For* $\lambda, \mu \in I^X$, $r, r_1 \in I_0$ *and* $s, s_1 \in I_1$, *the following properties hold:*

(1) $\mathcal{D}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) \leq \mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$.

(2) $\mathcal{C}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s) = \lambda \vee \mathcal{D}_{\hat{\mathcal{T}},\check{\mathcal{T}}}(\lambda, r, s)$.

(3) $\mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s) = \lambda$ *iff* $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \leq \lambda$.

(4) If $r \leq s$ *and* $r_1 \geq s_1$, *then* $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \leq \mathcal{D}_{\hat{T},\check{T}}(\lambda, r_1, s_1)$.

(5) $\mathcal{D}_{\hat{T},\check{T}}(\lambda \vee \mu, r, s) \leq \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \vee \mathcal{D}_{\hat{T},\check{T}}(\mu, r, s)$.

Proof. (1) Clear.

(2) Since $\lambda \leq \mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s)$ and $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \leq \mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s)$, we have

$$\lambda \vee \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \leq \mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s).$$

Conversely, suppose $\mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s) \not\leq \lambda \vee \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$\mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s)(x) > t > \lambda(x) \vee \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)(x).$$

Since $\lambda(x) \vee \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)(x) < t$, then $x_t \notin \lambda$ and x_t is not a fuzzy (r,s)-accumulation point of λ. Hence there exists $\mu \in \mathcal{N}(x_t, r, s)$ such that $\mu \bar{q} (\lambda \backslash x_t)$. Since $x_t \notin \lambda$, we have $(\lambda \backslash x_t) = \lambda$. Thus $\mu \bar{q} \lambda$. It implies $\lambda \leq \mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s) \leq \mu'$. Since $x_t q \mu$, that is, $\mu'(x) < t$,

$$\mathcal{C}_{\hat{T},\check{T}}(\lambda, r, s) < \mu'(x) < t.$$

It is a contradiction.

(3) It follows from (2).

(4) Suppose $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \not\leq \mathcal{D}_{\hat{T},\check{T}}(\lambda, r_1, s_1)$. Then there exists a fuzzy (r,s)-accumulation point $x_t \in Pt(X)$ of λ such that

$$\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)(x) \geq t > \mathcal{D}_{\hat{T},\check{T}}(\lambda, r_1, s_1)(x).$$

Since $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r_1, s_1)(x) < t$, then x_t is not a fuzzy (r_1, s_1)-accumulation point of λ. Hence there exists $\nu \in \mathcal{N}(x_t, r_1, s_1)$ such that $\nu \bar{q} (\lambda \backslash x_t)$. Since $\hat{T}(\nu) \geq r_1 \geq r$ and $\check{T}(\nu) \leq s_1 \leq s$. Then x_t is not a fuzzy (r,s)-accumulation point of λ. It is a contradiction.

(5) Suppose $\mathcal{D}_{\hat{T},\check{T}}(\lambda \vee \mu, r, s) \not\leq \mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s) \vee \mathcal{D}_{\hat{T},\check{T}}(\mu, r, s)$. Then there exists a fuzzy (r,s)-accumulation point $x_t \in Pt(X)$ of $\lambda \vee \mu$ such that

$$\begin{aligned} &\mathcal{D}_{\hat{T},\check{T}}(\lambda \vee \mu, r, s)(x) \geq t > \\ &\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)(x) \vee \mathcal{D}_{\hat{T},\check{T}}(\mu, r, s)(x). \end{aligned}$$

Since $\mathcal{D}_{\hat{T},\check{T}}(\lambda, r, s)(x) < t$ and $\mathcal{D}_{\hat{T},\check{T}}(\mu, r, s) < t$, then x_t is not a fuzzy (r,s)-accumulation point of either λ nor μ. Hence there exist $\nu_1, \nu_2 \in \mathcal{N}(x_t, r, s)$ such that $\nu_1 \bar{q} (\lambda \backslash x_t)$ and $\nu_2 \bar{q} (\mu \backslash x_t)$. By taking $\nu = \nu_1 \wedge \nu_2$. Then $x_t q \nu_1 \wedge \nu_2$ and $\hat{T}(\nu_1 \wedge \nu_2) \geq r$, $\check{T}(\nu_1 \wedge \nu_2) \leq s$, that is, $\nu_1 \wedge \nu_2 \in \mathcal{N}(x_t, r, s)$. Moreover, $(\lambda \vee \mu) \backslash x_t = \nu'$. Hence $\nu \bar{q} ((\lambda \vee \mu) \backslash x_t)$. Thus x_t is not a fuzzy (r,s)-accumulation point of $\lambda \vee \mu$. It is a contradiction. □

Example 2.1. *Let* $X = \{x, y\}$ *be a set. Define* $\mu, \nu \in I^X$ *as follows:*

$$\mu(x) = 0.3, \quad \mu(y) = 0.4;$$

$$\nu(x) = 0.6, \quad \nu(y) = 0.2.$$

We define a double fuzzy topology $\hat{T}, \check{T} : I^X \to I$ *as follows:*

$$\hat{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } \lambda = \mu; \\ \frac{2}{3}, & \text{if } \lambda = \nu; \\ \frac{3}{4}, & \text{if } \lambda = \mu \wedge \nu; \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \nu; \\ 0, & \text{otherwise.} \end{cases}, \quad \check{T}(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \mu; \\ \frac{1}{3}, & \text{if } \lambda = \nu; \\ \frac{1}{5}, & \text{if } \lambda = \mu \wedge \nu; \\ \frac{1}{4}, & \text{if } \lambda = \mu \vee \nu; \\ 1, & \text{otherwise.} \end{cases}$$

For $x_1 \in Pt(X)$, *let* x_t *with* $0 < t \leq 1$ *and* $r = \frac{1}{2}$, $s = \frac{1}{3}$. *Since* $(x_1 \backslash x_t) = \underline{0}$, *for all* $\lambda \in \mathcal{N}(x_t, \frac{1}{2}, \frac{1}{3})$, *we have*

$$\lambda\, \bar{q} \big((x_1 \backslash x_t) = \underline{0} \big).$$

Hence x_t *is not a fuzzy* $(\frac{1}{2}, \frac{1}{3})$*-accumulation point of* x_1.

Let y_s *with* $0 < s \leq 1$ *and* $r = \frac{1}{2}$, $s = \frac{1}{3}$. *Since* $(x_1 \backslash y_s) = x_1$, *for all* $\lambda \in \mathcal{N}((x_1 \backslash y_s) = x_1)$, *we have*

$$\lambda\, q \big((x_1 \backslash y_s) = x_1 \big).$$

Hence y_s *is a fuzzy* $(\frac{1}{2}, \frac{1}{3})$*-accumulation point of* x_1. *Therefore*

$$\mathcal{D}_{\hat{T},\check{T}}(x_1, \frac{1}{2}, \frac{1}{3}) = y_1.$$

For $x_{0.8} \in Pt(X)$, *let* x_t *with* $0.8 < t \leq 1$ *and* $r = \frac{1}{2}$, $s = \frac{1}{3}$. *Since* $(x_{0.8} \backslash x_t) = x_{0.8}$, *for all* $\lambda \in \mathcal{N}(x_t, \frac{1}{2}, \frac{1}{3})$, *we have*

$$\lambda\, q \big((x_{0.8} \backslash x_t) = x_{0.8} \big).$$

Hence x_t *is a fuzzy* $(\frac{1}{2}, \frac{1}{3})$*-accumulation point of* $x_{0.8}$.

Let y_s *with* $0 < s \leq 1$ *and* $r = \frac{1}{2}$, $s = \frac{1}{3}$. *Since* $(x_{0.8} \backslash y_s) = x_{0.8}$, *for all* $\lambda \in \mathcal{N}(y_s, \frac{1}{2}, \frac{1}{3})$, *we have*

$$\lambda\, q \big((x_{0.8} \backslash y_s) = x_{0.8} \big).$$

Thus y_s *is a fuzzy* $(\frac{1}{2}, \frac{1}{3})$*-accumulation point of* $x_{0.8}$. *Therefore*

$$\mathcal{D}_{\hat{T},\check{T}}(x_{0.8}, \frac{1}{2}, \frac{1}{3}) = \underline{1}.$$

Hence $x_{0.8} \leq x_1$ *but*

$$\mathcal{D}_{\hat{T},\check{T}}(x_{0.8}, \frac{1}{2}, \frac{1}{3}) \not\leq \mathcal{D}_{\hat{T},\check{T}}(x_1, \frac{1}{2}, \frac{1}{3}).$$

Furthermore,

$$\mathcal{D}_{\hat{T},\check{T}}(x_{0.8} \vee x_1, \frac{1}{2}, \frac{1}{3}) \neq \mathcal{D}_{\hat{T},\check{T}}(x_{0.8}, \frac{1}{2}, \frac{1}{3}) \vee \mathcal{D}_{\hat{T},\check{T}}(x_1, \frac{1}{2}, \frac{1}{3}).$$

Theorem 2.3. *Let $(\hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$ and $(\hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ be double fuzzy topologies on X. If $(\hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$ coarser than $(\hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$. Then, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have:*

(1) $\mathcal{C}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s) \leq \mathcal{C}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)$.

(2) $\mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s) \leq \mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)$.

Proof. (1) For each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$, we have the following:

$$\begin{aligned}\mathcal{C}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s) &= \bigwedge\{\mu \mid \mu \geq \lambda,\ \hat{\mathcal{T}}_2(\mu') \geq r,\ \check{\mathcal{T}}_2(\mu') \leq s\}\\ &\leq \bigwedge\{\mu \mid \mu \geq \lambda,\ \hat{\mathcal{T}}_1(\mu') \geq r,\ \check{\mathcal{T}}_1(\mu') \leq s\}\\ &= \mathcal{C}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s).\end{aligned}$$

(2) Suppose $\mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s) \not\leq \mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)$. Then there exists a fuzzy (r, s)-accumulation point $x_t \in Pt(X)$ of λ on $(\hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ such that

$$\mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s)(x) \geq t > \mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)(x).$$

Since $\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)(x) < t$, then x_t is not a fuzzy (r, s)-accumulation point of λ on $(\hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$. Hence there exists $\nu \in I^X$ with $x_t q \nu$ and $\hat{\mathcal{T}}_1(\nu) \geq r$, $\check{\mathcal{T}}_1(\nu) \leq s$ such that $\nu \bar{q} (\lambda \backslash x_t)$. Since $\hat{\mathcal{T}}_2(\nu) \geq \hat{\mathcal{T}}_1(\nu) \geq r$ and $\check{\mathcal{T}}_2(\nu) \leq \check{\mathcal{T}}_1(\nu) \leq s$. Then x_t is not a fuzzy (r, s)-accumulation point of λ on $(\hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$. It is a contradiction. □

Theorem 2.4. *Let $(X, \hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$ and $(Y, \hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ be double fuzzy topological spaces. Let $f : X \to Y$ be an injective function. Then the following statements are equivalent:*

(1) $f^{\to} : (X, \hat{\mathcal{T}}_1, \check{\mathcal{T}}_1) \to (Y, \hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ is double fuzzy continuous.

(2) $f^{\to}(\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)) \leq \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)$, for all $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

(3) $f^{\to}(\mathcal{C}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)) \leq \mathcal{C}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)$, for all $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

Proof. (1)⇒(2): Suppose there exist $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that

$$f^{\to}(\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)) \not\leq \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(\lambda, r, s).$$

Then there exists a $y \in Y$ such that

$$f^{\to}(\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s))(y) > \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)(y).$$

Since $f^{\to}$ is injective, there exists a unique $x \in f^{-1}(y)$ such that

$$\begin{aligned}f^{\to}(\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s))(y) &\geq \mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)(x)\\ &> \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)(y).\end{aligned}$$

There exists a fuzzy (r, s)-accumulation point x_t of λ on $(\hat{\mathcal{T}}_1, \check{\mathcal{T}}_1)$ such that

$$\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)(x) \geq t > \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)(f(x)).$$

Therefore $f^{\to}(x_t) = f(x)_t$ is not a fuzzy (r, s)-accumulation point of $f^{\to}(\lambda)$. Hence there exist $\nu \in I^X$ with $f^{\to}(x_t) q \nu$ and $\hat{\mathcal{T}}_2(\nu) \geq r$, $\check{\mathcal{T}}_2(\nu) \leq s$ such that $\nu \bar{(f^{\to}(\lambda) \backslash f^{\to}(x_t))}$. Since $f^{\to}$ is injective $f^{\to}(\lambda) \backslash f^{\to}(x_t) = f^{\to}(\lambda \backslash x_t)$. Since $f^{\to} : (X, \hat{\mathcal{T}}_1, \check{\mathcal{T}}_1) \to (Y, \hat{\mathcal{T}}_2, \check{\mathcal{T}}_2)$ is double fuzzy continuous, then $\hat{\mathcal{T}}_1(f^{\leftarrow}(\nu)) \geq \hat{\mathcal{T}}_2(\nu) \geq r$, $\check{\mathcal{T}}_1(f^{\leftarrow}(\nu)) \leq \check{\mathcal{T}}_2(\nu) \leq s$ and

$$f^{\to}(x_t) q \nu \Rightarrow x_t q f^{\leftarrow}(\nu),$$

$$\nu \bar{q} f^{\to}(\lambda \backslash x_t) \Rightarrow f^{\leftarrow}(\nu) \bar{q} (\lambda \backslash x_t).$$

Hence x_t is not a fuzzy (r, s)-accumulation point of λ. It is a contradiction.

(2)⇒(3): It easily proved from the following: for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$,

$$\begin{aligned}f^{\to}(\mathcal{C}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s)) &= f^{\to}(\lambda \vee \mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s))\\ &= f^{\to}(\lambda) \vee f^{\to}(\mathcal{D}_{\hat{\mathcal{T}}_1,\check{\mathcal{T}}_1}(\lambda, r, s))\\ &\leq f^{\to}(\lambda) \vee \mathcal{D}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s)\\ &= \mathcal{C}_{\hat{\mathcal{T}}_2,\check{\mathcal{T}}_2}(f^{\to}(\lambda), r, s).\end{aligned}$$

(3)⇒(1): Straightforward. □

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