J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 15, Number 1 (February 2008), Pages 93–101

ISSN 1226-0657

ON HEINZ-KATO-FURUTA INEQUALITY WITH BEST BOUNDS

C.-S. LIN

Dedicated to Professor Marc Kaltenbach on his retirement

ABSTRACT. In this article we shall characterize the Heinz-Kato-Furuta inequality in several ways, and the best bound for sharpening of the inequality is obtained by the method in [7].

1. INTRODUCTION

Throughout this note it is to be understood that the capital letters always mean bounded linear operators acting on a Hilbert space H into itself, and $T = U \mid T \mid$ is the polar decomposition of the operator T with U the partial isometry with $U^*U = I$, the identity operator, and $\mid T \mid$ the positive square root of the positive operator T^*T satisfying the kernel condition $N(\mid T \mid) = N(U)$. A conjecture about an inequality of possitive linear operators on a Hilbert space proposed by Chan and Kwong [1] was solved by Furuta [2] with more general form than the originally proposed, which we commonly call the Furuta inequality in the literature. More precisely,

Theorem F ([2]). *If $A \geq B \geq O$, then $(B^r A^p B^r)^{\frac{(1+2r)\theta}{p+2r}} \geq B^{(1+2r)\theta}$, or equivalently, $A^{(1+2r)\theta} \geq (A^r B^p A^r)^{\frac{(1+2r)\theta}{p+2r}}$ for all $r \geq 0$, $p \geq 1$, and $\theta \in [0,1]$.*

The fact that the Furuta inequality is equivalent to the Heinz-Kato type inequality was proved by Furuta himself in [3], which is precisely the inequality (2.1) in Theorem 2.1 below, and is called the Heinz-Kato-Furuta inequality in the literature. In this article we shall further more characterize this inequality in several ways, and show

Received by the editors August 8, 2007.
2000 *Mathematics Subject Classification.* 47A63.
Key words and phrases. Furuta inequality, Heinz-Kato-Furuta inequality, polar decomposition of operator, partial isometry, Cauchy-Schwarz inequality.

that one of which is a simple Cauchy-Schwarz inequality. Finally, we shall determine the best bound for sharpening of the Heinz-Kato-Furuta inequality.

2. MAIN RESULT

We present in this section some characterizations of the Heinz-Kato-Furuta inequality, and show that validity of each inequality is due to the Cauchy-Schwarz inequality.

Theorem 2.1. *Suppose that* $A, B \geq O, T = U \mid T \mid, \parallel Tx \parallel \leq \parallel Ax \parallel$, *and* $\parallel T^*y \parallel \leq \parallel By \parallel$ *for all* $x, y \in H$. *Then the following are equivalent, where* $r, s \geq 0,\ p, q \geq 1$, $\alpha, \beta \in [0,1]$ *such that* $(1+2r)\alpha + (1+2s)\beta \geq 1$ (*this last condition is unnecessary if* T *is positive, or if* T *is invertible* [3]), *and* $p + 2r \neq 0 \neq q + 2s$.

$$\text{(2.1)} \qquad \mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2 \\ \leq \left((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x\right)\left((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y\right) \ ([3]);$$

$$\text{(2.2)} \qquad \left((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y\right)\left|\left(x, (\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} z\right)\right|^2 \\ \leq \left((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x\right)\left((\mid T^* \mid^{2s} B^{2q} \mid T^* \mid^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y\right) \\ - \mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2$$

for some $z \in H$ *for which* $\mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} T^*y$ *and* z *are orthogonal, and* $(\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}} z$ *is a unit vector;*

$$\text{(2.3)} \qquad \left|\left(x, (\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} z\right)\right|^2 \leq \left((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x\right)$$

for some $z \in H$ *for which* $(\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}} z$ *is a unit vector;*

$$\text{(2.4)} \qquad \mid (T \mid T \mid^{(1+2r)\alpha+(1+2s)\beta-1} x, y) \mid^2 \\ \leq \left((\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x\right) \parallel \mid T^* \mid^{(1+2s)\beta} y \parallel^2 .$$

Moreover, each inequality in above holds true.

Proof. (2.1)⇒(2.2): The proof rests on the vector u which is defined by

$$u = x - \left(x, (\mid T \mid^{2r} A^{2p} \mid T \mid^{2r})^{\frac{(1+2r)\alpha}{p+2r}} z\right) z$$

for some $z \in H$ such that $|T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*y$ and z are orthogonal, and $(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}z$ is a unit vector. Let us write

$$M=(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} \text{ and } N=(|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}$$

for the convenience of computation. Then,

$$(u,Mz)=(x-(x,Mz)z,Mz)=(x,Mz)-(x,Mz)\|M^{1/2}z\|^2=0,$$

as $\|M^{1/2}z\|=1$. It follows that

(a) $(Mx,x)=(Mu+(x,Mz)Mz,u+(x,Mz)z)=(Mu,u)+|(x,Mz)|^2$;

and

(b) $$\begin{aligned}&(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)\\&=(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}u+(x,Mz)T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z,y)\\&=(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}u,y)+(x,Mz)(z,|T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*y)\\&=(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}u,y).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x\right)\left((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y\right)\\&\quad-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2\\&=(Mx,x)(Ny,y)-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2\\&=(Ny,y)[(Mu,u)+|(x,Mz)|^2]-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}u,y)|^2 \text{ by (a)}\\&=(Ny,y)|(x,Mz)|^2+[(Ny,y)(Mu,u)-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}u,y)|^2]\\&\geq(Ny,y)|(x,Mz)|^2\\&=\left((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y\right)\left|\left(x,(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}z\right)\right|^2.\end{aligned}$$

The inequality above is due to (b) and (2.1), so that (2.2) holds.

(2.2)⇒(2.3): Dividing the inequality (2.2) by the term

$$\left((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y\right)\neq 0,$$

and we arrive at the inequality (2.3).

(2.3)⇒(2.4): We first assert that if $(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}z$ is a unit vector (same as to say that if $(|T|^{2s}A^{2q}|T|^{2s})^{\frac{(1+2s)\beta}{2(q+2s)}}z$ is a unit vector) for some $z \in$H, then inequality (2.3), i.e.,

(c) $$\begin{aligned}&\left|\left((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}x,(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}z\right)\right|\\&\leq\left\|(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}x\right\|\end{aligned}$$

implies inequality (2.4), i.e.,

(d) $$\begin{aligned} &|(|T|^{(1+2r)\alpha+(1+2s)\beta}x, U^*y)| \\ &\le \left\|(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}x\right\| \| |T^*|^{(1+2s)\beta}y\| \end{aligned}$$

for all $x, y \in$H. To this end, we may assume that $A = |T|$ in particular, since $\|Tx\| \le \|Ax\|$ for all $x \in$H, so that we may replace the operator $(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}$ by the operator $|T|^{(1+2r)\alpha}$, and $|T|^{(1+2s)\beta}z$ for the second component of the inner product in the inequality (c). We may also assume without loss of generality that A(so is $|T|$) is an invertible operator. It follows from the inequality (c) (here, we do not assume Cauchy-Schwarz inequality) that

$$\left|\left(|T|^{(1+2r)\alpha}x, \frac{|T|^{(1+2s)\beta}U^*y}{\||T|^{(1+2s)\beta}U^*y\|}\right)\right| \le \||T|^{(1+2r)\alpha}x\|$$

for all $x, y \in$H for which $|T|^{(1+2s)\beta}U^*y \ne 0$. But then

$$\begin{aligned} \||T|^{(1+2s)\beta}U^*y\|^2 &= (U|T|^{2(1+2s)\beta}U^*y, y) \\ &= (|T^*|^{2(1+2s)\beta}y, y) = \||T^*|^{(1+2s)\beta}y\|^2 \end{aligned}$$

by a well-known relation $U|T|^tU^* = |T^*|^t$ for $t \ge 0$ [3]. In view of assumption $\|Tx\| \le \|Ax\|$ for all $x \in$H, i.e., $|T|^2 \le A^2$, and by Theorem F we have

$$(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} \ge |T|^{2(1+2r)\alpha}$$

for $r \ge 0$, $p \ge 1$ and $\alpha \in [0, 1]$. It follows that

$$\begin{aligned} \||T|^{(1+2r)\alpha}x\|^2 &= (|T|^{2(1+2r)\alpha}x, x) \\ &\le \left((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x, x\right) \\ &= \left\|(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}x\right\|^2. \end{aligned}$$

Now, we are ready to show the desired inequality as follows.

$$\begin{aligned} |(|T|^{(1+2r)\alpha+(1+2s)\beta}x, U^*y)| &= |(|T|^{(1+2r)\alpha}x, |T|^{(1+2s)\beta}U^*y)| \\ &\le \||T|^{(1+2r)\alpha}x\| \||T|^{(1+2s)\beta}U^*y\| \\ &\le \|(|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}}x\| \||T^*|^{(1+2s)\beta}y\|, \end{aligned}$$

and we have the inequality (d).

(2.4)⇒(2.1): The condition $\||T^*|y\| \le \|By\|$ for all $y \in$H means that $|T^*|^2 \le B^2$. If we apply Theorem F to this, then

$$|T^*|^{2(1+2s)\beta} \le (|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}$$

for $s \geq 0$, $q \geq 1$, and $\beta \in [0,1]$. Thus,

$$\| |T^*|^{(1+2s)\beta} y \|^2 = (|T^*|^{2(1+2s)\beta} y, y) \leq \left((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right)$$

for all $y \in$H, and (2.1) follows.

Finally, each inequality in above holds true since (2.3) is nothing but a Cauchy-Schwarz inequality. The proof of the theorem is now finished. □

Corresponding to the inequality (2.4) we may add one more inequality

$$\text{(2.4)'} \qquad |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1} x, y)|^2 \leq \| |T|^{(1+2r)\alpha} x \|^2 \left((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right)$$

to Theorem 2.1. The proof of the implication (2.3)⇒(2.4)'⇒(2.1) are quite similar as the proof in Theorem 2.1, and we leave the it to the reader.

Before proceeding, it is noteworthy that the following statement is valid which is a natural generalization of inequality (2.2) in Theorem 2.1, cf. [6], and, again, we leave the proof to the reader.

Theorem 2.2. *If* $|T|^{(1+2r)\alpha+(1+2s)\beta-1} T^*y$ *is orthogonal to a set* $\{z_i\}_{i=1}^n$ *of vectors, and* $\left\{ (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{2(p+2r)}} z_i \right\}_{i=1}^n$ *is a set of unit vectors . Then*

$$\left((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right) \left[\sum_{i=1}^n \left| \left(u_{i-1}, (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} z_i \right) \right|^2 \right]$$
$$\leq ((|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x) \left((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right) - |(T|T|^{(1+2r)\alpha+(1+2s)\beta-1} x, y)|^2$$

for all $x, y \in H$, *where* $\{u_i\}_{i=1}^n$ *is a sequence of vectors recursively defined by*

$$u_i = u_{i-1} - \left(u_{i-1}, (|T|^{2r} A^{2p} |T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}} z_i \right) z_i$$

for which $u_0 = x$, $i = 1, ..., n$.

It is interesting to observe that the bound of inequality is retained as in (2.2) of Theorem 2.1 which is a special case of the above when $n = 1$.

3. Best Bounds

Recall from the proof in Theorem 2.1 that for $p, q \geq 1$, $\alpha, \beta \in [0,1]$, and for all $x, y \in$H, we have

$$\| |T|^{(1+2s)\beta} U^*y \|^2 = \| |T^*|^{(1+2s)\beta} y \|^2 \leq \left((|T^*|^{2s} B^{2q} |T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}} y, y \right);$$

and

$$\| | T |^{(1+2r)\alpha} x \|^2 \leq \left((| T |^{2r} A^{2p} | T |^{2r})^{\frac{(1+2r)\alpha}{p+2r}} x, x \right).$$

Since

$$| (T | T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) | = | (| T |^{(1+2r)\alpha+(1+2s)\beta} x, U^* y) |$$

$$= | (| T |^{(1+2r)\alpha} x, | T |^{(1+2s)\beta} U^* y) |,$$

we arrive at

$$(*) \quad | (T | T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) | \leq \| | T |^{(1+2r)\alpha} x \| \| | T^* |^{(1+2s)\beta} y \| .$$

From above consideration (*) is obviously a better inequality than inequality (2.1) in Theorem 2.1, and so we are going to find its bounds next. Of course, the former is also a special case of the latter which is obtained by letting $A = | T |$ and $B = | T^* |$. Before finding bounds of the inequality (*) we require the next crucial lemma which we proved in our previous paper [7]. Let us state the results without proof.

Theorem 3.1 ([7]). *For any x and y in a pre-Hilbert space we have*

$$(3.1) \qquad \| x \|^2 \| y - \xi x \|^2 - | (x, y - \xi x) |^2 = \| x \|^2 \| y \|^2 - | (x, y) |^2$$
$$\leq \frac{1}{| \mu - \nu |^2} \| y - \mu x \|^2 \| y - \nu x \|^2$$

for any real numbers ξ, μ, and ν for which $\mu \neq \nu$.

Moreover, if $(\mu - \nu)(x, y - \mu x)$ is a nonzero real number, then inequality (3.1) *becomes equality if and only if*

$$\nu - \mu = \frac{1}{(x, y - \mu x)} \| y - \mu x \|^2 .$$

Theorem 3.2. *If $r, s \geq 0$, and $\alpha, \beta \in [0,1]$ such that $(1+2r)\alpha + (1+2s)\beta \geq 1$. Then for $x, y \in H$ we have*

$$(3.2) \qquad \| | T |^{(1+2r)\alpha} x \|^2 \| | T |^{(1+2s)\beta} U^* y - \xi | T |^{(1+2r)\alpha} x \|^2$$
$$- | (| T |^{(1+2r)\alpha} x, | T |^{(1+2s)\beta} U^* y - \xi | T |^{(1+2r)\alpha} x) |^2$$
$$= \| | T |^{(1+2r)\alpha} x \|^2 \| | T^* |^{(1+2s)\beta} y \|^2 - | (T | T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2$$
$$\leq \frac{1}{| \mu - \nu |^2} \| | T |^{(1+2s)\beta} U^* y - \mu | T |^{(1+2r)\alpha} x \|^2$$
$$\cdot \| | T |^{(1+2s)\beta} U^* y - \nu | T |^{(1+2r)\alpha} x \|^2$$

for any real numbers ξ, μ, and ν for which $\mu \neq \nu$.

Moreover, if $(\mu-\nu)(|T|^{(1+2r)\alpha}x, |T|^{(1+2s)\beta}U^*y-\mu|T|^{(1+2r)\alpha}x)$ *is a nonzero real number, then inequality* (3.2) *becomes equality if and only if*

$$\nu-\mu=\frac{1}{a}\,\||\,T\,|^{(1+2s)\beta}U^*y-\mu\,|\,T\,|^{(1+2r)\alpha}x\,\|^2,$$

where $a=(|T|^{(1+2r)\alpha}x, |T|^{(1+2s)\beta}U^*y-\mu|T|^{(1+2r)\alpha}x)$.

Proof. From the foregoing all we have to do is replacing x by $|T|^{(1+2r)\alpha}x$, and y by $|T|^{(1+2s)\beta}U^*y$ in Lemma 3.1. □

In Theorem 2.1 we see that inequality (2.2) is evidently a sharpening of inequality (2.1), and naturally we are interested in determining bounds of the sharpening. Recall that inequality (*) may be obtained by substituting $A=|T|$ and $B=|T^*|$ in the inequality (2.1) of Theorem 2.1. Here, we likewise let $A=|T|$ and $B=|T^*|$ in the inequality (2.2) of Theorem 2.1, and write it in the form

$$(**)\qquad \begin{aligned}&|(x,|T|^{2(1+2r)\alpha}z)|^2\\ &\le\frac{1}{\||\,T^*|^{(1+2s)\beta}y\|^2}[\||\,T\,|^{(1+2r)\alpha}x\|^2\||\,T^*|^{(1+2s)\beta}y\|^2\\ &\quad-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2].\end{aligned}$$

Then (**) is a better inequality than inequality (2.2) of Theorem 2.1. Because of the equivalence of (2.1) and (2.2) in Theorem 2.1, and that of (*) and (**), we shall consider bounds of inequality (**). We will show that the bound in (**) is indeed the best of the bounds that could be obtained from a class of squares of ratios of shifted norm of vectors to the number shifted by the same amount. More precisely, we have

Theorem 3.3. *For any real number* $\delta\neq0$, $r,s\ge1$, *and* $\alpha,\beta\in[0,1]$ *such that* $(1+2r)\alpha+(1+2s)\beta\ge1$, *we have*

$$(3.3)\qquad \begin{aligned}&\frac{1}{\||\,T^*|^{(1+2s)\beta}y\|^2}[\||\,T\,|^{(1+2r)\alpha}x\|^2\||\,T^*|^{(1+2s)\beta}y\|^2\\ &\quad-|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2]\\ &\le\frac{1}{\delta^2}\||\,T\,|^{(1+2s)\beta}U^*y-\delta\,|\,T\,|^{(1+2r)\alpha}x\|^2\end{aligned}$$

for all $x,y\in H$ *with* $|T^*|^{(1+2s)\beta}y\neq0$. *Moreover, if* $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)$ *is a nonzero real number, then equality holds if and only if*

$$\delta=\||\,T^*|^{(1+2s)\beta}y\|^2/(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y).$$

Proof. Since $\||\,T^*|^{(1+2s)\beta}y\|=\||\,T\,|^{(1+2s)\beta}U^*y\|$ as was mentioned before, and if we put $|T|^{(1+2r)\alpha}x=a$, and $|T|^{(1+2s)\beta}U^*y=b$ for the convenience of computation, then, in short we are going to show inequality

$$\frac{1}{\| b \|^2}[\| a \|^2 \| b \|^2 - | (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2] \leq \frac{1}{\delta^2} \| b - \delta a \|^2$$

holds. Now,

$$\begin{aligned}
&\| b \|^2 \| b - \delta a \|^2 - \delta^2 [\| a \|^2 \| b \|^2 - | (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2] \\
=& \| b \|^2 \, [\| b \|^2 - 2\delta Re(a,b) + \delta^2 \| a \|^2] - \delta^2 [\| a \|^2 \| b \|^2 \\
&- | (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2] \\
=& \delta^2 \, | (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2 - 2\delta Re(a,b) \| b \|^2 + \| b \|^4 \\
\geq& [\delta \, | (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) | - \| b \|^2]^2 \geq 0,
\end{aligned}$$

because $\mathrm{Re}(a,b) = \mathrm{Re}(T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y)$, and $\mathrm{Re}(u,v) \leq | (u,v) |$ holds true for any vectors u and v. Hence, the desired conclusions follow easily. □

In conclusion we mention that although the inequality (2.1) in Theorem 2.1 was proved in [3] by using the Furuta inequality, the next proof is much simpler and direct. We first assume that conditions in Theorem 2.1 hold. Replace x by $U \mid T |^{(1+2r)\alpha} x$, and y by $| T^* |^{(1+2s)\beta} y$ in the Cauchy-Schwarz inequality $| (x,y) | \leq \| x \| \| y \|$ for $x, y \in$H (and the equality holds if and only if x and y are proportional). Then,

$$\begin{aligned}
&| (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2 \\
&\leq (| T |^{2(1+2r)\alpha} x, x)(| T^* |^{2(1+2s)\beta} y, y) \\
&\leq ((| T |^{2r} A^{2p} | T |^{2r})^{(1+2r)\alpha/(p+2r)} x, x)((| T^* |^{2s} B^{2q} | T^* |^{2s})^{(1+2s)\beta/(q+2s)} y, y).
\end{aligned}$$

The last inequality is due to Theorem F, of course. The first inequality becomes an equality if and only if $U \mid T |^{(1+2r)\alpha} x$ and $| T^* |^{(1+2s)\beta} y$ are proportional.

On the other hand, we may use a different replacement to get the same result. Replace x by $| T |^{(1+2r)\alpha} x$, and y by $U^* \mid T^* |^{(1+2s)\beta} y$ which is $| T |^{(1+2s)\beta} U^* y$ in the Cauchy-Schwarz inequality. Then,

$$\begin{aligned}
&| (T \mid T |^{(1+2r)\alpha+(1+2s)\beta-1} x, y) |^2 \\
&\leq (| T |^{2(1+2r)\alpha} x, x)(| T |^{2(1+2s)\beta} U^* y, U^* y) \\
&\leq ((| T |^{2r} A^{2p} | T |^{2r})^{(1+2r)\alpha/(p+2r)} x, x)(| T |^{2(1+2s)\beta} U^* y, U^* y).
\end{aligned}$$

The first inequality becomes an equality if and only if $| T |^{(1+2r)\alpha} x$ and $U^* \mid T^* |^{(1+2s)\beta} y$ are proportional.

REFERENCES

1. N.N. Chan & M.K. Kwong: Hermitian matrix inequalities and a conjecture. *Amer. Math. Month.* **92** (1985), 533-541.
2. T. Furuta: $A \geq B \geq O$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$. *Proc. Amer. Math. Soc.* **101** (1987), 85-88.
3. T. Furuta: Generalization of Heinz-Kato theorem via Furuta inequality. *Operator Theory* **62** (1993), 77-83.
4. E. Heinz: Beiträge zur Störungstheorie der Spektralzerlegung. *Math. Ann.* **123** (1951), 415-438.
5. T. Kato: Notes on some inequalities for linear operators. *Math. Ann.* **125** (1952), 208-212.
6. C.-S. Lin: Heinz's inequality and Bernstein's inequality. *Proc. Amer. Math. Soc.* **125** (1997), 2319-2325.
7. C.-S. Lin: Operator versions of inequalities and equalities on a Hilbert space. *Linear Algebra Appl.* **268** (1998), 365-374.
8. C.-S. Lin: On inequalities of Heinz and Kato, and Furuta for Linear operators. *Math. Japonica* **50** (1999), 463-468.
9. K. Löwner: Über monotone Matrixfunktionen. *Math. Z.* **38** (1934), 177-216.

DEPARTMENT OF MATHEMATICS, BISHOP'S UNIVERSITY, 2600 COLLEGE STREET, SHERBROOKE, QUEBEC, J1M 1Z7, CANADA
Email address: plin@ubishops.ca