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New Family of the t Distributions for Modeling Semicircular Data†

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Abstract

We develop new family of the t distributions for modeling semicircular data. It has convenient mathematical features. It is extended to the l-axial t distribution and a generalized semicircular t distribution.

Keywords: Projection; axial data; heavy tail; transformation.

1. Introduction

Angular data are very common in the areas of geology, meteorology, biology, *etc.* Sometimes it is not necessary to model angular data on a full circle. However most of works devoted to the modeling of circular data (Fisher, 1993; Jammalamadaka and SenGupta, 2001; Mardia and Jupp, 2000). This fact is emphasized on Guardiola (2004) and Jones (1968). Guardiola (2004) obtained a semicircular normal distribution using a simple projection. Similarly we project a t distribution over a semicircular segment and obtain the semicircular t(SCT) distribution. Often a t distribution is an alternative to a normal distribution if the errors are heavier than those of a normal distribution in linear statistics. Similarly a SCT distribution has more heavier tail than a semicircular normal distribution. So a SCT distribution is an alternative to a semicircular normal distribution for the heavy tail semicircular data.

This paper is organized as follows. Section 2 defines the distribution and lists some of its basic properties. We estimate parameters of the SCT distribution by a maximum likelihood method in Section 3. We concludes in Section 4 with two extensions of the SCT distribution: the l-axial distribution for modeling any arc of arbitrary length, say $2\pi/l,\ l = 1, 2, \cdots$ and a generalized SCT distribution.

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2. New Family of the t Distributions

2.1. Definition and some basic properties

The SCT distribution is obtained by projecting a t distribution over a semicircular segment. Let X have a t distribution with a *df* ν, *i.e.*, the density of X is

$$\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)}\frac{1}{\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty,\ \nu = 1, 2, \cdots. \tag{2.1}$$

For brevity, we shall also say that X is $t(\nu)$. For a positive real number r, define the angle θ by $\theta = \tan^{-1}(x/r)$. Hence, $x = r\tan(\theta)$. Obviously, the *pdf* of θ is given by

$$\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)}\frac{\sec^2(\theta)}{\sqrt{\varphi\pi}}\left(1+\frac{\tan^2(\theta)}{\varphi}\right)^{-\frac{\nu+1}{2}}, \quad \varphi = \frac{\nu}{r^2},\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{2.2}$$

More generally, we introduce the parameter μ as the location parameter for the SCT distribution and write the *pdf* as

$$\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\frac{\sec^2(\theta-\mu)}{\sqrt{\varphi\pi}}\left(1+\frac{\tan^2(\theta-\mu)}{\varphi}\right)^{-\frac{\nu+1}{2}}, \quad -\frac{\pi}{2}+\mu<\theta<\frac{\pi}{2}+\mu,\ -\pi<\mu<\pi. \tag{2.3}$$

Then, we say that θ is an SCT random variable with parameters μ, ν and φ; for brevity, we shall also say that θ is SCT(μ,ν,φ). Figure 2.1 shows the shape of (2.3) for four values of ν with $\mu = 0$ and $r^2 = 10$. Geometrically r is the distance between the radius and the support of the t density. The closer the support is to the radius, the larger φ. One special case of (2.3) is the semicircular Uniform distribution over $(-\pi/2,\ \pi/2)$ when $\nu = r^2 = 1$. To compare the tail behavior of a semicircular normal distribution and a SCT distribution, Figure 2.2 shows two *pdf*s. SCT distribution is the *pdf* of Figure 2.1 with $\nu = 1$. And a semicircular normal distribution is obtained from transforming the standard normal distribution with $r^2 = 10$. From the Figure 2.2, we denote that a SCT distribution has more heavier tail than a semicircular normal distribution. Similar relationship also appears at linear statistics, *i.e.* a relationship between a normal distribution and a t distribution.

It is straightforward to generate samples from an SCT distribution using the transformation. First, generate samples from a t distribution, and then use the inverse transformation, $\theta = \mu + \tan^{-1}(x/r)$, $r^2 = \nu/\varphi$. Similarly, the *cdf* of an SCT distribution is $F(\theta;\ \mu,\ \nu,\ \varphi) = T(r\tan(\theta-\mu))$, where the function $T(\cdot)$ is a *cdf* of a t distribution.

We consider the behavior of the SCT distribution when the degree of freedom ν goes to ∞. Suppose θ follows SCT$(\mu,\ \nu,\ \varphi)$. W.L.O.G., we assume that $\mu = 0$, $r = 1$, then the *pdf* is as follows:

$$\left[\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)\sqrt{\nu\pi}}\right]\sec^2(\theta)\left(1+\frac{\tan^2(\theta)}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{2.4}$$

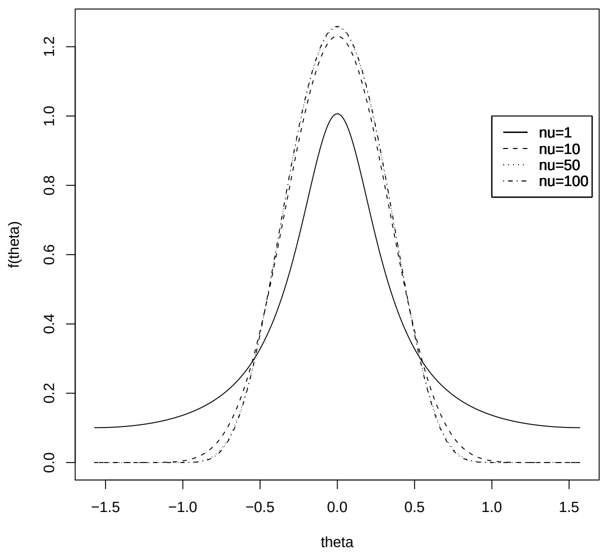

Figure 2.1: The density functions SCT(0, ν, φ)

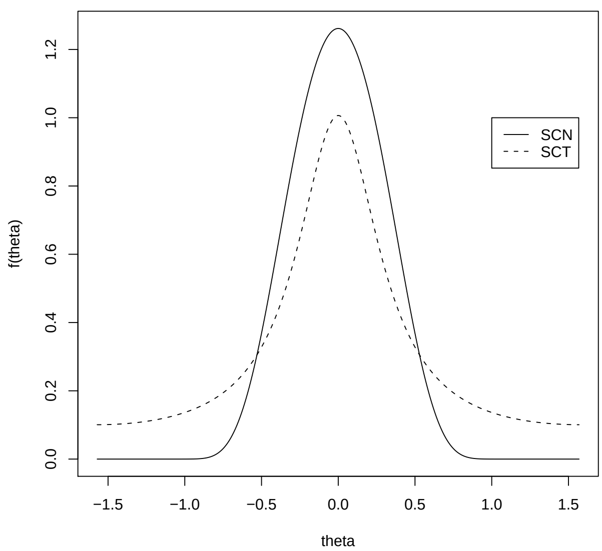

Figure 2.2: A SCT *pdf* and a semicircular normal(SCN) *pdf*

Denote the quantity in square brackets by C_ν. From an extension of Stirling's formula and an application of the following Lemma (Casella and Berger, 2002), we have

$$\lim_{\nu\to\infty} C_\nu = \frac{1}{\sqrt{2\pi}}. \tag{2.5}$$

Lemma 2.1 Let $a_1, a_2, \cdots$ be a sequence of numbers converging to a, that is, $\lim_{n\to\infty} a_n = a$. Then $\lim_{n\to\infty}(1 + a_n/n)^n = e^a$.

Furthermore

$$\lim_{\nu\to\infty} \sec^2(\theta)\left(1+\frac{\tan^2(\theta)}{\nu}\right)^{-\frac{\nu+1}{2}} = \sec^2(\theta)e^{-\frac{\tan^2(\theta)}{2}} \tag{2.6}$$

using the above Lemma again. So the SCT distribution converges to the semicircular normal distribution as ν goes to ∞. Similar phenomenon also occurs in linear statistics, *i.e.* the t distribution converges to the standard normal distribution as the degree of freedom goes to ∞.

2.2. Trigonometric moments

Similar to those of any circular density, trigonometric moments of the SCT distribution are defined as follows: $\phi_p = Ee^{ip\theta} = \alpha_p + i\beta_p = E\cos(p\theta) + iE\sin(p\theta),\ p = 0, \pm1, \pm2, \cdots$. We assumed that the location parameter $\mu = 0$ without loss of generality.

The first and the second α_p, $p = 1, 2$ are given as follows:

$$\begin{aligned} \alpha_1 &= \frac{\nu\varphi^{\frac{\nu}{2}}}{2\pi}\ {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+1}{2}; \frac{\nu+2}{2}; 1-\varphi\right), \\ \alpha_2 &= \frac{2\varphi^{\frac{\nu}{2}}(\nu+1)\Gamma^2((\nu+1)/2)}{\nu\pi\Gamma^2(\nu/2)}\ {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{\nu+3}{2}; 1-\varphi\right), \end{aligned} \tag{2.7}$$

where ${}_2F_1$ is the hypergeometric function (Abramowitz and Stegun, 1972). For example, $\alpha_1 = 2/\pi$ when $\nu = 1$ and $r = 1$. The hypergeometric function has a series expansion

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.8}$$

where $(a)_k$ is the Pochhammer symbol or rising factorial

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1). \tag{2.9}$$

The function is a solution of the hypergeometric differential equation $z(1-z)y^{''} + [c - (a+b+1)z]y^{'} - aby = 0$.

The proof is the process of using some transformations. For the first cosine moment, use the transformation $x = \tan(\theta)$ and use a property of even function. So

$$\alpha_1 = 2\int_0^{\infty} \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\varphi\pi}}(1+x^2)^{-\frac{1}{2}}\left(1+\frac{x^2}{\varphi}\right)^{-\frac{\nu+1}{2}} dx. \tag{2.10}$$

Finally a transformation $y = x^2$ is applied. Then

$$\alpha_1 = \frac{\Gamma((\nu+1)/2)\varphi^{\frac{\nu}{2}}}{\Gamma(\nu/2)\sqrt{\pi}}\int_0^{\infty} y^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}}(\varphi+y)^{-\frac{\nu+1}{2}} dy. \tag{2.11}$$

The result α_1 follows by the integral formula 3.197.9 (Gradshteyn and Ryzhik, 1994). To obtain α_2, we use the transformation $x = \tan(\theta)$ and a property of a generalized t (Arellano-Valle and Bolfarine, 1995) *pdf* is used. So

$$\alpha_2 = \frac{2\Gamma((\nu+1)/2)\varphi^{\frac{\nu}{2}}}{\Gamma(\nu/2)\sqrt{\pi}}\int_{-\infty}^{\infty} (1+x^2)^{-1}(\varphi+x^2)^{-\frac{\nu+1}{2}} dx. \tag{2.12}$$

After using a property of even function followed by $y = x^2$ and the result is immediate by the same integral formula of α_1. Like any other symmetric circular density, $\beta_p = E\sin(p\theta)$, are 0 as the density is symmetric about 0.

3. Parameter Estimation and Example

The minus log-likelihood for a random sample of size n, $\theta = (\theta_1, \ldots, \theta_n)$, from the SCT distribution is given by

$$\begin{aligned} -l(\mu, \nu, \varphi; \theta) = &- n\log\Gamma\left(\frac{\nu+1}{2}\right) + n\log\Gamma\left(\frac{\nu}{2}\right) + \frac{n}{2}\log(\varphi\pi) \\ &+ \sum_{i=1}^{n}\log\cos^2(\theta_i-\mu) + \frac{\nu+1}{2}\sum_{i=1}^{n}\log\left(1+\frac{\tan^2(\theta_i-\mu)}{\varphi}\right). \end{aligned} \tag{3.1}$$

For the following example, the corresponding estimates have been computed by direct minimization of the minus log-likelihood itself.

Example 3.1 We simulated a dataset of size 5,000 from an SCT distribution with $\mu = 0$, $\varphi = 5$ and $\nu = 50$. For this dataset, the corresponding maximum likelihood estimates are given by $\hat{\mu} = -0.007$, $\hat{\varphi} = 4.972$ and $\hat{\nu} = 51$. Histogram with *pdfs* and Healy's plot (Healy, 1968) are shown in Figure 3.1. A visual inspection of Figure 3.1 indicates a satisfactory fit of the density to the data. Healy's plot is based on

$$d_i = r\tan(\theta_i - \mu), \quad (i = 1, \ldots, n) \tag{3.2}$$

and is sampled from $t(\nu)$ distribution if the fitted model is appropriate. In practice, estimates must replace the exact parameter values in equation (3.2). d_i above must

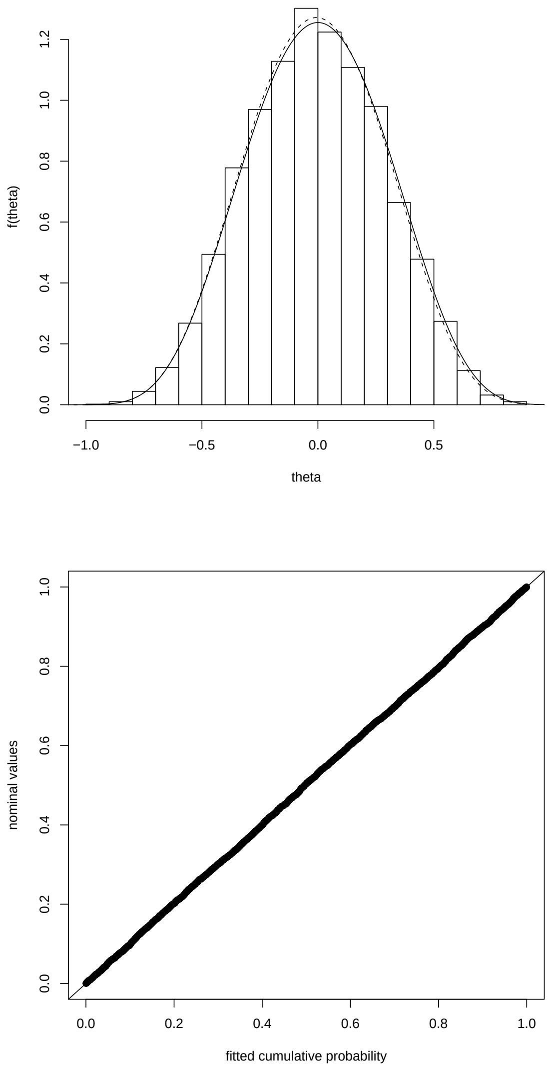

Figure 3.1: Histogram with *pdf*s(the solid line represents the original *pdf* and the dashed line corresponds to the fitted *pdf*) and Healy's plot

be sorted and plotted against the $t(\nu)$ percentage points. Equivalently, the cumulative $t(\nu)$ probabilities can be plotted against their nominal values $1/n$, $2/n, \ldots, 1$; the points should lie on the bisection line of the quadrant. This diagnostic method is a natural analogue of a well-known diagnostic used in the normal theory context (Healy, 1968).

4. Extensions

We extend the suggested model to the l-axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l = 1, 2, \ldots$. Occasionally, measurements result in any arc of arbitrary length, say $2\pi/l$, $l = 1, 2, \ldots$, so it is desirable to extend the SCT distribution. To construct the l-axial t distribution, we consider the *pdf* (2.2) and use the transformation $\theta^* = 2\theta/l,\ l = 1, 2, \ldots$. The *pdf* of θ^* is given by

$$\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)} \frac{\sec^2(l\theta^*/2)}{\sqrt{\varphi\pi}} \left(1 + \frac{\tan^2(l\theta^*/2)}{\varphi}\right)^{-\frac{\nu+1}{2}}, \quad \varphi = \frac{\nu}{r^2}, \ -\frac{\pi}{l} < \theta^* < \frac{\pi}{l}. \tag{4.1}$$

Please note that when $l = 2$, the *pdf* (4.1) is the same as the *pdf* (2.2), the SCT *pdf*. When $l = 1$, it becomes the *pdf* of a circular t distribution.

Arellano-Valle and Bolfarine (1995) extended a t distribution to a generalized t distribution since the usual t distribution does not retain its conditional distributions. Let X have a generalized t distribution with *dfs* λ and ν, *i.e.*, the density of X is

$$\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)} \frac{1}{\sqrt{\lambda\pi}} \left(1 + \frac{x^2}{\lambda}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty, \ \nu \ \& \ \lambda = 1, 2, \cdots. \tag{4.2}$$

For brevity, we shall also say that X is $gt(\lambda, \nu)$. When $\lambda = \nu$, it is a usual t distribution. Similarly we use the same transformation of an SCT distribution, then obtained a generalized SCT distribution as follows:

$$\frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)} \frac{\sec^2(\theta)}{\sqrt{\varphi\pi}} \left(1 + \frac{\tan^2(\theta)}{\varphi}\right)^{-\frac{\nu+1}{2}}, \quad \varphi = \frac{\lambda}{r^2}, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{4.3}$$

Please note that $\varphi = \lambda/r^2$ instead of $\varphi = \nu/r^2$ since it has two degree of freedom parameters.

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