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A Note on Comparing Multistage Procedures for Fixed-Width Confidence Interval†

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Abstract

Application of the bootstrap to problems in multistage inference procedures are discussed in normal and other related models. After a general introduction to these procedures, here we explore in multistage fixed precision inference in models. We present numerical comparisons of these procedures based on bootstrap critical points for small and moderate sample sizes obtained via extensive sets of simulated experiments. It is expected that the procedure based on bootstrap leads to better results.

Keywords: Bootstrap; fixed-width confidence interval; multistage procedures.

1. Introduction

This paper is intended to provide a brief overview of the multistage procedures based on bootstrapping that are used for drawing inferences in normal and other related models. Since Wald's sequential probability ratio test, sequential and multistage procedures have come a long way.

Suppose that the observations are independent identically distributed(iid) normal variables with unknown mean μ and unknown variance and one wants a confidence interval for μ of given length and given coverage probability $1-\alpha$. This is the so called fixed-width confidence interval problem. Such an interval does not exist if one consider the fixed data $X_1, X_2, \ldots, X_m$ for any $m(\geq 1)$. It was solved by Stein (1945) using a two-stage sampling procedure. Stein (1945) proposed a kind of sequential inference procedure which is relevant only when the underlying distribution involves, besides the parameter of interest, some nuisance parameter. In such a procedure a pilot sample size is taken to estimate the nuisance parameter. The final number of observations is determined on the basis of this estimate, so that the procedure achieves a stipulated level of performance irrespective of the nuisance parameter. These procedures are generally two or multistage procedures and are naturally convenient when the cost of experiment depends on the number of stages. Sequential and multistage procedures are relatively

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not occurring regularly employed in practice. The reason is that the user does not know in advance how many observations will be required to reach a decision. When applying sequential and multistage procedures, prediction of quantities such as sample sizes is of great importance. The aim of this paper is to show that the bootstrap can be applied at any point during sequential and multistage procedures to predict the final values of these unknown quantities. How bootstrap methods can be applied at any stage of the multistage procedures to predict unknown quantities encountered. This method will be explained by considering the stopping rule. Such multistage procedures based on stopping rule are discussed in Section 2. All these procedures are approximates ones and the prescribed coverage probability $1-\alpha$ can only be guaranteed asymptotically as $d \to 0$. We address the issue of fixed precision inference in models and multistage methodologies based on bootstrapping with a predetermined measure of accuracy in each case. Finally, The performance of the method for small and moderate sample sizes is investigated in Section 3, which contains the results of some simulations. We will simulate random samples from standard normal, a contaminated normal and from double exponential distribution. We want to see how well do the confidence intervals based on the normal approximation compare to the bootstrap approximation. An exhaustive overview of multistage procedures is given in Ghosh *et al.* (1997).

2. Multistage Procedures for Fixed-Width Confidence Interval

2.1. Multistage procedures

Let $X_1, X_2, \ldots$ be *iid* normally distributed random variables with unknown mean μ, $-\infty < \mu < \infty$. Now consider the problem of finding a confidence interval of width at most $2d$, $d > 0$ and confidence coefficient α, $0 < \alpha < 1$. Let $u_{1-\alpha/2}$ denote the $1-\alpha/2$ quantile of the standard normal distribution. If σ^2 were known, then we could take a sample of size $n \geq c(d) = u^2_{1-\alpha/2}\sigma^2/d^2$ and use $I_n = (\bar{X}_n - d, \bar{X}_n + d)$ for $P_{\mu,\sigma}\{\mu \in I_n\} \geq \alpha$. However, Lehmann (1959) showed that if σ^2 were unknown, then it is impossible to find a confidence interval of width at most $2d$ and confidence coefficient $\alpha > 0$. Chow and Robbins (1965) proposed fully sequential procedure with the unknown variance. The stopping rule is given as the smallest integer exceeding $c(d)$, *i.e.*,

$$N = \inf\left\{ n \geq m,\, n \geq \left(\frac{u_{1-\frac{\alpha}{2}} S_n}{d} \right)^2 \right\},$$

where $m \geq 2$ is the initial sample size and $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$ is the sample variance of X_i's. They showed that the asymptotic coverage probability of the resulting confidence interval is really $1-\alpha$. The fully sequential procedure involves a lot of computational effort. Also, in some situations, the data are obtained in batches and the price is more proportional to the number of batches instead of observations. Stein (1945) introduced a sampling procedure requiring only two batches of data. The total sample size to

$$N = N(d) = \max\left\{m, \left\langle\left(\frac{t_{m-1}(\alpha)S_m}{d}\right)^2\right\rangle + 1\right\},$$

where $t_{m-1}(\alpha)$ denotes the $(1+\alpha)/2$ quantile of the t distribution on $m-1$ degrees of freedom and $< x >$ denotes the greatest integer which does not exceed x. If $N = m$, we do not take any more samples at the second stage. On the other hand, if $N > m$, then we sample the difference $N - m$ at the second stage. It is easily seen that the distribution of N and the coverage probabilities depend on μ, σ and d only through $c(d)$. Stein showed that I_N covers μ with probability of at least α for all $c(d) > 0$ and that $E_{c(d)}(N) \sim c(d)t^2_{m-1}/u^2_{1-\alpha/2}$ as $d \to 0$. But, the two-stage procedure is less efficient than the fully sequential procedure. Stein's two-stage procedure tends to overestimate the optimal sample size, even asymptotically. This negative feature is due to the fixed starting sample size m, because the small d can cause a small change in the estimate of σ^2, which causes big differences in the final sample size N. In order to circumvent some of these undesirable properties, Mukhopadhyay (1980) proposed a type of modification in two-stage procedure. We choose a real number $\gamma > 0$ and define the stopping rule

$$m = m(d) = \max\left\{m_0, \left\langle\left(\frac{u_{1-\frac{\alpha}{2}}}{d}\right)^{\frac{2}{1+\gamma}}\right\rangle + 1\right\}.$$

We start the experiment with this number of observation and continue as in the Stein's two-stage procedure to obtain N

$$N = N(d) = \max\left\{m, \left\langle\left(\frac{t_{m-1}(\alpha)S_m}{d}\right)^2\right\rangle + 1\right\}.$$

The stopping rule is valid for both Stein's two-stage procedure and modified two-stage procedure. However in Stein's case m is a predetermined fixed value where as in modified two-stage. The asymptotic properties of this procedure shows the first order asymptotic efficiency, namely, $\lim_{d\to 0} E(N/c(d)) = 1$. But, $\lim_{d\to 0+} \inf E(N - c(d)) = \infty$. If $\lim_{d\to 0+} \inf E(N - c(d))$ is a finite constant. Then the procedure is called asymptotically second order efficient. Later, Mukhopadhyay and Duggan (1997) suggested to improve the asymptotic properties of the two-stage procedure by adding the assumption that there exists a prior knowledge to justify that $\sigma > \sigma_L$ and $\sigma_L > 0$ is known. In the first stage, we draw

$$m = m(d) = \max\left\{m_0, \left\langle\left(\frac{u_{1-\frac{\alpha}{2}}\sigma_L}{d}\right)^2\right\rangle + 1\right\}.$$

Then, let

$$N = N(d) = \max\left\{m, \left\langle\left(\frac{t_{m-1}(\alpha)S_m}{d}\right)^2\right\rangle + 1\right\}.$$

In the second stage, we add $N - m$ observations and obtain the confidence interval $(\bar{X}_N - d, \bar{X}_N + d)$. The corresponding procedure still enjoyed the exact consistency property and asymptotically second order efficient. A drawback of this procedure might be the assumption concerning the lower bound σ_L. It is clearly desirable to choose σ_L as close to the real value of σ as possible. On the other hand, we have to be very careful that we do not choose σ_L too large as we might easily overestimate the optimal sample size. The trade-off between asymptotic second order efficiency and operational convenience came in the form of three-stage procedure. Hall (1981) introduced the three-stage procedure for normal mean which cuts the number of sampling operations to bare bones without sacrificing second order optimality.

A general three stage procedure can be defined as follows: We choose a real number $\gamma > 0$ and take an initial sample m with the stopping rule

$$m = m(d) = \max\left\{m_0, \left\langle \left(\frac{u_{1-\frac{\alpha}{2}}}{d}\right)^{\frac{2}{1+\gamma}} \right\rangle + 1\right\}.$$

Next, for $0 < k < 1$, we take additional $N_1 - m$ observations to get an intermediate sample with

$$N_1 = N_1(d) = \max\left\{m, \left\langle k\left(\frac{u_{1-\frac{\alpha}{2}} S_m}{d}\right)^2 \right\rangle + 1\right\}.$$

Finally, we take $N_2 - N_1$ observations with

$$N_2 = N_2(d) = \max\left\{N_1, \left\langle \left(\frac{u_{1-\frac{\alpha}{2}} S_{N_1}}{d}\right)^2 \right\rangle + 1\right\}$$

and confidence interval $(\bar{X}_{N_2} - d, \bar{X}_{N_2} + d)$.

2.2. Multistage procedures based on bootstrap

The multistage procedure rely on the normal approximation. So, the accuracy of those procedures depends on how good this normal approximation is. Hence, it is useful to consider approach based on bootstrap which was introduced by Swanepoel *et al.* (1983). We briefly describe their procedure. Let $F_n(x)$ denote the empirical distribution function of $X_1, X_2, \ldots, X_n$. Given $X_1, X_2, \ldots, X_n$, let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed random variables with common distribution function $F_n(x)$. And $X_n^* = X_n(Y_1, \ldots, Y_n)$ is the sample mean. Let P_n^* denote the conditional joint distribution of $Y_1, Y_2, \ldots, Y_n$ given $X_1, X_2, \ldots, X_n$. Then

$$P_n^* = P(X_n^* - d < X_n < X_n^* + d).$$

We have the bootstrap stopping rule

$$N = \inf\{n \geq m, P_n^* \geq 1 - \alpha\}.$$

Multistage procedure based on bootstrap can be defined as follows.

Table 3.1: Stein's two-stage procedure with $m_o = 10$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	15	N	21.83	0.961
0.3	43	N	57.95	0.957
0.1	384	N	524.77	0.958

Table 3.2: Modified two-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	22.85 (23.99)	0.964 (0.957)
0.5	15	SA	15.62 (14.90)	0.918 (0.925)
		SU	27.98 (31.50)	0.955 (0.965)
		N	51.33 (53.62)	0.961 (0.963)
0.3	43	SA	41.78 (40.98)	0.932 (0.925)
		SU	52.00 (57.46)	0.958 (0.968)
		N	398.34 (402.90)	0.939 (0.951)
0.1	384	SA	374.39 (377.12)	0.957 (0.943)
		SU	396.55 (400.53)	0.943 (0.935)

1. Using the standardized bootstrap, the normal quantile $u_{1-\alpha/2}$ is substituted by the $1-\alpha$ quantile of the conditional distribution of $|X_n^* - \bar{X}_n|/(S_n/\sqrt{n})$.

2. Using the studentized bootstrap, the normal quantile $u_{1-\alpha/2}$ is substituted by the $1-\alpha$ quantile of the conditional distribution of $|X_n^* - \bar{X}_n|/(S_n^*/\sqrt{n})$.

The intermediate sample size and the final sample size are calculated similarly as in the multistage procedure with the bootstrap critical points replacing the quantile $u_{1-\alpha/2}$. The generalization of three-stage procedures using bootstrap approximation of critical points was proposed in Aerts and Gijbels (1993).

3. Simulation

We have investigated the performance of the Stein's two-stage procedure, modified two-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$, modified two-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$ three-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$ and three-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$. We implement the bootstrap ideas in those procedures. A natural question could be raised as to how one may choose γ at the beginning of the experimentation. At this point, we can only give some empirical answers obtained by means of simulations. For analogous problems and a wide range of values of μ and σ, the reasonable choices of γ seem to lie between 0.1 and 0.5.

We considered $\alpha = 0.05$. That is, we required a 95% fixed-width confidence interval for μ. Three major types of distribution functions F were considered, the standard distribution, contaminated normal distribution and double exponential distribution. We obtained random samples from those distributions via random number generation. For

Table 3.3: Modified two-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	22.20 (20.94)	0.965 (0.968)
0.5	15	SA	15.52 (17.47)	0.929 (0.950)
		SU	25.94 (21.44)	0.981 (0.970)
		N	48.34 (48.58)	0.952 (0.970)
0.3	43	SA	43.35 (45.58)	0.946 (0.959)
		SU	49.44 (48.35)	0.958 (0.957)
		N	389.90 (388.71)	0.936 (0.938)
0.1	384	SA	381.86 (386.65)	0.946 (0.943)
		SU	386.92 (391.10)	0.939 (0.945)

Table 3.4: Three-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	16.93 (16.93)	0.945 (0.932)
0.5	15	SA	15.62 (15.03)	0.918 (0.909)
		SU	28.74 (32.30)	0.975 (0.969)
		N	45.18 (45.52)	0.947 (0.949)
0.3	43	SA	41.78 (41.88)	0.932 (0.926)
		SU	52.18 (57.78)	0.953 (0.963)
		N	384.43 (383.84)	0.956 (0.949)
0.1	384	SA	374.39 (379.04)	0.957 (0.944)
		SU	394.37 (401.49)	0.950 (0.942)

Table 3.5: Three-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	16.95 (17.96)	0.926 (0.962)
0.5	15	SA	16.83 (18.30)	0.953 (0.970)
		SU	16.82 (18.32)	0.934 (0.957)
		N	43.98 (45.93)	0.936 (0.947)
0.3	43	SA	44.05 (45.98)	0.943 (0.959)
		SU	44.08 (45.74)	0.941 (0.962)
		N	385.06 (385.95)	0.952 (0.960)
0.1	384	SA	384.54 (386.04)	0.948 (0.937)
		SU	386.34 (385.66)	0.962 (0.955)

each of the choices of F, we present simulation results for value of d 0.1, 0.3 and 0.5. For a value of d, suppose that N and $\bar{X}_N$ have the observed values n_i and $\bar{x}_n$, respectively, in the i^{th} replication and also let $p_i = 1$ or 0 according as $\mu \in (\notin)[\bar{x}_{n_i} \pm d]$ in the i^{th} replication, $i = 1, \ldots, k$. Let $\bar{n} = k^{-1}\sum_{i=1}^{k} n_i$, $\bar{p} = k^{-1}\sum_{i=1}^{k} p_i$. In the following Tables, we report the values of d, $c(d)$, critical point, $\bar{n}$ and $\bar{p}$ in the first block. It clear that $\bar{n}$ and $\bar{p}$ values estimate respectively $E(N)$ and $P\{\mu \in I_N\}$. Since asymptotics here means $d \to 0$ it is expected that all procedures perform better for smaller values of d. In all Tables, the third column specifies the critical points which were used for calculations.

Table 3.6: Stein's Two-stage Procedure with $m_o = 10$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	22	N	30.33	0.968
0.3	60	N	82.93	0.939
0.1	538	N	716.36	0.942

Table 3.7: Modified two-stage procedure with $m_0 = 2$, $\sigma_L = 1/3$, $\sigma_L = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	22	N	32.02 (31.99)	0.945 (0.957)
		SA	20.39 (20.10)	0.917 (0.920)
		SU	60.50 (65.32)	0.976 (0.964)
0.3	60	N	70.12 (74.01)	0.952 (0.945)
		SA	57.50 (58.11)	0.927 (0.916)
		SU	90.49 (107.42)	0.942 (0.947)
0.1	538	N	567.64 (565.68)	0.950 (0.935)
		SA	525.72 (519.97)	0.945 (0.913)
		SU	558.12 (579.56)	0.935 (0.945)

N denotes the normal critical points, SA stand for the standardized bootstrap critical points and SU denotes the studentized bootstrap critical points.

3.1. Simulation results for standard normal distribution

The coverage probabilities and stopping rules given Table 3.1–3.5 suggest that three-stage procedure tends to stop too early. The standardized bootstrap method tends to underestimate the optimal sample size. From the point of view of the coverage probability obtained for bootstrap method are not much worse. But, there is no advantage in using a bootstrap critical points instead of a normal critical points. Underneath the column for mean and coverage probability, in the fourth block and fifth block of Table 3.2, 3.4, the numbers within $(\cdot)$ correspond to $\gamma = 1/2$. Similarly, underneath the column for mean and coverage probability, in the fourth block and fifth block of Table 3.3, 3.5, the numbers within $(\cdot)$ correspond to $\sigma_L = 0.95$.

3.2. Simulation results for contaminated normal distributions

Now we wish to explore the behavior of the proposed procedures when the population distributions are slightly different from normal. Consider a contaminated normal distribution with

$$0.95N(0,1) + 0.05N(0,3).$$

From the Table 3.6–3.10, we see that the coverage probabilities of the three-stage procedure based on the standardized bootstrap method lie below the desired value. But the coverage probabilities of normal approximation and the studentized bootstrap based

Table 3.8: Modified two-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	30.52 (27.36)	0.948 (0.967)
0.5	22	SA	20.58 (22.45)	0.927 (0.960)
		SU	36.90 (29.28)	0.968 (0.963)
		N	68.05 (65.01)	0.957 (0.943)
0.3	60	SA	60.44 (60.17)	0.935 (0.947)
		SU	68.69 (65.66)	0.954 (0.955)
		N	544.34 (542.01)	0.949 (0.957)
0.1	538	SA	534.15 (535.26)	0.955 (0.939)
		SU	547.82 (541.26)	0.956 (0.937)

Table 3.9: Three-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	23.79 (24.49)	0.936 (0.943)
0.5	22	SA	20.98 (20.77)	0.934 (0.915)
		SU	45.67 (51.90)	0.972 (0.976)
		N	60.44 (60.51)	0.932 (0.943)
0.3	60	SA	56.62 (55.94)	0.935 (0.915)
		SU	80.71 (87.75)	0.961 (0.958)
		N	543.98 (548.29)	0.956 (0.953)
0.1	538	SA	530.91 (528.64)	0.949 (0.930)
		SU	563.04 (575.93)	0.957 (0.949)

Table 3.10: Three-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
		N	23.28 (24.01)	0.942 (0.963)
0.5	22	SA	24.62 (24.32)	0.938 (0.966)
		SU	23.00 (23.83)	0.930 (0.956)
		N	62.01 (62.27)	0.950 (0.954)
0.3	60	SA	61.87 (62.05)	0.939 (0.960)
		SU	62.94 (62.83)	0.942 (0.959)
		N	539.66 (536.74)	0.949 (0.953)
0.1	538	SA	536.05 (541.01)	0.955 (0.940)
		SU	540.57 (539.05)	0.945 (0.955)

procedures lie in all cases closer to 0.95. Hence the procedure based on studentized bootstrap method can be recommended. Underneath the column for mean and coverage probability, in the fourth block and fifth block of Table 3.7, 3.9, 3.12 and 3.14, the numbers within $(\cdot)$ correspond to $\gamma = 1/2$. Similarly, underneath the column for mean and coverage probability, in the fourth block and fifth block of Table 3.8, 3.10, 3.13 and 3.15, the numbers within $(\cdot)$ correspond to $\sigma_L = 0.95$.

Table 3.11: Stein's Two-stage Procedure with $m_0 = 10$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	31	N	41.07	0.948
0.3	85	N	119.98	0.944
0.1	768	N	1020.92	0.933

Table 3.12: Modified two-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	31	N	43.57 (44.24)	0.954 (0.951)
		SA	28.82 (28.88)	0.921 (0.911)
		SU	78.13 (86.32)	0.970 (0.967)
0.3	85	N	101.93 (106.67)	0.954 (0.950)
		SA	80.31 (79.17)	0.932 (0.895)
		SU	124.85 (134.01)	0.959 (0.948)
0.1	768	N	793.68 (802.00)	0.951 (0.943)
		SA	761.79 (754.58)	0.947 (0.937)
		SU	798.94 (823.65)	0.950 (0.944)

Table 3.13: Modified two-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	31	N	43.62 (39.08)	0.956 (0.962)
		SA	28.05 (30.58)	0.927 (0.934)
		SU	54.43 (46.10)	0.969 (0.970)
0.3	85	N	96.38 (92.65)	0.958 (0.958)
		SA	83.44 (84.20)	0.929 (0.929)
		SU	147.86 (93.81)	0.947 (0.953)
0.1	768	N	778.43 (771.38)	0.947 (0.967)
		SA	760.34 (763.17)	0.954 (0.954)
		SU	788.41 (770.42)	0.940 (0.933)

3.3. Simulation results for double exponential distribution

We simulated observations from the double exponential probability density function which is given by

$$f(x) = \frac{1}{2}e^{-|x|}, \quad x \in R.$$

From the Table 3.11–3.15, we see that the three-stage procedure are better than the other procedures. The coverage probabilities for normal approximation and the bootstrap based on studentized critical points lie closer to the 0.95. Bootstrap based on standardized critical points seems to underestimate the optimal sample size. Underneath the column for mean and coverage probability, in the fourth block and fifth block of Table 3.12 and 3.14, the numbers within $(\cdot)$ correspond to $\gamma = 1/2$. Similarly, underneath the column for mean and coverage probability, in the fourth block and fifth block

Table 3.14: Three-stage procedure with $m_0 = 2$, $\gamma = 1/3$, $\gamma = 1/2$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	31	N	33.10 (32.49)	0.942 (0.936)
		SA	28.45 (28.21)	0.916 (0.908)
		SU	68.70 (78.53)	0.971 (0.978)
0.3	85	N	86.93 (89.78)	0.932 (0.929)
		SA	80.05 (80.37)	0.933 (0.911)
		SU	112.21 (129.74)	0.947 (0.964)
0.1	768	N	759.94 (777.28)	0.947 (0.954)
		SA	758.53 (747.49)	0.947 (0.960)
		SU	790.44 (816.38)	0.956 (0.946)

Table 3.15: Three-stage procedure with $m_0 = 2$, $\sigma_L = 0.75$, $\sigma_L = 0.95$

d	$c(d)$	critical points	mean($\bar{n}$)	coverage probability($\bar{p}$)
0.5	31	N	32.85 (31.74)	0.943 (0.954)
		SA	32.33 (32.00)	0.960 (0.931)
		SU	31.58 (32.55)	0.927 (0.957)
0.3	85	N	87.67 (85.99)	0.941 (0.943)
		SA	86.04 (86.22)	0.922 (0.940)
		SU	85.69 (85.90)	0.944 (0.947)
0.1	768	N	762.75 (774.70)	0.960 (0.942)
		SA	772.18 (765.26)	0.937 (0.950)
		SU	769.70 (766.01)	0.938 (0.951)

of Table 3.13 and 3.15, the numbers within $(\cdot)$ correspond to $\sigma_L = 0.95$.

3.4. Concluding remarks

We recommend a three-stage procedure based on the studentized bootstrap critical points. Same recommendation is put forth even when the population distributions are somewhat nonnormal.

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