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GENERALIZATION OF A TRANSFORMATION FORMULA FOUND BY BAILLON AND BRUCK

Arjun K. Rathie and Yong Sup Kim

Abstract. We aim mainly at presenting a generalization of a transformation formula found by Baillon and Bruck. The result is derived with the help of the well-known quadratic transformation formula due to Gauss.

1. Introduction

There was an open problem posed by Baillon and Bruck [1, Eq.(9.10)] who needed to verify the following hypergeometric identity

$$\begin{aligned} &{}_2F_1\left[\begin{matrix}\frac{1}{2}, -m \\ 2\end{matrix}; 4x(1-x)\right] \\ &= (m+1)(1-x)x^{2m-1}{}_2F_1\left[\begin{matrix}-m,\ -m \\ 2\end{matrix}; \left(\frac{1-x}{x}\right)^2\right] \\ &\quad + (2x-1)x^{2m-1}{}_2F_1\left[\begin{matrix}-m,\ -m \\ 1\end{matrix}; \left(\frac{1-x}{x}\right)^2\right] \end{aligned} \tag{1.1}$$

in order to derive a quantitative form of the Ishikawa-Edelstein-O'Brain asymptotic regularity theorem. Using Zeilberger's algorithm [4], Baillon and Bruck [1] gave a computer proof of this identity which is the key to the integral representation [1, Eq.(2.1)] of their main theorem. In 1995, Paule [2] gave the proof of (1.1) by using classical hypergeometric machinery by means of the following contiguous relations:

$$\frac{abz}{c(c-1)}{}_2F_1\left[\begin{matrix}a+1,\ b+1 \\ c+1\end{matrix}; z\right] = {}_2F_1\left[\begin{matrix}a,\ b \\ c-1\end{matrix}; z\right] - {}_2F_1\left[\begin{matrix}a,\ b \\ c\end{matrix}; z\right], \tag{1.2}$$

$${}_2F_1\left[\begin{matrix}a+1,\ b \\ c+1\end{matrix}; z\right] = \frac{a-c}{a}{}_2F_1\left[\begin{matrix}a,\ b \\ c+1\end{matrix}; z\right] + \frac{c}{a}{}_2F_1\left[\begin{matrix}a,\ b \\ c\end{matrix}; z\right], \tag{1.3}$$

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and the following well-known quadratic transformation formula [3] due to Gauss

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ 2b \end{matrix};\ \frac{4z}{(1+z)^2}\right] = (1+z)^{2a}\,{}_2F_1\left[\begin{matrix} a,\ a-b+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix};\ z^2\right]. \tag{1.4} $$

The aim of this short paper is to provide a generalization of (1.1) by employing the transformation formula (1.4).

2. Main result

The following a generalization of the result (1.1) will be established:

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a,\ b \\ 2b+1 \end{matrix};\ 4z(1-z)\right] \\ &= z^{-2a}\left\{{}_2F_1\left[\begin{matrix} a,\ a-b+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix};\ \left(\frac{1-z}{z}\right)^2\right]\right. \\ &\quad \left. + \frac{2a(1-z)}{(2b+1)z}\,{}_2F_1\left[\begin{matrix} a+1,\ a-b+\frac{1}{2} \\ b+\frac{3}{2} \end{matrix};\ \left(\frac{1-z}{z}\right)^2\right]\right\}. \end{aligned} \tag{2.1} $$

3. Proof of (2.1)

In order to prove the main result (2.1), we proceed as follows. From (1.2), we obtain the following relation:

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ 2b+1 \end{matrix};\ x\right] = {}_2F_1\left[\begin{matrix} a,\ b \\ 2b \end{matrix};\ x\right] - \frac{ax}{2(2b+1)}\,{}_2F_1\left[\begin{matrix} a+1,\ b+1 \\ 2b+2 \end{matrix};\ x\right]. \tag{3.1} $$

Now, put $x = -\frac{4y}{(1-y)^2}$, we get

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a,\ b \\ 2b+1 \end{matrix};\ -\frac{4y}{(1-y)^2}\right] &= {}_2F_1\left[\begin{matrix} a,\ b \\ 2b \end{matrix};\ -\frac{4y}{(1-y)^2}\right] \\ &+ \frac{2ay}{(2b+1)(1-y)^2}\,{}_2F_1\left[\begin{matrix} a+1,\ b+1 \\ 2b+2 \end{matrix};\ -\frac{4y}{(1-y)^2}\right]. \end{aligned} \tag{3.2} $$

Multiplying both sides of (3.2) by $(1-y)^{-2a}$, we get

$$ \begin{aligned} &(1-y)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\ b \\ 2b+1 \end{matrix};\ -\frac{4y}{(1-y)^2}\right] = (1-y)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\ b \\ 2b \end{matrix};\ -\frac{4y}{(1-y)^2}\right] \\ &\quad + \frac{2ay}{(2b+1)}(1-y)^{-2(a+1)}\,{}_2F_1\left[\begin{matrix} a+1,\ b+1 \\ 2b+2 \end{matrix};\ -\frac{4y}{(1-y)^2}\right]. \end{aligned} \tag{3.3} $$

Now it is easy to see that the two ${}_2F_1$'s on the right hand side of (3.3) can be evaluated with the help of the Gauss' quadratic transformation formula (1.4), we get

$$(3.4)\quad \begin{aligned}(1-y)^{-2a}{}_2F_1\left[\begin{matrix}a,\ b\\ 2b+1\end{matrix};-\frac{4y}{(1-y)^2}\right] &= {}_2F_1\left[\begin{matrix}a,\ a-b+\frac{1}{2}\\ b+\frac{1}{2}\end{matrix};y^2\right]\\ &+\frac{2ay}{(2b+1)}{}_2F_1\left[\begin{matrix}a+1,\ a-b+\frac{1}{2}\\ b+\frac{3}{2}\end{matrix};y^2\right],\end{aligned}$$

which can be written as

$$(3.5)\quad \begin{aligned}{}_2F_1\left[\begin{matrix}a,\ b\\ 2b+1\end{matrix};-\frac{4y}{(1-y)^2}\right] &= (1-y)^{2a}\Bigg\{{}_2F_1\left[\begin{matrix}a,\ a-b+\frac{1}{2}\\ b+\frac{1}{2}\end{matrix};y^2\right]\\ &+\frac{2ay}{(2b+1)}{}_2F_1\left[\begin{matrix}a+1,\ a-b+\frac{1}{2}\\ b+\frac{3}{2}\end{matrix};y^2\right]\Bigg\}.\end{aligned}$$

Now, changing y to $-y$, we get

$$(3.6)\quad \begin{aligned}{}_2F_1\left[\begin{matrix}a,\ b\\ 2b+1\end{matrix};\frac{4y}{(1+y)^2}\right] &= (1+y)^{2a}\Bigg\{{}_2F_1\left[\begin{matrix}a,\ a-b+\frac{1}{2}\\ b+\frac{1}{2}\end{matrix};y^2\right]\\ &-\frac{2ay}{(2b+1)}{}_2F_1\left[\begin{matrix}a+1,\ a-b+\frac{1}{2}\\ b+\frac{3}{2}\end{matrix};y^2\right]\Bigg\}.\end{aligned}$$

Finally, taking $y=\frac{1-z}{z}$ and we, after a little simplification, have

$$(3.7)\quad \begin{aligned}{}_2F_1\left[\begin{matrix}a,\ b\\ 2b+1\end{matrix};4z(1-z)\right] &= z^{-2a}\Bigg\{{}_2F_1\left[\begin{matrix}a,\ a-b+\frac{1}{2}\\ b+\frac{1}{2}\end{matrix};\left(\frac{1-z}{z}\right)^2\right]\\ &-\frac{2a(1-z)}{(2b+1)z}{}_2F_1\left[\begin{matrix}a+1,\ a-b+\frac{1}{2}\\ b+\frac{3}{2}\end{matrix};\left(\frac{1-z}{z}\right)^2\right]\Bigg\}.\end{aligned}$$

This completes the proof of (2.1).

4. Special case

In our main transformation formula (2.1), if we take $a=-m$ and $b=\frac{1}{2}$, we obtain

$$(4.1)\quad \begin{aligned}{}_2F_1\left[\begin{matrix}\frac{1}{2},\ -m\\ 2\end{matrix};4z(1-z)\right] &= z^{2m}\Bigg\{{}_2F_1\left[\begin{matrix}-m,\ -m\\ 1\end{matrix};\left(\frac{1-z}{z}\right)^2\right]\\ &+m\left(\frac{1-z}{z}\right){}_2F_1\left[\begin{matrix}-m,\ -m+1\\ 2\end{matrix};\left(\frac{1-z}{z}\right)^2\right]\Bigg\}.\end{aligned}$$

Equation (4.1) is an alternate form of the result (1.1) due to Baillon and Bruck.

Remark. The result (1.1) in its exact form can be obtained from (4.1) by using (1.3) with $a=b=-m$ and $c=1$.

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Arjun K. Rathie
Department of Mathematics
Vedant College of Engineering and Technology
Tulsi-323021, Dist. Bundi, Rajasthan State, India
E-mail address: akrathie@rediffmail.com

Yong Sup Kim
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail address: yspkim@wonkwang.ac.kr