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ON THE COMPLETE MOMENT CONVERGENCE OF MOVING AVERAGE PROCESSES GENERATED BY ρ^*-MIXING SEQUENCES

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Abstract. Let $\{Y_i; -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed and ρ^*-mixing random variables with zero means and finite variances and $\{a_i; -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete moment convergence of $\{\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_{i+k}Y_i/n^{1/p}; n \geq 1\}$ under some suitable conditions. We extend Theorem 1.1 of Li and Zhang [Y. X. Li and L. X. Zhang, *Complete moment convergence of moving average processes under dependence assumptions,* Statist. Probab. Lett. **70** (2004), 191–197.] to the ρ^*-mixing case.

1. Introduction

We assume that $\{Y_i; -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables with zero means and finite variances. Let $\{a_i; -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k}Y_i, k \geq 1. \tag{1.1}$$

Under independence assumptions, i.e., $\{Y_i; -\infty < i < \infty\}$ is a sequence of independent random variables, many limiting results have been obtained for moving average process $\{X_k; k \geq 1\}$. For examples, Ibragimov [6] has established the central limit theorem for $\{X_k; k \geq 1\}$, Burton and Dehling [4] have obtained a large deviation principle for $\{X_k; k \geq 1\}$ assuming $E\exp(tY_1) < \infty$ for all t, and Li et al. [7] have obtained the following result on complete convergence.

Theorem A. *Suppose $\{Y_i; -\infty < i < \infty\}$ is a sequence of independent and identically distributed random variables. Let $\{X_k; k \geq 1\}$ be defined as* (1.1)

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and $1 \le p < 2$. *Then* $EY_1 = 0$ *and* $E|Y_1|^{2p} < \infty$ *imply*

$$\sum_{n=1}^{\infty} P\{|\sum_{k=1}^{n} X_k| \ge n^{1/p}\epsilon\} < \infty \ for\ all\ \epsilon > 0.$$

Zhang [12] extended Theorem A to the ϕ-mixing case and Baek, Kim, and Liang [1] discussed the complete convergence of moving average processes under negative association assumption and Liang [9] obtained some general results on the complete convergence of weighted sums of negatively associated random variables, including moving average processes.

When $\{X_k; k \ge 1\}$ is a sequence of i.i.d random variables with mean zeros and positive finite variances, Chow [5] obtained the following result on the complete moment convergence:

Theorem B. *Suppose that* $\{X_k; k \ge 1\}$ *is a sequence of i.i.d random variables with* $EX_1 = 0$. *For* $1 \le p < 2$ *and* $r > p$, *if* $E\{|X_1|^r + |X_1| \log(1+|X_1|)\} < \infty$, *then for any* $\epsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} E\{|\sum_{k=1}^{\infty} X_k| - \epsilon n^{1/p}\}^+ < \infty.$$

Recently Li and Zhang [8] showed that this kind of result also holds for moving average processes under negative association as follows:

Theorem C. *Suppose* $\{X_k; k \ge 1\}$ *is defined as* (1.1), *where* $\{a_i; -\infty < i < \infty\}$ *is a sequence of real numbers with* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ *and* $\{Y_i; -\infty < i < \infty\}$ *is a sequence of identically distributed and negatively associated random variables with* $EY_1 = 0, EY_1^2 < \infty$. *Let* $h(x) > 0 (x > 0)$ *be a slowly varying function and* $1 \le p < 2$, $r > 1 + p/2$. *Then* $E|Y_1|^r h(|Y_1|^p) < \infty$ *implies* $\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ < \infty$, *where* $S_n = \sum_{k=1}^{n} X_k, n \ge 1$.

Let $\{Y_n, n \ge 1\}$ be a sequence of random variables. Let S be a subset of natural number set N and $F_S = \sigma(Y_k, k \in S)$. Define $\rho_n^* = \sup\{\text{corr}(f,g) :$ For all $S \times T \subset N \times N, \text{dist}(S,T) \ge n,\ \forall f \in L^2(F_S), g \in L^2(F_T)\}$, where

$$\text{corr}(f,g) = \frac{\text{Cov}\{f(Y_i, i \in S), g(Y_j, j \in T)\}}{[\text{Var}\{f(X_i, i \in S)\}\text{Var}\{g(X_j, j \in T)\}]^{\frac{1}{2}}}.$$

We call $\{Y_n, n \ge 1\}$ is a ρ^*-mixing sequence if

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.2}$$

Let us note that, since $0 \le \cdots \le \rho_n^* \le \rho_{n-1}^* \le \cdots \le \rho_1^* \le 1$, (1.2) is equivalent to

$$\rho_N^* < 1 \text{ for some } N > 1. \tag{1.3}$$

Bryc and Smolenski [3] and Peligrad and Gut [11] pointed out the importance of condition (1.2) in estimating the moments of partial sums or of maximum of partial sums. Various limit properties under the condition $\lim \rho_n^* < 1$ were studied by Bradley [2] and Miller [10].

In this paper we shall extend Theorem C to the ρ^*-mixing case.

2. Results

The following lemma comes from Burton and Dehling [4].

Lemma 2.1. *Let $\sum_{-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{-\infty}^{\infty} a_i$ and $k \geq 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} |\sum_{j=i+1}^{i+n} a_j|^k = |a|^k. \tag{2.1}$$

The following lemma will be useful. A proof appears in Peligrad and Gut [11].

Lemma 2.2. *Let $\{Y_n; n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $EY_i = 0$ and $E|Y_i|^q < \infty$ for some $q \geq 2$. Assume that $\lim_{n\to\infty} \rho_n^* < 1$. Then there exists a constant $C(q, N, \rho_N^*)$, depending on q, N, and ρ_N^*, with N and ρ_N^* defined via (1.3), such that*

$$E|S_n|^q \leq C(q, N, \rho_N^*)(\sum_{i=1}^{n} E|Y_i|^q + (\sum_{i=1}^{n} EY_i^2)^{\frac{q}{2}}) \ \forall q \geq 2. \tag{2.2}$$

Our main result is as follows:

Theorem 2.3. *Set $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, where $\{X_k; k \geq 1\}$ is defined as (1.1). Suppose that $\{Y_i; -\infty < i < \infty\}$ is a sequence of identically distributed and ρ_n^*-mixing random variables with $EY_1 = 0$, $E|Y_1|^q < \infty$ for some $q \geq 2$ and $\lim_{n\to\infty} \rho_n^* < 1$ and that $\{a_i; -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Let $h(x) > 0 (x > 0)$ be a slowly varying function and $1 \leq p < 2$, $r > p$. Then $E|Y_1|^r h(|Y_1|^p) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ < \infty \ for\ all\ \epsilon > 0. \tag{2.3}$$

Remark. Let $a_{i+k} = 1, i = k; a_{i+k} = 0, i \neq k, 1 \leq k \leq n$. Then $X_k = Y_k$, $S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} Y_k$. Hence Theorem 2.3 holds when $\{X_k; k \geq 1\}$ is a sequence of identically distributed and ρ^*-mixing random variables.

Corollary 2.4. *Under the conditions of Theorem 2.3 $E|Y_1|^r h(|Y_1|^p) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{r/p-2} h(n) P\{|S_n| > \epsilon n^{1/p}\} < \infty \ for\ all\ \epsilon > 0. \tag{2.4}$$

Proof. By Theorem 2.3 we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ \\
&= \sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) \int_0^{\infty} P\{|S_n| - \epsilon n^{\frac{1}{p}} > x\} dx \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) P\{|S_n| > (\epsilon + y) n^{\frac{1}{p}}\} n^{\frac{1}{p}} dy \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} n^{r/p-2} h(n) P\{|S_n| > (\epsilon + y) n^{\frac{1}{p}}\} dy < \infty.
\end{aligned} \tag{2.5}$$

Hence from (2.5) the result (2.4) follows. □

3. Proof of Theorem 2.3

Recall that

$$\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{i+k} Y_i = \sum_{i=-\infty}^{\infty} a_{ni} Y_i,$$

where $a_{ni} = \sum_{k=1}^{n} a_{i+k}$.

From Lemma 2.1, we can assume, without loss of generality, that

$$\sum_{i=-\infty}^{\infty} |a_{ni}| \leq n, n \geq 1 \text{ and } \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \leq 1.$$

Let $S_n = \sum_{-\infty}^{\infty} a_{ni} Y_i I\{|a_{ni} Y_i| \leq x\}$. First note that for $x > n^{\frac{1}{p}}$,

$$\begin{aligned}
x^{-1}|ES_n| &= x^{-1} \Big| \sum_{i=-\infty}^{\infty} a_{ni} E Y_i I\{|a_{ni} Y_i| > x\} \Big| \\
&\leq x^{-1} \sum_{i=-\infty}^{\infty} |a_{ni}| E|Y_1| I\{|a_{ni} Y_1| > x\} \\
&\leq x^{-1} n E|Y_1| I\{\tilde{a}|Y_1| > x\} \\
&\leq x^{-1} n E|Y_1| I\{|Y_1| > x\} \\
&\leq x^{-1} x^p E|Y_1| I\{\tilde{a}|Y_1| > x\} \\
&\leq E|Y_1|^p I\{|Y_1| > x\} \to 0 \text{ as } x \to \infty.
\end{aligned}$$

So, for x large enough we have $x^{-1} E|S_n| < \epsilon/2$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) E\{|\sum_{k=1}^{n} X_k| - \epsilon n^{\frac{1}{p}}\}^+ \\
&= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{\epsilon n^{\frac{1}{p}}}^{\infty} P\{|\sum_{k=1}^{n} X_k| \geq x\} dx \text{ (letting } x = \epsilon x')
\end{aligned}$$

$$= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\epsilon \int_{n^{\frac{1}{p}}}^{\infty} P\{|\sum_{k=1}^{n} X_k| \geq \epsilon x^{'}\}dx^{'} \quad \text{(letting } x' = x\text{)}$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\epsilon \int_{n^{\frac{1}{p}}}^{\infty} (P\{\sup_i |a_{ni}Y_i| \geq x\} + P\{|S_n - ES_n| \geq x\frac{\epsilon}{2}\})dx$$

$$= C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\epsilon \int_{n^{\frac{1}{p}}}^{\infty} (I_1 + I_2)dx,$$

where $I_1 = P\{\sup_i |a_{ni}Y_i| > x\}$ and $I_2 = P\{|S_n - ES_n| \geq x\frac{\epsilon}{2}\}$.

Set $I_{nj} = \{i \in \mathcal{I}; (j+1)^{-\frac{1}{p}} < |a_{ni}| \leq j^{-\frac{1}{p}}\}, j = 1, 2, \ldots$. Then $\cup_{j\geq 1} I_{nj} = \mathcal{I}$. Note that (cf. Li et al. [7])

$$\sum_{j=1}^{k} \# I_{nj} \leq n(k+1)^{\frac{1}{p}}.$$

For I_1 and $1 \leq p < 2, r \geq p$ noting that $E|Y_1|^r h(|Y_1|^p) < \infty$, we get

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_1 dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{i=-\infty}^{\infty} P\{|a_{ni}Y_i| > x\}dx$$

$$= C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{i=-\infty}^{\infty} P\{|a_{ni}Y_1| > x\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} P\{|Y_1| > j^{\frac{1}{p}} x\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{j=1}^{\infty} (\# I_{nj}) \sum_{k\geq jx^p} P\{k \leq |Y_1|^p < k+1\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} \sum_{j=1}^{[k/x^p]} (\# I_{nj}) P\{k \leq |Y_1|^p < k+1\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} n(\frac{k}{x^p} + 1)^{\frac{1}{p}} P\{k \leq |Y_1|^p < k+1\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} k^{\frac{1}{p}} x^{-1} P\{k \leq |Y_1|^p < k+1\}dx$$

$$\leq C\int_{1}^{\infty} t^{\frac{r}{p}-1-\frac{1}{p}} h(t) \int_{t^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} k^{\frac{1}{p}} x^{-1} P\{k \leq |Y_1|^p < k+1\}dxdt$$

(letting $y = t^{\frac{1}{p}}$)

$$\le C\int_1^\infty y^{r-2}h(y^p)\int_y^\infty x^{-1}\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dxdy$$
$$\le C\int_1^\infty (\int_1^x y^{r-2}h(y^p)dy)x^{-1}\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dx$$
$$\le C\int_1^\infty x^{r-2}h(x^p)\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dx$$
$$\le C\sum_{k=1}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}\int_1^{(k+1)^{\frac{1}{p}}} x^{r-2}h(x^p)dx$$
$$\le C\sum_{k=1}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}(k+1)^{\frac{r-1}{p}}h(k+1)$$
$$\le C\sum_{k=0}^\infty (k+1)^{\frac{r}{p}}h(k+1)P\{k\le |Y_1|^p<k+1\}$$
$$\le CE|Y_1|^r h(|Y_1|^p)+1<\infty.$$

Now we estimate I_2, for $1\le p<2,\ r>1+\frac{p}{2}$. By Lemma 2.2 and Markov's inequality, we have for $q\ge 2$

$$P\{|S_n-ES_n|\ge \frac{\epsilon}{2}x\}\le Cx^{-q}E|S_n-ES_n|^q$$
$$\le Cx^{-q}((\sum_{i=-\infty}^\infty a_{ni}^2EY_1^2I\{|a_{ni}Y_1|\le x\})^{q/2}+\sum_{i=-\infty}^\infty E|a_{ni}Y_1|^qI\{|a_{ni}Y_1|\le x\}).$$

Then

$$\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty I_2dx \tag{3.1}$$
$$\le C\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty x^{-q}(\sum_{i=-\infty}^\infty a_{ni}^2EY_1^2I\{|a_{ni}Y_1|\le x\})^{\frac{q}{2}}dx$$
$$+\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty x^{-q}\sum_{i=-\infty}^\infty E|a_{ni}Y_1|^qI\{|a_{ni}Y_1|\le x\}dx$$
$$=C\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty (I_3+I_4)dx.$$

If $q\ge 2$ is large enough such that $q(\frac{1}{p}-\frac{1}{2})>\frac{r}{p}-1$, then for I_3 we get

$$\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty I_3dx \tag{3.2}$$

$$= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \Big(\sum_{i=-\infty}^{\infty} a_{ni}^2 E Y_1^2 I\{|a_{ni}Y_1| \le x\}\Big)^{\frac{q}{2}} dx$$

$$\le C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}+\frac{q}{2}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} dx = C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-q(\frac{1}{p}-\frac{1}{2})} h(n) < \infty.$$

For I_4 and $r \ge 2$ we get

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_4 dx$$

$$= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{i=-\infty}^{\infty} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \le x\} dx$$

$$\le C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \le x\} dx$$

$$\le C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\#I_{nj}) j^{-\frac{q}{p}} E|Y_1|^q I\{|Y_1|^p \le x^p (j+1)\} dx$$

$$\le C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\#I_{nj}) j^{-\frac{q}{p}}$$

$$\times \sum_{0 \le k \le (j+1)x^p} E|Y_1|^q I\{k \le |Y_1|^p < k+1\} dx.$$

$$\le C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \Big[\int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\#I_{nj}) j^{-\frac{q}{p}} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\} dx$$

$$+ \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\#I_{nj}) j^{-\frac{q}{p}} \sum_{k=[2x^p]+1}^{[(j+1)x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\} dx \Big]$$

$$= C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} (I_5 + I_6) dx.$$

Note that for $q \ge 1$ and $m \ge 1$, we have

$$n \ge \sum_{i=-\infty}^{\infty} |a_{ni}| = \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}| \ge \sum_{j=1}^{\infty} (\#I_{nj}) (j+1)^{-\frac{1}{p}}$$

$$\ge \sum_{j=m}^{\infty} (\#I_{nj}) (j+1)^{-\frac{1}{p}} \ge \sum_{j=m}^{\infty} (\#I_{nj}) (j+1)^{-\frac{q}{p}} (m+1)^{\frac{q}{p}-\frac{1}{p}}.$$

So

$$\sum_{j=m}^{\infty} (\#I_{nj}) j^{-q/p} \le C n m^{-(q-1)/p}.$$

If $r \geq 2$ and $q > r$, for I_5 we get

(3.3)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_5 dx \\
&= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\# I_{nj}) j^{-\frac{q}{p}} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) n \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx \\
&\leq C \int_1^{\infty} t^{\frac{r}{p}-1-\frac{1}{p}} h(t) \int_{t^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx dt \\
&\quad \text{letting } t = y^p \\
&\leq C \int_1^{\infty} y^{r-2} h(y^p) \int_{y}^{\infty} x^{-q} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx dy \\
&\leq C \int_1^{\infty} \left(\int_1^{x} y^{r-2} h(y^p) dy\right) x^{-q} \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx \\
&\leq C \int_1^{\infty} x^{r-1-q} h(x^p) \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx \\
&\leq C \sum_{k=1}^{\infty} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} \int_{(\frac{k}{2})^{\frac{1}{p}}}^{\infty} x^{r-1-q} h(x^p) dx \\
&\leq C \sum_{k=1}^{\infty} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} k^{\frac{r-q}{p}} h(k) \\
&\leq C \sum_{k=1}^{\infty} (k+1)^{\frac{r}{p}} h(k+1) P\{k \leq |Y_1|^p < k+1\} \\
&\leq C E|Y_1|^r h(|Y_1|^p) + 1 < \infty.
\end{aligned}$$

If $r \geq 2$, then for I_6, $1 \leq p < 2$, and $p > 1 + \frac{p}{2}$, we also get

(3.4)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_6 dx \\
&= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\# I_{nj}) j^{-\frac{q}{p}} \sum_{k=[2x^p]+1}^{[(j+1)x^p]} E|Y_1|^q I\{k \leq |Y_1|^p < k+1\} dx
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{k=[2x^p]+1}^{\infty}\sum_{j\geq[\frac{k}{x^p}]-1}(\sharp I_{nj})j^{-\frac{q}{p}}E|Y_1|^q I\{k\leq|Y_1|^q<k+1\}dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{k=[2x^p]+1}^{\infty} n(\frac{k}{x^p})^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}dx$$

$$\leq C\int_1^{\infty} t^{\frac{r}{p}-1-\frac{1}{p}}h(t)\int_{t^{\frac{1}{p}}}^{\infty} x^{-1}\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}dxdt$$

letting $t=y^p$

$$\leq C\int_1^{\infty} y^{r-2}h(y^p)\int_y^{\infty} x^{-1}\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}dxdy$$

$$\leq C\int_1^{\infty}(\int_1^x y^{r-2}h(y^p)dy)x^{-1}\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}dx$$

$$\leq C\int_1^{\infty} x^{r-2}h(x^p)\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}dx$$

$$\leq C\sum_{k=1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}\int_0^{(\frac{k}{2})^{\frac{1}{p}}} x^{r-2}h(x^p)dx$$

$$\leq C\sum_{k=1}^{\infty} k^{\frac{r-q}{p}}h(k)E|Y_1|^q I\{k\leq|Y_1|^p<k+1\}$$

$$\leq C\sum_{k=1}^{\infty}(k+1)^{\frac{r}{p}}h(k+1)P\{k\leq|Y_1|^p<k+1\}$$

$$\leq CE|Y_1|^r h(|Y_1|^p)+1<\infty.$$

So by (3.3) and (3.4) we get

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} I_4 dx<\infty \tag{3.5}$$

for $r\geq 2$ and $q>r$.

Note that $r\geq 2,\ q>2$ and $q(\frac{1}{p}-\frac{1}{2})>\frac{r}{p}-1$ imply $q>r$. Hence, (3.2) and (3.5) yield

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} I_2 dx<\infty \tag{3.6}$$

for $r\geq 2$ and $q>2$ such that $q(\frac{1}{p}-\frac{1}{2})>\frac{r}{p}-1$.

If $1+\frac{p}{2}<r<2$ and $q=2$, then (3.6) follows from (3.1) and (3.2) since $I_3=I_4$. Thus we have $\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)E\{|S_n|-\epsilon n^{\frac{1}{p}}\}^+<\infty$ for all $\epsilon>0$.

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