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A NOTE ON THE VOLUME COMPARISON OF TUBES AROUND GEODESICS

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Abstract. In this paper, we shall calculate the volume of normal tubes around geodesics under a curvature perturbation to establish a theorem of volume comparison type.

1. Introduction

Let M be an n-dimensional Riemannian manifold with a submanifold N of dimension k $(< n)$. The *normal tube of radius* r around N is defined as $T(N,r) = \{x \in M \mid x = \exp_N(tv),\ \text{where } |t| < r \text{ and } v \in \nu(N)\}$. Here, $\nu(N)$ is the normal bundle of N in M consisting of vectors perpendicular to N and $\exp_N : \nu(N) \to M$ is the normal exponential map. Some formulas for the volume of tubes around N, where N is a hypersurface or a point are studied in [3], [10] and a new method was found for estimating the volume of tubes around *closed* geodesics in [6].

When N is any geodesic $\gamma : [0,a] \to M$, it was also shown in [8] that an analogue of the result of Heintz and Karcher [6] can be obtained in the situation where one has L^p-curvature bounds, which measure the quantities of the sectional curvature lying below a given number λ in the L^p-norm. More specifically, if $f(x)$ is the smallest sectional curvature of a plane in T_xM, then we consider

$$K(\lambda,p) = \int_M \{\max\{-f(x)+\lambda,\ 0\}\}^p d\text{vol}.$$

Indeed, in [8], the following theorem was obtained.

Theorem 1.1 ([8]). *Let $N \subset M$ be a geodesic and $\lambda \le 0$. Then the volume of the normal tube around N satisfies* $\text{vol}T(N,r) \le F(n,p,a,b,c,r)$, *where*

$$a = l(N) = \text{length of } N,$$

$$b = |\lambda|^p,\ c = K(\lambda,p),\ p > n-1.$$

Furthermore, as $a,\ c \to 0$ we have that $F(n,p,a,b,c,r) \to 0$.

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In this paper, we shall show that if the sectional curvature K_M of M is bounded below by a real number, then we obtain a comparison result analogous to the classical Bishop-Gromov volume comparison for the volumes of tubes around a geodesic in M and in the standard sphere $S^n(1)$ with constant curvature 1 by pinching $K(1,p)$ for $p \geq 1$.

To state our main theorem more specifically, we need some notations as following. Note first that using the fact that N is a geodesic we can use a parallel frame along N to coordinatize the normal tubes $T(N,r)$ as (s,t,θ), where s is the arclength parameter on N, t measures the distance to N and $\theta \in \mathrm{S}^{n-2}(1)$ is the angular parameter from the unit normal bundle. Now write the Riemannian volume element as

$$d\mathrm{vol} = \omega(s,t,\theta) dt \wedge ds \wedge d\theta_{n-2},$$

where $d\theta_{n-2}$ is the standard volume form on $\mathrm{S}^{n-2}(1)$. As t increases ω may become undefined but we can just define it to be zero for these t for our purpose to estimate the volume.

We also define J by $J^{n-2} = \omega$ to obtain the initial conditions $J(0) = 0,\ J'(0) = 1$, where $'$ denotes the differentiation with respect to t.

In case of the space form $\mathrm{S}^n(\kappa)$ with constant curvature $\kappa \in \mathbb{R}$, an immediate calculation (refer to Lemma 2.2 and p. 39 in [2]) shows that the metric is given by

$$g = C_\kappa^2(t) ds^2 + dt^2 + S_\kappa^2(t) d\theta_{n-2},$$

where $C_\kappa(t)$ (resp. $S_\kappa(t)$) is the unique solution of the Jacobi equation for constant curvature κ, namely

$$x''(t) + \kappa x(t) = 0$$

with initial conditions $x(0) = 1,\ x'(0) = 0$ (resp. $x(0) = 0,\ x'(0) = 1$).

So in this case we have that

$$\omega_\kappa(t) = C_\kappa(t) S_\kappa^{n-2}(t).$$

Furthermore, if we let $\omega' = h\omega$, then we obtain the differential inequality as following (for details, refer to [10]).

$$h' + \frac{h^2}{n-1} \leq -\mathrm{Ric}(\partial_t, \partial_t).$$

Here, ∂_t is the unit gradient vector field for the distance function to N. We know that h is the mean curvature of the level set of distance function to N. It is also easy to check that for the case of $\mathrm{S}^n(\kappa)$, we have

$$h_\kappa(t) = \frac{C'_\kappa(t)}{C_\kappa(t)} + (n-2)\frac{S'_\kappa(t)}{S_\kappa(t)}.$$

Now we are in a position to state our result as follows.

Theorem 1.2. *Let N be a given geodesic $\gamma : [0,\ a] \to M$ and $0 < R < \frac{\pi}{2}$ be given. We also assume that $n > 2,\ \kappa \in \mathbb{R},\ 0 < r < R$. Then for every*

$\bar{\epsilon} > 0$, *there exists a* $\delta = \delta(n, \kappa, a, r, R, \bar{\epsilon})$ *such that if* M *is an* n*-dimensional Riemannian manifold with* $K_M \geq \kappa$ *and* $K(1,p) < \delta$, $(p \geq 1)$, *then we have*

$$\frac{\mathrm{vol}T(N,R)}{\mathrm{vol}\overline{T}(R)} < \frac{\mathrm{vol}T(N,z)}{\mathrm{vol}\overline{T}(z)} + \bar{\epsilon}$$

for all z *with* $r < z < R$, *where* $\mathrm{vol}\overline{T}(r)$ *is the volume of a normal* r*-tube around a geodesic* $\bar{\gamma} : [0,a] \to \mathrm{S}^n(1)$.

Remark 1.1. When compared with the similar results in [5], the curvature condition in the above theorem has been weakened to a situation where one has 'deep wells of sectional curvature' below a positive real number by considering the standard arguments for metric rescaling.

But it should still be pointed out that the above theorem holds only for $r > 0$, so it is not the complete generalization of those results in [5].

2. Preliminaries

Note first that the smallness of L^1-norm of the sectional curvature which is bounded below implies the smallness of L^p-norm of the curvature for any $p \geq 1$. So without loss of generality, we may assume that $p = 1$ in Theorem 1.2 and for this reason, throughout this section we denote by M an n-dimensional Riemannian manifold satisfying $K(1,1) = \int_M \{\max\{1 - f(x),\ 0\}\}d\mathrm{vol} < \delta$ and $K_M \geq \kappa$ for any real number κ and a positive number δ.

We also let N and $R > 0$ be given as in the previous section and define, for any $\eta > 0$, $E_\eta := \{x \in T(N,R) |\ \max\{1 - f(x), 0\} > \eta\}$.

Note first that $\mathrm{vol}(E_\eta)$ converges to zero as $\delta \to 0$. This follows immediately since

$$\begin{aligned}\int_M \{\max\{1 - f(x),\ 0\}\}d\mathrm{vol} &> \int_{T(N,R)} \{\max\{1 - f(x),\ 0\}\}d\mathrm{vol} \\ &> \int_{E_\eta} \eta d\mathrm{vol} = \eta \mathrm{vol}(E_\eta).\end{aligned}$$

Now we let $\mu_{s,\theta}$ be the measure on $\exp_{\gamma(s)} t\theta =: C_{s,\theta}(t)$ for each s $(0 \leq s \leq a)$, $\theta \in \mathrm{S}^{n-2}(1)$ and we define $\nu > 0$ so that $\omega_\kappa(\nu) = \sqrt{\epsilon}$, where $\epsilon = \mathrm{vol}(E_\eta)$.

From now on, we proceed similar arguments as in [7] and define the explicit quantities in our case as follows.

$$S_{\sqrt[4]{\epsilon},\nu}(s,\theta) = \inf\{c \mid c > \nu, (s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c, \mu_{s,\theta}(C_{s,\theta}([\nu, c]) \cap E_\eta) \geq \sqrt[4]{\epsilon}\},$$

where

$$\Phi_{\sqrt[4]{\epsilon},\nu} = \{(s,\theta) \in [0,a] \times \mathrm{S}^{n-2} \mid \mu_{s,\theta}(C_{s,\theta}([\nu, R]) \cap E_\eta) < \sqrt[4]{\epsilon}\}.$$

Then we obtain the similar lemma as in [7] which will be used later in Section 4 to estimate the volume of normal tubes.

Lemma 2.1. $\mathrm{vol}\{\exp_{\gamma(s)} t\theta \mid (s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c,\ S_{\sqrt[4]{\epsilon},\nu}(s,\theta) \leq t \leq R\}$ *converges to zero as* $\delta \to 0$.

The proof of this lemma basically follows the arguments in [7]. Let first

$$\Psi_{\sqrt[4]{\epsilon},\nu} = \{(s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c \mid \int_{C_{s,\theta}([\nu,R])\cap E_\eta} \omega(s,t,\theta)dt \geq \sqrt{\epsilon}\}.$$

Since

$$\begin{aligned}\epsilon = \mathrm{vol}(E_\eta) &= \int_{\mathrm{S}^{n-2}\times[0,a]} (\int_{C_{s,\theta}([0,R])\cap E_\eta} \omega(s,t,\theta)dt)dsd\theta \\ &\geq \int_{\Psi_{\sqrt[4]{\epsilon},\nu}} (\int_{C_{s,\theta}([\nu,R])\cap E_\eta} \omega(s,t,\theta)dt)dsd\theta \\ &\geq \sqrt{\epsilon}\mathrm{vol}(\Psi_{\sqrt[4]{\epsilon},\nu}),\end{aligned}$$

$\mathrm{vol}(\Psi_{\sqrt[4]{\epsilon},\nu})$ converges to zero as $\delta \to 0$.

Thus we may assume that for every $(s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c$,

$$\int_{C_{s,\theta}([\nu,R])\cap E_\eta} \omega(s,t,\theta)dt < \sqrt{\epsilon}.$$

We then know that there exists a $d > \nu$ such that $\omega(s,d,\theta) < \sqrt[4]{\epsilon}$ and $\mu_{s,\theta}(C_{s,\theta}([\nu,d]) \cap E_\eta) \leq \sqrt[4]{\epsilon}$. Of course, we know that $d \leq S_{\sqrt[4]{\epsilon},\nu}(s,\theta)$.

Furthermore, from the condition $K_M \geq \kappa$, it is easy to show the following inequality using the standard arguments.

$$\frac{\omega'}{\omega} \leq \frac{\omega'_\kappa}{\omega_\kappa}.$$

Thus we have for any t with $S_{\sqrt[4]{\epsilon},\nu}(s,\theta) \leq t \leq R$ and $(s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c$,

$$\begin{aligned}\omega(s,t,\theta) &\leq \frac{\omega_\kappa(t)}{\omega_\kappa(d)}\omega(s,d,\theta) \\ &\leq \frac{\max\{\omega_\kappa(r) \mid 0 \leq r \leq R\}}{\omega_\kappa(\nu)}\sqrt[4]{\epsilon} \\ &= \max\{\omega_\kappa(r) \mid 0 \leq r \leq R\}\sqrt[8]{\epsilon},\end{aligned}$$

which converges to zero as $\delta \to 0$. This completes our proof.

3. Mean curvature comparison

In this section, we compare the mean curvature $h(s,t,\theta)$ in M with $h_1(t)$ in $\mathrm{S}^n(1)$. Recall that we have two assumptions on the curvature of M as follows:

$$K_M \geq \kappa, \int_M \{\max\{1 - f(x),\ 0\}\}d\mathrm{vol} < \delta$$

for any real number κ and a positive number δ.

We will use the same notations as in the previous section and provide the comparison theorem as following.

Theorem 3.1. *For any $(s,\theta) \in \Phi_{\sqrt[4]{\epsilon},\nu}$, we have $h(s,r,\theta) \leq h_1(r) + \tau(\epsilon)$ for all $r,\ R$ with $0 \leq r \leq R < \frac{\pi}{2}$, where $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.*

We first take an orthonormal parallel vector fields $\{E_i(t)\}_{i=1}^n$ along $C_{s,\theta}(t)$ such that $E_1(0) = C'_{s,\theta}(0),\ E_2(0) = \gamma'(s)$.

Now we recall that by the standard theory of index form we have the following.

$$\begin{aligned} h(s,r,\theta) \le & \int_0^r \left\{ \left(\frac{C'(t)}{C(r)}\right)^2 - \left(\frac{C(t)}{C(r)}\right)^2 K(E_2, C'_{s,\theta}(t)) \right\} dt - \left[\frac{C(t)}{C(r)} \cdot \frac{C'(t)}{C(r)}\right]_0^r \\ & + \sum_{i=3}^n \int_0^r \left\{ \left(\frac{S'(t)}{S(r)}\right)^2 - \left(\frac{S(t)}{S(r)}\right)^2 K(E_i, C'_{s,\theta}(t)) \right\} dt \\ & - (n-2) \left[\frac{S(t)}{S(r)} \cdot \frac{S'(t)}{S(r)}\right]_0^r, \end{aligned}$$

where $S(t) = \frac{1}{\sqrt{1-\eta}} \sin(\sqrt{1-\eta}\ t),\ C(t) = \cos(\sqrt{1-\eta}\ t)$.

Note that the right hand side of the above inequality can be rewritten as follows (refer to p. 142 in [4]).

$$\begin{aligned} & \int_0^r \left(\frac{C'(t)}{C(r)}\right)^2 ((1-\eta) - K(E_2, C'_{s,\theta}(t)))dt \\ & + \sum_{i=3}^n \int_0^r \left(\frac{S'(t)}{S(r)}\right)^2 ((1-\eta) - K(E_i, C'_{s,\theta}(t)))dt \\ & + \left[\frac{C(t)}{C(r)} \cdot \frac{C'(t)}{C(r)}\right]_0^r + (n-2) \left[\frac{S(t)}{S(r)} \cdot \frac{S'(t)}{S(r)}\right]_0^r. \end{aligned}$$

We first observe that the third and fourth terms in the above sum turn into $h_{1-\eta}(r) = (n-2)\sqrt{1-\eta}\cot(\sqrt{1-\eta}\ r) - \sqrt{1-\eta}\tan(\sqrt{1-\eta}\ r)$.

By letting η as small as we please, we can express this as $h_1(r) + \tau(\eta)$ ($r < R < \frac{\pi}{2}$), where $\tau(\eta)$ converges to zero as $\eta \to 0$.

Next, in the first and second terms in the above sum, we break the interval $[0, r]$ of integration into three parts as follows.

$$C_{s,\theta}([0,r]) = C_{s,\theta}([0,\nu]) \cup \{C_{s,\theta}([\nu,r]) \cap E_\eta\} \cup \{C_{s,\theta}([\nu,r]) \cap E_\eta^c\}.$$

First of all, note that $(1-\eta) - K(E_i, C'_{s,\theta}(t)) \le 0$ on $C_{s,\theta}([\nu,r]) \cap E_\eta^c$ for $i = 2, \ldots, n$. So the integration in this part is negative.

On $C_{s,\theta}([\nu,r]) \cap E_\eta$ on the other hand, we have $(1-\eta) - K(E_i, C'_{s,\theta}(t)) \le 1 - \eta - \kappa$ and $\mu_{s,\theta}(C_{s,\theta}([\nu,R]) \cap E_\eta) < \sqrt[4]{\epsilon}$. Thus we can say that

$$\begin{aligned} & \int_{C_{s,\theta}([\nu,R]) \cap E_\eta} \left(\frac{C'(t)}{C(r)}\right)^2 ((1-\eta) - K(E_2, C'_{s,\theta}(t)))dt \\ & + \sum_{i=3}^n \int_{C_{s,\theta}([\nu,R]) \cap E_\eta} \left(\frac{S'(t)}{S(r)}\right)^2 ((1-\eta) - K(E_i, C'_{s,\theta}(t)))dt \le \tau(\epsilon), \end{aligned}$$

where $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.

Finally on $C_{s,\theta}([0,\nu])$, we break it into $C_{s,\theta}([0,\nu])\cap E_\eta$ and $C_{s,\theta}([0,\nu])\cap E_\eta^c$ and apply the same arguments as on $C_{s,\theta}([\nu,r])\cap E_\eta$ and $C_{s,\theta}([\nu,r])\cap E_\eta^c$ to obtain the similar estimates noting that ν can be chosen arbitrarily small as $\epsilon\to 0$. Consequently, by putting all these estimates together, we conclude that $h(r,s,\theta)\le h(r)+\tau(\epsilon)$, which completes the proof.

4. Volume comparison

We finally estimate the volume of the normal tube around a geodesic in our case and obtain a comparison theorem which is analogous to the classical Bishop-Gromov volume comparison. Every notations in this section also follow those in Section 2.

Recall first that we declared $\omega(s,t,\theta)$ to be zero whenever it is undefined since t goes too far from N.

Now let us analyze the volume of $T(N,R)$ as following.

$$\begin{aligned}\mathrm{vol}T(N,R) = &\int_{\Phi_{\sqrt[4]{\epsilon},\nu}}\left(\int_0^R \omega(s,t,\theta)dt\right)dsd\theta_{n-2}\\ &+\int_{(\Phi_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^{S_{\sqrt[4]{\epsilon},\nu}(\theta,s)}\omega(s,t,\theta)dt\right)dsd\theta_{n-2}\\ &+\int_{(\Phi_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_{S_{\sqrt[4]{\epsilon},\nu}(\theta,s)}^R\omega(s,t,\theta)dt\right)dsd\theta_{n-2}.\end{aligned}\tag{4.1}$$

Here, note that the last term above can be written as $\xi(\delta)$, which converges to 0 as $\delta\to 0$ by Lemma 2.1.

By the same arguments as in the proof of Theorem 3.1, we can say that on $\Psi := \Psi_1\cup\Psi_2$, where

$$\Psi_1 = \{(s,t,\theta)\mid (s,\theta)\in\Phi_{\sqrt[4]{\epsilon},\nu},\ 0\le t\le R\}$$

and

$$\Psi_2 = \{(s,t,\theta)\mid (s,\theta)\in(\Phi_{\sqrt[4]{\epsilon},\nu})^c,\ 0\le t\le S_{\sqrt[4]{\epsilon},\nu}(\theta,s)\},$$

we have

$$h(s,t,\theta)-h_1(t)<\tau(\epsilon)$$

for some $\tau(\epsilon)>0$ where $\tau(\epsilon)$ can be arbitrarily small as $\epsilon\to 0$.

From this fact, a straightforward calculation of integration shows that

$$\frac{\omega(s,t_2,\theta)}{\omega_1(t_2)}<\exp(\alpha(\epsilon))\frac{\omega(s,t_1,\theta)}{\omega_1(t_1)}\tag{4.2}$$

for any $(s,t_1,\theta),\ (s,t_2,\theta)\in\Psi$ with $t_1<t_2$, where $\alpha(\epsilon)$ converges to zero as $\epsilon\to 0$. Now we reproduce Lemma 2.1 in [9] to proceed our arguments and just sketch the proof below.

Lemma 4.1. *Let f, g be two positive continuous functions defined on $[0,\infty]$. If $\frac{f(b)}{g(b)} \le \exp(\alpha)\frac{f(a)}{g(a)}$ for some $\alpha > 0$ and for all a, b with $0 < a < b$, then for any given $R > 0$, $r > 0$ and with $R > r > 0$ we have*

$$\frac{\int_0^R f(t)dt}{\int_0^R g(t)dt} \le \frac{\int_0^z f(t)dt}{\int_0^z g(t)dt} + \tau(\alpha)$$

for all $z > 0$ with $R \ge z \ge r > 0$ and for some $\tau(\alpha) > 0$ satisfying $\lim_{\alpha\to 0} \tau(\alpha) =$ 0.

It suffices to show that the function $F(y) = \frac{\int_0^y f(t)dt}{\int_0^y g(t)dt}$ is almost nonincreasing with respect to $y \in [r, R]$.

Specifically, we compute

$$F'(y) \le \frac{g(y)\int_0^y g(t)dt}{(\int_0^y g(t)dt)^2}\frac{\int_0^y f(t)dt}{\int_0^y g(t)dt}(\exp(\alpha) - 1) \tag{4.3}$$

for all y with $r \le y \le R$.

Since the right hand of the above inequality tends to zero as $\alpha \to 0$, we can express $F'(y) \le \mu(\alpha)$ for some $\mu(\alpha) > 0$ satisfying $\lim_{\alpha\to 0}\mu(\alpha) = 0$. Then by integrating this inequality from z to R, we get $F(R) - F(z) \le (R - z)\mu(\alpha)$. So if we let $\tau(\alpha) := (R - z)\mu(\alpha) < R\mu(\alpha)$, then we have $F(R) \le F(z) + \tau(\alpha)$, which is our desired result.

For the $\alpha = \alpha(\epsilon)$ in (4.2), we define y_0 so that $\int_0^{y_0} \omega_1(t)dt = \sqrt{\alpha}$. Then from (4.3) in the proof of the above Lemma 4.1 and (4.2), it is easy to check

$$\left(\frac{\int_0^y \omega dt}{\int_0^y \omega_1 dt}\right)' |_{y_0\le y\le R} \le \frac{\exp(\alpha) - 1}{\sqrt{\alpha}} C(k, n, R),$$

which converges to zero as $\alpha \to 0$. So we have

$$\frac{\int_0^R \omega dt}{\int_0^R \omega_1 dt} \le \frac{\int_0^z \omega dt}{\int_0^z \omega_1 dt} + \tau(\alpha(\epsilon))$$

for all z with $y_0 \le z \le R$, where $\tau(\alpha(\epsilon)) > 0$ goes to zero as $\epsilon \to 0$.

From the above inequality, we can easily obtain the following.

$$\frac{\int_{\Phi_{\sqrt[4]{\epsilon},\nu}} \left(\int_0^R \omega dt\right) ds d\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]} \left(\int_0^R \omega_1 dt\right) ds d\theta_{n-2}} \le \frac{\int_{\Phi_{\sqrt[4]{\epsilon},\nu}} \left(\int_0^z \omega dt\right) ds d\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]} \left(\int_0^z \omega_1 dt\right) ds d\theta_{n-2}} + \tau(\alpha(\epsilon)) \tag{4.4}$$

for all z with $y_0 \le z \le R$.

Next, we shall estimate the volume ratio for the case $(s, t, \theta) \in \Psi_2$ in the similar way. Note first that $(\Phi_{\sqrt[4]{\epsilon},\nu})^c$ can be divided into the following three subsets:

$$(\Phi^1_{\sqrt[4]{\epsilon},\nu})^c = \{(s, \theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c \mid S_{\sqrt[4]{\epsilon},\nu}(s, \theta) < y_0 < R\},$$

$$(\Phi^2_{\sqrt[4]{\epsilon},\nu})^c = \{(s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c \mid y_0 < S_{\sqrt[4]{\epsilon},\nu}(s,\theta) < R\},$$

and

$$(\Phi^3_{\sqrt[4]{\epsilon},\nu})^c = \{(s,\theta) \in (\Phi_{\sqrt[4]{\epsilon},\nu})^c \mid y_0 < R < S_{\sqrt[4]{\epsilon},\nu}(s,\theta)\}.$$

For the case $(s,t,\theta) \in \Psi_2$ and $(s,\theta) \in (\Phi^1_{\sqrt[4]{\epsilon},\nu})^c$, we get obviously for all z with $y_0 \le z \le R$ that

$$\frac{\int_{(\Phi^1_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^{S_{\sqrt[4]{\epsilon},\nu}(s,\theta)} \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^R \omega_1 dt\right) dsd\theta_{n-2}} \le \frac{\int_{(\Phi^1_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^z \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^z \omega_1 dt\right) dsd\theta_{n-2}}. \tag{4.5}$$

For the case $(s,t,\theta) \in \Psi_2$ and $(s,\theta) \in (\Phi^2_{\sqrt[4]{\epsilon,\nu}})^c$, we use Lemma 4.1 and obtain

$$\begin{aligned} &\frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^{S_{\sqrt[4]{\epsilon},\nu}(s,\theta)} \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^R \omega_1 dt\right) dsd\theta_{n-2}} \\ &\le \frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^z \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^z \omega_1 dt\right) dsd\theta_{n-2}} + \tau(\alpha(\epsilon)) \end{aligned} \tag{4.6}$$

for all z with $y_0 \le z \le S_{\sqrt[4]{\epsilon},\nu}(s,\theta)$.

Furthermore, in case $S_{\sqrt[4]{\epsilon},\nu}(s,\theta) < z \le R$, we clearly have

$$\frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^{S_{\sqrt[4]{\epsilon},\nu}(s,\theta)} \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^R \omega_1 dt\right) dsd\theta_{n-2}} \le \frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^z \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^z \omega_1 dt\right) dsd\theta_{n-2}}.$$

So we may say that (4.6) holds for any z with $y_0 \le z \le R$.

Finally, we obtain the similar estimate for the case $(s,t,\theta) \in \Psi_2$ and $(s,\theta) \in (\Phi^3_{\sqrt[4]{\epsilon},\nu})^c$ using the same method as above. That is, we have

$$\frac{\int_{(\Phi^3_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^R \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^R \omega_1 dt\right) dsd\theta_{n-2}} \le \frac{\int_{(\Phi^3_{\sqrt[4]{\epsilon},\nu})^c}\left(\int_0^z \omega dt\right) dsd\theta_{n-2}}{\int_{\mathbb{S}^{n-2}(1)\times[0,a]}\left(\int_0^z \omega_1 dt\right) dsd\theta_{n-2}} + \tau(\alpha(\epsilon)) \tag{4.7}$$

for any z with $y_0 \le z \le R$.

Now we put together all the above four inequalities (4.4)–(4.7) and recall the analysis of $\mathrm{vol}T(N,R)$ in (4.1), which gives the following inequality:

$$\frac{\mathrm{vol}T(N,R) - \xi(\delta)}{\mathrm{vol}\overline{T}(R)} \le \frac{\mathrm{vol}T(N,z)}{\mathrm{vol}\overline{T}(z)} + \tau(\alpha(\epsilon)).$$

If we choose a sufficiently small $\delta > 0$ so that $\frac{\xi(\delta)}{\mathrm{vol}\overline{T}(R)}$ can be as small as we please, (consequently, δ depends on a and R) then we can say that for every

$\bar{\epsilon} > 0$, there exists a $\delta > 0$ such that

$$\frac{\mathrm{vol}T(N,R)}{\mathrm{vol}\overline{T}(R)} \leq \frac{\mathrm{vol}T(N,z)}{\mathrm{vol}\overline{T}(z)} + \bar{\epsilon}$$

for all z with $y_0 \leq z \leq R$. Since we can adjust y_0 so that $y_0 < r$ by requiring δ to be sufficiently small enough, we complete the proof of Theorem 1.2.

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