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HYERS-ULAM STABILITY OF TRIGONOMETRIC FUNCTIONAL EQUATIONS

Jeongwook Chang and Jaeyoung Chung

Abstract. In this article we prove the Hyers–Ulam stability of trigonometric functional equations.

1. Introduction

In 1940, S. M. Ulam proposed the following problem [18]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

Then does there exist a group homomorphism L and $\delta_\epsilon > 0$ such that

$$d(f(x),\, L(x)) \le \delta_\epsilon$$

for all $x \in G_1$?

This problem was solved affirmatively by D. H. Hyers [11] under the assumption that G_2 is a Banach space. In 1978, Th. M. Rassias [16] firstly generalized the above result and since then, stability problems of many other functional equations have been investigated [4, 5, 6, 7, 8, 9, 12, 13, 14, 15]. In 1990, L. Székelyhidi [17] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for sine and cosine functional equations. In this paper, employing the idea of L. Székelyhidi [17] we consider the Hyers-Ulam stability problem of the following two trigonometric functional equations

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0, \tag{1.1}$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0, \tag{1.2}$$

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where $f, g : G \to \mathbb{C}$ and G is an abelian group divisible by 2. We call $A : G \to \mathbb{C}$ *additive* provided that $A(x+y) = A(x) + A(y)$ for all $x, y \in G$ and call $m : G \to \mathbb{C}$ *exponential* provided that $m(x+y) = m(x)m(y)$ for all $x, y \in G$. We prove as results that if $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \leq M$$

for all $x, y \in G$, then f, g satisfy one of the followings:
(i) $f = 0, \quad g :$ arbitrary,
(ii) f and g are bounded functions,
(iii) $f(x) = A(x) + B(x)$ and $g(x) = \lambda A(x) + \mu B(x) + 1$,
(iv) $f(x) = \frac{\lambda}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x)) + \frac{\mu}{2}(m(x) - m(-x))$,
where $\lambda, \mu \in \mathbb{C}$, A is an additive function, m is an exponential function and B is a bounded function.

Also we prove that if $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \leq M$$

for all $x, y \in G$, then f, g satisfy one of the followings:
(i) f and g are bounded functions,
(ii) $f(x) = \frac{1}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x))$, where m is an exponential function.

2. Stability of the equations

We first discuss the general solutions of the equations (1.1) and (1.2). We refer the reader to Aczél ([1], p. 180) and Aczél–Dombres ([2], pp. 209–217) for the proofs.

Lemma 2.1. *Let G be an abelian group divisible by 2. Then the general solutions f, g of the equation (1.1) are given by*

$$f(x) = \frac{\lambda}{2}(m(x) - m(-x)),$$
$$g(x) = \frac{1}{2}(m(x) + m(-x)) + \frac{\mu}{2}(m(x) - m(-x)),$$

or

$$f(x) = A(x), \ \ g(x) = \lambda A(x) + 1,$$

and the nonconstant general solutions f, g of (1.2) are given by

$$f(x) = \frac{1}{2}(m(x) - m(-x)),$$
$$g(x) = \frac{1}{2}(m(x) + m(-x)),$$

where $\mu, \lambda \in \mathbb{C}$, A is an additive function and m is an exponential function.

Lemma 2.2. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality; there exists a positive constant M such that*

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \le M \tag{2.1}$$

for all $x, y \in G$. Then either there exist λ, $\nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\lambda f(x) - \nu g(x)| \le L, \tag{2.2}$$

or else

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \tag{2.3}$$

for all x, $y \in G$.

Proof. We prove that the equation (2.3) satisfied if the condition (2.2) fails. Assume that $|\lambda f(x) - \nu g(y)| \le L$ for some $L > 0$ implies $\lambda = \nu = 0$. Let

$$F(x, y) = f(x+y) - f(x)g(-y) + g(x)f(-y).$$

Then we can choose y_1 satisfying $f(-y_1) \neq 0$. It is easy to show that

$$g(x) = \lambda_0 f(x) + \lambda_1 f(x + y_1) - \lambda_1 F(x, y_1), \tag{2.4}$$

where $\lambda_0 = \frac{g(-y_1)}{f(-y_1)}$ and $\lambda_1 = -\frac{1}{f(-y_1)}$.

By the definition of F and the use of (2.4) we have

$$\begin{aligned}
&f\big((x+y)+z\big)\\
&= f(x+y)g(-z) - g(x+y)f(-z) + F(x+y, z)\\
&= \Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)g(-z)\\
&\quad - \Big(\lambda_0 f(x+y) + \lambda_1 f(x+y+y_1) - \lambda_1 F(x+y, y_1)\Big)f(-z)\\
&\quad + F(x+y, z)\\
&= \Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)g(-z)\\
&\quad - \lambda_0\Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)f(-z)\\
&\quad - \lambda_1\Big(f(x)g(-y-y_1) - g(x)f(-y-y_1) + F(x, y+y_1)\Big)f(-z)\\
&\quad + \lambda_1 F(x+y, y_1)f(-z) + F(x+y, z),
\end{aligned} \tag{2.5}$$

and

$$f\big(x + (y+z)\big) = f(x)g(-y-z) - g(x)f(-y-z) + F(x, y+z). \tag{2.6}$$

It follows from the equations (2.5) and (2.6),

$$\begin{aligned} & f(x)\Big(g(-y)g(-z) - \lambda_0 g(-y)f(-z) \\ & - \lambda_1 g(-y-y_1)f(-z) - g(-y-z)\Big) \\ & + g(x)\Big(- f(-y)g(-z) + \lambda_0 f(-y)f(-z) \\ & + \lambda_1 f(-y-y_1)f(-z) + f(-y-z)\Big) \\ = & - F(x,y)g(-z) + \lambda_0 F(x,y)f(-z) + \lambda_1 F(x,y+y_1)f(-z) \\ & - \lambda_1 F(x+y,y_1)f(-z) - F(x+y,z) + F(x,y+z). \end{aligned}$$

Since F is a bounded function, if we fix y, z the right hand side of the above equation is bounded function of x. Thus by the assumption that $|\lambda f(x) - \nu g(y)| \le L$ for some $L > 0$ implies $\lambda = \nu = 0$, the both sides of the above equation become zero. Consequently we have

(2.7)
$$\begin{aligned} & |-F(x,y)g(-z) + \Big(\lambda_0 F(x,y) + \lambda_1 F(x,y+y_1) - \lambda_1 F(x+y,y_1)\Big)f(-z)| \\ & = |F(x+y,z) - F(x,y+z)| \le M. \end{aligned}$$

Again by the assumption, we have $F(x,y) \equiv 0$. This completes the proof. □

Theorem 2.3. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.1). *Then f, g satisfy one of the followings*:

(i) $f = 0$, g : arbitrary,
(ii) *f and g are bounded functions,*
(iii) $f(x) = A(x) + B(x)$ *and* $g(x) = \lambda A(x) + \mu B(x) + 1$
(iv) $f(x) = \frac{\lambda}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x)+m(-x)) + \frac{\mu}{2}(m(x)-m(-x))$,

where $\lambda, \mu \in \mathbb{C}$, A is an additive function, m is an exponential function and B is a bounded function.

Proof. First we assume that the inequality (2.2) holds. If $f = 0$, g is arbitrary which is the case (i). If f is a nontrivial bounded function, in view of (2.1) g is also bounded which is the case (ii). If f is unbounded, it follows from (2.2) that $\nu \ne 0$ and

$$g(x) = \mu f(x) + B(x) \tag{2.8}$$

for some $\mu \in \mathbb{C}$ and a bounded function B. Putting (2.8) in (2.1) we have

$$|f(x-y) - f(x)B(y) + B(x)f(y)| \le M. \tag{2.9}$$

Replacing x by y and y by x and using the triangle inequality we have

$$|f(x) + f(-x)| \le 2M \tag{2.10}$$

for all $x \in G$. Replacing x by $-x$, y by $-y$ in (2.9) and using the inequality (2.10) we have for some $M_1 > 0$,

$$|f(-x+y) + f(x)B(-y) - B(-x)f(y)| \le M_1. \tag{2.11}$$

Using (2.9), (2.10), (2.11) and the triangle inequality we have

$$|f(x)(B(y) - B(-y)) - f(y)(B(x) - B(-x))| \leq M_1 + 3M. \tag{2.12}$$

Since f is unbounded it follows from (2.12) that $B(y) = B(-y)$ for all $y \in G$. Also, in view of (2.9), for fixed $y \in G$, $x \to f(x + y) - f(x)B(-y)$ is a bounded function of x. Thus it follows from [10, p. 104, Theorem 5.2] that $B(y)$ is an exponential function. Since G is divisible by 2 we can write $B(x) = B(\frac{x}{2})B(\frac{x}{2}) = B(\frac{x}{2})B(-\frac{x}{2}) = B(0)$ and that $B(y) \equiv 1$ or 0. Since f is unbounded, we have $B \equiv 1$. Replacing y by $-y$ in (2.9) and using (2.10), we have

$$|f(x + y) - f(x) - f(y)| \leq 3M. \tag{2.13}$$

By the well known Hyers-Ulam stability theorem [11], there exists an additive function $A(x)$ such that

$$|f(x) - A(x)| \leq 3M, \tag{2.14}$$

which gives the case (iii). Now if the equality (2.3) holds, then by Lemma 2.1, f, g satisfies (iii) or (iv). This completes the proof. □

As a direct consequence of Theorem 2.3 we have the following.

Corollary 2.4. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be continuous functions satisfying* (2.1). *Then f and g satisfy one of the followings*:

(i) $f \equiv 0$ *and g is arbitrary,*
(ii) *f and g are bounded functions,*
(iii) $f(x) = c \cdot x + r(x)$, $g(x) = \lambda(c \cdot x + r(x)) + 1$,
(iv) $f(x) = \lambda \sin(c \cdot x)$, $g(x) = \cos(c \cdot x) + \lambda \sin(c \cdot x)$ *for some* $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ *and a bounded function* $r(x)$.

Proof. The continuous solutions of the equation (1.1) are given by (iv) or $f(x) = c \cdot x$, $g(x) = 1 + \lambda c \cdot x$. This completes the proof. □

Now we prove the stability of the equation (1.2).

Lemma 2.5. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality; there exists a positive constant M such that*

$$|g(x - y) - g(x)g(y) - f(x)f(y)| \leq M \tag{2.15}$$

for all $x, y \in G$. Then either there exist λ, $\nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\lambda f(x) - \nu g(x)| \leq L, \tag{2.16}$$

or else

$$g(x - y) - g(x)g(y) - f(x)f(y) = 0 \tag{2.17}$$

for all x, $y \in G$.

Proof. Suppose that, for $L > 0$, $|\lambda f(x) - \nu g(y)| \leq L$ does not hold unless $\lambda = \nu = 0$. Note that both f and g are unbounded. Let

$$F(x, -y) = g(x - y) - g(x)g(y) - f(x)f(y). \tag{2.18}$$

Just for convenience, we consider the following equation which is equivalent to (2.18).

$$F(x, y) = g(x + y) - g(x)g(-y) - f(x)f(-y). \tag{2.19}$$

Since f is nonconstant, we can choose y_1 satisfying $f(-y_1) \neq 0$. It is easy to show that

$$f(x) = \lambda_0 g(x) + \lambda_1 g(x + y_1) - \lambda_1 F(x, y_1), \tag{2.20}$$

where $\lambda_0 = -\frac{g(-y_1)}{f(-y_1)}$ and $\lambda_1 = \frac{1}{f(-y_1)}$.

By the definition of F and the use of (2.20), we have

$$\begin{aligned}
& g\big((x+y)+z\big) \\
&= g(x+y)g(-z) + f(x+y)f(-z) + F(x+y, z) \\
&= \Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad + \Big(\lambda_0 g(x+y) + \lambda_1 g(x+y+y_1) - \lambda_1 F(x+y, y_1)\Big)f(-z) \\
&\quad + F(x+y, z) \\
&= \Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad + \lambda_0\Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)f(-z) \\
&\quad + \lambda_1\Big(g(x)g(-y-y_1) + f(x)f(-y-y_1) + F(x, y+y_1)\Big)f(-z) \\
&\quad - \lambda_1 F(x+y, y_1)f(-z) + F(x+y, z),
\end{aligned} \tag{2.21}$$

and

$$g\big(x + (y+z)\big) = g(x)g(-y-z) + f(x)f(-y-z) + F(x, y+z). \tag{2.22}$$

By equating the above two equations we have

$$\begin{aligned}
& g(x)\Big(g(-y)g(-z) + \lambda_0 g(-y)f(-z) \\
&\quad + \lambda_1 g(-y-y_1)f(-z) - g(-y-z)\Big) \\
&\quad + f(x)\Big(f(-y)g(-z) + \lambda_0 f(-y)f(-z) \\
&\quad + \lambda_1 f(-y-y_1)f(-z) - f(-y-z)\Big) \\
= &\ - F(x,y)g(-z) - \lambda_0 F(x,y)f(-z) - \lambda_1 F(x, y+y_1)f(-z) \\
&\quad + \lambda_1 F(x+y, y_1)f(-z) - F(x+y, z) + F(x, y+z).
\end{aligned}$$

When y, z are fixed, the right hand side of the above equality is bounded, so we have

(2.23)
$$\begin{aligned} &F(x, y+z) - F(x+y, z) \\ &= F(x,y)g(-z) + \Big(\lambda_0 F(x,y) + \lambda_1 F(x, y+y_1) - \lambda_1 F(x+y, y_1)\Big) f(-z). \end{aligned}$$

Again considering (2.23) as a function of z for all fixed x, y, we have $F(x,y) \equiv 0$ which is equivalent to (2.17). □

Theorem 2.6. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.15). *Then f, g satisfy one of the followings*:

(i) *f and g are bounded functions,*

(ii) *$f(x) = \frac{1}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x))$, where m is an exponential function.*

Proof. First we prove that if the inequalities (2.15) and (2.16) hold, then both f and g are bounded functions. From (2.15), it is impossible that only one of f and g is unbounded. Assume that both f and g are unbounded. In view of (2.16), we can write

$$g = \mu f + B \tag{2.24}$$

for some $\mu \neq 0$ and a bounded function B. Putting (2.24) in (2.15) we have

$$|\mu f(x+y) + B(x+y) - \Big(\mu f(x) + B(x)\Big)\Big(\mu f(-y) + B(-y)\Big) + f(x)f(-y)| \leq M.$$

Since B is bounded, we have

$$f(x+y) - \mu^{-1}\Big((\mu^2+1)f(-y) + \mu B(-y)\Big)f(x)$$

is bounded for fixed $y \in G$. Thus it follows from [10, p. 104, Theorem 5.2] that

$$\mu^{-1}\Big((\mu^2+1)f(y) + \mu B(y)\Big) = m(y)$$

for some exponential m. Thus if $\mu^2 = -1$, we can write

$$f = \pm i(g - m), \tag{2.25}$$

where m is a bounded exponential function. Putting (2.25) in (2.15) we have

$$|g(x-y) - g(x)m(y) - g(y)m(x)| \leq M \tag{2.26}$$

for all x, $y \in G$.

Replacing x by y and y by x and using the triangle inequality we have

$$|g(x) - g(-x)| \leq 2M \tag{2.27}$$

for all $x \in G$. Replacing x by $-x$, y by $-y$ and using the inequality (2.27) we have for some $M_1 > 0$,

$$|g(-x+y) - g(x)m(-y) - m(-x)g(y)| \leq M_1. \tag{2.28}$$

Using (2.26), (2.27), (2.28) and the triangle inequality we have for some $M^* > 0$,

$$|g(x)(m(y) - m(-y)) - g(y)(m(x) - m(-x))| \le M^* + 3M. \tag{2.29}$$

Since g is unbounded and m is bounded, it follows from (2.29) that $m(y) = m(-y)$ for all $y \in G$. Since G is divisible by 2 we have $m \equiv 1$. Putting $y = x$ in (2.26), using the triangle inequality we have $|g(x)| \le \frac{1}{2}(M + |g(0)|)$ for all $x \in G$, which contradicts to the assumption that f and g are unbounded. If $\mu \ne -1$, we have

$$f = \frac{\mu(m-b)}{\mu^2+1}, \quad g = \frac{\mu^2 m + b}{\mu^2+1},$$

which contradicts to the assumption that both f and g are unbounded. If the equation (2.17) holds, then by Lemma 2.1, we have the case (ii). This completes the proof. □

Since every continuous exponential function $m : \mathbb{R}^n \to \mathbb{C}$ is given by $m(x) = e^{c\cdot x}$ for some $c \in \mathbb{C}^n$, we have the following as a direct consequence of Theorem 2.6:

Corollary 2.7. *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be continuous functions satisfying* (2.15). *Then f and g satisfy one of the followings*:

(i) *f and g are bounded measurable functions,*

(ii) $f(x) = \cosh(c \cdot x)$, $g(x) = \sinh(c \cdot x)$, *where $c \in \mathbb{C}^n$.*

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Jeongwook Chang
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail address: jchang@kunsan.ac.kr

Jaeyoung Chung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail address: jychung@kunsan.ac.kr