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BLOW-UP FOR A NON-NEWTON POLYTROPIC FILTRATION SYSTEM WITH NONLINEAR NONLOCAL SOURCE

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Abstract. This paper deals the global existence and blow-up properties of the following non-Newton polytropic filtration system,

$$u_t - \triangle_{m,p}u = u^{\alpha_1}\int_\Omega v^{\beta_1}(x,t)dx, \quad v_t - \triangle_{n,q}v = v^{\alpha_2}\int_\Omega u^{\beta_2}(x,t)dx,$$

with homogeneous Dirichlet boundary condition. Under appropriate hypotheses, we prove that the solution either exists globally or blows up in finite time depends on the initial data and the relations of the parameters in the system.

1. Introduction

In this paper, we consider the following doubly degenerate parabolic system with nonlocal source,

$$\begin{aligned}
&u_t - \triangle_{m,p}u = u^{\alpha_1}\int_\Omega v^{\beta_1}(x,t)dx, && (x,t)\in\Omega\times(0,T],\\
&v_t - \triangle_{n,q}v = v^{\alpha_2}\int_\Omega u^{\beta_2}(x,t)dx, && (x,t)\in\Omega\times(0,T],\\
&u(x,t)=0,\quad v(x,t)=0, && (x,t)\in\partial\Omega\times(0,T],\\
&u(x,0)=u_0(x),\quad v(x,0)=v_0(x), && x\in\Omega,
\end{aligned}\tag{1.1}$$

where $\triangle_{k,\gamma}\Theta = \nabla\cdot(|\nabla\Theta^k|^{\gamma-2}\cdots\nabla\Theta^k)$, $\nabla\Theta^k = k\Theta^{k-1}(\Theta_{x_1},\ldots,\Theta_{x_N})$ $k>0$, $\gamma>2$, $N\geq 1$. $\Omega\subset\mathbb{R}^N(N\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $m,n\geq 1$, $p,q>2$, $\alpha_i,\beta_i>0$ $(i=1,2)$ are parameters.

Throughout this paper, we denote $Q_T=\Omega\times(0,T]$, $S_T=\partial\Omega\times[0,T]$, $T>0$, and make the following assumption on initial data.

(H) The nonnegative initial data satisfies compatibility conditions and $u_0^m(x)\in C(\bar\Omega)\cap W_0^{1,p}(\Omega)$, $v_0^n(x)\in C(\bar\Omega)\cap W_0^{1,q}(\Omega)$, and $\nabla u_0^m\cdot\nu<0$,

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$\bigtriangledown v_0^n \cdot \nu < 0$ on the boundary $\partial\Omega$, where ν is unite outer normal vector on $\partial\Omega$.

Parabolic systems like (1.1) appear in population dynamics, chemical reactions, heat transfer, and so on. In particular, equations (1.1) may be used to describe the non-stationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, equations (1.1) are called the non-Newtonian polytropic filtration equations (see [15, 28, 30] and references therein). The problems with nonlinear reaction term and nonlinear diffusion include blow-up and global existence conditions of solutions, blow-up rates and blow-up sets, etc. (see the surveys [3, 13, 16, 25]). Here, we say solution blows up in finite time if the solution becomes unbounded (in the sense of maximum norm) at that time.

System (1.1) has been studied by many authors. For $p = q = 2$, it is called porous medium equations (see [23, 24, 32] for nonlinear boundary conditions, see [6, 10, 11] for local nonlinear reaction terms, see [1, 7, 8, 9, 19] for nonlocal nonlinear reaction terms).

In [19], Li and Xie consider the following problem,

$$\begin{array}{ll} u_t - \triangle u^m = u^q \int_\Omega u^p dx, & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, & x \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{array} \tag{1.2}$$

and obtain that the solution either exists globally or blows up in finite time under appropriate assumptions. Furthermore, if $p + q > m$, they also get the blow-up rate.

Recently, in [8], Du generalize (1.2) to system, and study the following problem,

$$\begin{array}{ll} u_t - \triangle u^m = u^{p_1} \int_\Omega v^{q_1} dx, & (x,t) \in \Omega \times (0,T], \\ v_t - \triangle v^n = v^{p_2} \int_\Omega u^{q_2} dx, & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, \quad v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in \Omega. \end{array} \tag{1.3}$$

Similar to [19], he also get the criteria for solution exists globally or blows up in finite time. Moreover, if $p_1 = 0$ or $p_1 > m$; $p_2 = 0$ or $p_2 > n$; $q_1 > n$, $q_2 > m$ and satisfy $q_2 > p_1 - 1$, $q_1 > p_2 - 1$, he also get the blow-up rates under the monotone assumption for initial data.

When $m = n = 1$, problem (1.1) is called p-Laplace equations (see [12, 33] for nonlinear boundary conditions, see [14, 17, 27, 31] for local nonlinear reaction terms, see [18] for nonlocal nonlinear reaction terms).

In [18] Li and Xie consider the following problem,

$$(1.4)\qquad \begin{array}{ll} u_t - \bigtriangledown\cdot(|\bigtriangledown u|^{p-2}\bigtriangledown u) = \int_\Omega u^q dx, & (x,t)\in\Omega\times(0,T],\\ u(x,t)=0, & x\in\partial\Omega\times(0,T],\\ u(x,0)=u_0(x), & x\in\Omega, \end{array}$$

and they obtain that the solution u exists globally if $q < p-1$; u blows up in finite time if $q > p-1$ and $u_0(x)$ is large enough.

For general $m,\ n\geq 1,\ p,q>2,\ \alpha_i,\beta_i>0\ (i=1,2)$, problem (1.1) is called non-Newton polytropic filtration system: for the case of nonlinear boundary condition see [26, 29, 34], for the case of local (nonlocal) nonlinear reaction terms see [35, 36].

In [26], Sun and Wang consider the following doubly degenerate equation,

$$(1.5)\qquad \begin{array}{ll} (u^m)_t = \triangle_{1,p}u, & (x,t)\in\Omega\times(0,T],\\ |\bigtriangledown u|^{p-2}\bigtriangledown u\cdot\nu = u^\alpha, & x\in\partial\Omega\times(0,T],\\ u(x,0)=u_0(x), & x\in\Omega, \end{array}$$

by using upper and lower solution methods, they proved that all positive solutions of (1.5) exist globally if and only if $\alpha\leq m$ when $m\leq p-1$, or $\alpha\leq\frac{(p-1)(m+1)}{p}$ when $m>p-1$.

In [29], Wang consider the system of (1.5) in one-dimension,

$$(1.6)\qquad \begin{array}{lll} (u^m)_t = (|u_x|^{p-2}u_x)_x,\ (v^n)_t=(|v_x|^{q-2}v_x)_x, & & (x,t)\in(0,1)\times(0,T],\\ u_x(0,t)=0, & u_x(1,t)=au^\alpha v^r(1,t), & t\in(0,T],\\ v_x(0,t)=0, & v_x(1,t)=bu^\beta v^s(1,t), & t\in(0,T],\\ u(x,0)=u_0(x), & v(x,0)=v_0(x), & x\in(0,1). \end{array}$$

He obtain that all positive solutions of (1.6) exist globally if and only if

$$\alpha<\min\left\{\frac{m}{p-1},\ \frac{m+1}{p}\right\},\ s<\min\left\{\frac{n}{q-1},\ \frac{n+1}{q}\right\},$$
$$\beta r\leq\left(\min\left\{\frac{m}{p-1},\ \frac{m+1}{p}\right\}-\alpha\right)\left(\min\left\{\frac{n}{q-1},\ \frac{n+1}{q}\right\}-s\right).$$

Motivated by the above cited paper, in this paper, we investigate the blow-up properties of solutions of the problem (1.1) and extend the results of [8, 18, 19] to more general cases. Our main results are stated as follows.

Theorem 1.1. *Suppose the initial data* $(u_0(x),v_0(x))$ *satisfies the assumption* (H)*, the solution of problem* (1.1) *exists globally if one of the following conditions holds,*

(i) $m(p-1)>\alpha_1,\ n(q-1)>\alpha_2,\ \beta_1\beta_2<(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$;

(ii) $m(p-1)>\alpha_1,\ n(q-1)>\alpha_2,\ \beta_1\beta_2=(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$ *and the measure of the domain* ($\|\Omega\|$) *is small*;

(iii) $m(p-1)>\alpha_1,\ n(q-1)>\alpha_2,\ \beta_1\beta_2>(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$ *and the initial values are small.*

Theorem 1.2. *Suppose the initial data* $(u_0(x), v_0(x))$ *satisfies the assumption* (H), *if* $m(p-1) < \alpha_1$ *or* $n(q-1) < \alpha_2$ *or* $\beta_1\beta_2 > (m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$, *then the solution of problem* (1.1) *blows up in finite time for sufficiently large initial values.*

This paper is organized as follows. In Section 2, we give some preliminaries, which is the basement to prove our theorems. The proof of Theorem 1.1 is the subject of Section 3. In Section 4, we consider the blow-up properties of problem (1.1) and give the proof of Theorem 1.2.

2. Preliminaries

As it is well known that degenerate equations need not have classical solutions, we give a precise definition of a weak solution for problem (1.1).

Definition of weak solution. A pair of functions $(u(x,t), v(x,t))$ is called a upper (*lower respectively*) solution of problem (1.1) in $\bar{Q}_T \times \bar{Q}_T$ if and only if $u^m(x,t) \in C(0,T; L^\infty(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$, $v^n(x,t) \in C(0,T; L^\infty(\Omega)) \cap L^q(0,T; W_0^{1,q}(\Omega))$, $u_t \in L^2(0,T; L^2(\Omega))$, $v_t \in L^2(0,T; L^2(\Omega))$, $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, and the following inequalities

$$\begin{aligned}
&\int_\Omega u(x,t_2)\psi(x,t_2)dx - \int_\Omega u(x,t_1)\psi(x,t_1)dx \\
\geq (\leq) &\int_{t_1}^{t_2}\int_\Omega u\psi_t dxdt - \int_{t_1}^{t_2}\int_\Omega |\bigtriangledown u^m|^{p-2} \bigtriangledown u^m \cdot \bigtriangledown\psi dxdt \\
&+ \int_{t_1}^{t_2}\int_\Omega \psi(x,t)u^{\alpha_1}(x,t)\left(\int_\Omega v^{\beta_1}(x,t)dx\right)dxdt,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega v(x,t_2)\psi(x,t_2)dx - \int_\Omega v(x,t_1)\psi(x,t_1)dx \\
\geq (\leq) &\int_{t_1}^{t_2}\int_\Omega v\psi_t dxdt - \int_{t_1}^{t_2}\int_\Omega |\bigtriangledown v^n|^{q-2} \bigtriangledown v^n \cdot \bigtriangledown\psi dxdt \\
&+ \int_{t_1}^{t_2}\int_\Omega \psi(x,t)v^{\alpha_2}(x,t)\left(\int_\Omega u^{\beta_2}(x,t)dx\right)dxdt,
\end{aligned}$$

hold for all $0 < t_1 < t_2 < T$, where nonnegative function $\psi(x,t) \in C^{1,1}(\bar{Q}_T)$ such that $\psi(x,T) = 0$ and $\psi(x,t) = 0$ on S_T. In particular, $(u(x,t), v(x,t))$ is called a weak solution of (1.1) if it is both a weak upper and a weak lower solution.

The local existence of weak solutions to problem (1.1) under the assumption (H) and the following comparison principle is standard (see [4, 15, 28, 30]).

Comparison Principle. *Suppose that* $(\underline{u}(x,t), \underline{v}(x,t))$ *and* $(\bar{u}(x,t), \bar{v}(x,t))$ *are lower and upper solution of problem* (1.1) *on* $\bar{Q}_T \times \bar{Q}_T$, *respectively. Then* $(\underline{u}(x,t), \underline{v}(x,t)) \leq (\bar{u}(x,t), \bar{v}(x,t))$ *a.e. on* $\bar{Q}_T \times \bar{Q}_T$.

In order to study the globally existing solutions to problem (1.1), we need to study the following elliptic system

$$(2.1)\qquad \begin{aligned} -\triangle_{k,\gamma}\Theta &= \lambda_{k,\gamma}\Theta^{k(\gamma-1)}, && x\in\Omega,\\ \Theta &= 1, && x\in\partial\Omega, \end{aligned}$$

where $\triangle_{k,\gamma}\Theta$ is defined in (1.1), and we get the following lemma:

Lemma 2.1. *Problem* (2.1) *has a solution* $\Theta(x)$ *with* $\lambda_{k,\gamma} = \lambda > 0$*, which satisfies the following relations,*

$$\Theta(x) > 1 \text{ in } \Omega,\ \bigtriangledown\Theta\cdot\nu < 0 \text{ on } \partial\Omega,\ \lim_{\mathrm{diam}(\Omega)\to 0}\lambda\to+\infty,\ \sup_{x\in\Omega}\Theta = M < +\infty,$$

where M is a positive constant.

Proof. Set $\Theta^k = \Phi$, then Φ satisfies the following equation

$$(2.2)\qquad \begin{aligned} -\bigtriangledown\cdot(|\bigtriangledown\Phi|^{\gamma-2}\bigtriangledown\Phi) &= \lambda_{k,\gamma}\Phi^{\gamma-1}, && x\in\Omega,\\ \Phi &= 1, && x\in\partial\Omega. \end{aligned}$$

Then [20, 21, 22] tells us problem (2.2) has a solution $\Phi(x)$ with first eigenvalue $\lambda > 0$, and satisfies the following relations,

$$\Phi > 1 \text{ in } \Omega,\ \bigtriangledown\Phi\cdot\nu < 0 \text{ on } \partial\Omega,\ \lim_{\mathrm{diam}(\Omega)\to 0}\lambda\to+\infty,\ \sup_{x\in\Omega}\Phi = M' < +\infty,$$

where M' is a positive constant. Since $\Theta = \Phi^{1/k}$, the conclusion of Lemma 2.1 comes directly. □

Denote

$$A = \begin{pmatrix} m(p-1)-\alpha_1 & -\beta_1\\ -\beta_2 & n(q-1)-\alpha_2 \end{pmatrix},\quad \gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}.$$

Then we get following lemma form paper [6].

Lemma 2.2. *If* $m(p-1) < \alpha_1$ *or* $n(q-1) < \alpha_2$ *or* $\beta_1\beta_2 > (m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$*, then there exist two positive constants* $\gamma_1,\ \gamma_2$ *such that* $A\gamma < (0,0)^T$*. Moreover,* $A(c\gamma) < (0,0)^T$ *for any* $c > 0$*.*

3. Global existence of solution

In this section, we investigate the global existence property of the solutions to problem (1.1) and prove Theorem 1.1. The main method is constructing a globally upper solution and using comparison principle to achieve our purpose.

Proof of Theorem 1.1. Let $\varphi(x)$ and $\psi(x)$ be the solution of the following elliptic problem with first eigenvalue $\lambda_1 > 0,\ \lambda_2 > 0$ respectively,

$$\begin{cases} -\triangle_{m,p}\varphi = \lambda_1\varphi^{m(p-1)}, & x\in\Omega,\\ \varphi = 1, & x\in\partial\Omega, \end{cases}\qquad \begin{cases} -\triangle_{n,q}\psi = \lambda_2\psi^{n(q-1)}, & x\in\Omega,\\ \psi = 1, & x\in\partial\Omega. \end{cases}$$

Then from Lemma 2.1, we get following relations

$$\varphi(x),\psi(x)>1 \text{ in } \Omega,\ \triangledown\varphi\cdot\nu,\triangledown\psi\cdot\nu<0 \text{ on } \partial\Omega \lim_{\operatorname{diam}(\Omega)\to 0}\lambda_i\to+\infty (i=1,2),$$

$$\sup_{x\in\Omega}\varphi=M_1<+\infty,\ \sup_{x\in\Omega}\psi=M_2<+\infty,$$

where M_1, M_2 are positive constant.

Let $\bar{u}(x,t)=\Lambda_1\varphi(x)$, $\bar{v}(x,t)=\Lambda_2\psi(x)$, where $\Lambda_1,\Lambda_2>0$ will be determined later.

Then a direct computation we get

$$\bar{u}_t-\triangle_{m,p}\bar{u}=\lambda_1(\Lambda_1\varphi)^{m(p-1)}\ge\lambda_1\Lambda_1^{m(p-1)},$$
$$\bar{v}_t-\triangle_{n,q}\bar{v}=\lambda_2(\Lambda_2\psi)^{n(q-1)}\ge\lambda_2\Lambda_2^{n(q-1)},$$

and

$$\bar{u}^{\alpha_1}\int_\Omega\bar{v}^{\beta_1}dx\le\|\Omega\|M_1^{\alpha_1}M_2^{\beta_1}\Lambda_1^{\alpha_1}\Lambda_2^{\beta_1},\quad \bar{v}^{\alpha_2}\int_\Omega\bar{u}^{\beta_2}dx\le\|\Omega\|M_1^{\beta_2}M_2^{\alpha_2}\Lambda_1^{\beta_2}\Lambda_2^{\alpha_2}.$$

So, $(\bar{u}(x,t),\bar{v}(x,t))$ is a upper solution of problem (1.1), if

(3.1)
$$\begin{aligned}&\lambda_1\Lambda_1^{m(p-1)}\ge\|\Omega\|M_1^{\alpha_1}M_2^{\beta_1}\Lambda_1^{\alpha_1}\Lambda_2^{\beta_1},\quad \lambda_2\Lambda_2^{n(q-1)}\ge\|\Omega\|M_1^{\beta_2}M_2^{\alpha_2}\Lambda_1^{\beta_2}\Lambda_2^{\alpha_2},\\ &\bar{u}(x,t)|_{\partial\Omega}\ge 0,\ \bar{v}(x,t)|_{\partial\Omega}\ge 0,\ \bar{u}(x,0)\ge u_0(x),\ \bar{v}(x,0)\ge v_0(x).\end{aligned}$$

Next we will prove (3.1) in three cases.

(i) When $m(p-1)>\alpha_1$, $n(q-1)>\alpha_2$, $\beta_1\beta_2<(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$, if we choose Λ_1,Λ_2 large enough such that

$$\Lambda_1^{\frac{(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)-\beta_1\beta_2}{n(q-1)-\alpha_2}}$$
$$>\frac{1}{\lambda_1}\left(\frac{1}{\lambda_2}\right)^{\frac{\beta_1}{n(q-1)-\alpha_2}}\|\Omega\|^{\frac{n(q-1)-\alpha_2+\beta_1}{n(q-1)-\alpha_2}}M_1^{\frac{n(q-1)\alpha_1-\alpha_1\alpha_2+\beta_1\beta_2}{n(q-1)-\alpha_2}}M_2^{\frac{n(q-1)\beta_1}{n(q-1)-\alpha_2}},$$
$$\Lambda_2^{\frac{(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)-\beta_1\beta_2}{m(p-1)-\alpha_1}}$$
$$>\frac{1}{\lambda_2}\left(\frac{1}{\lambda_1}\right)^{\frac{\beta_2}{m(p-1)-\alpha_1}}\|\Omega\|^{\frac{m(p-1)-\alpha_1+\beta_2}{m(p-1)-\alpha_1}}M_2^{\frac{m(p-1)\alpha_2-\alpha_1\alpha_2+\beta_1\beta_2}{m(p-1)-\alpha_1}}M_1^{\frac{m(p-1)\beta_2}{m(p-1)-\alpha_1}},$$

and

$$\Lambda_1>\max_{x\in\bar\Omega}u_0(x),\quad \Lambda_2>\max_{x\in\bar\Omega}v_0(x),$$

then (3.1) holds.

(ii) When $m(p-1)>\alpha_1$, $n(q-1)>\alpha_2$, $\beta_1\beta_2=(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$, we can choose Λ_1,Λ_2 large enough such that

$$\Lambda_1>\max_{x\in\bar\Omega}u_0(x),\quad \Lambda_2>\max_{x\in\bar\Omega}v_0(x),$$

then (3.1) holds if $\|\Omega\|$ is small enough such that $\lambda_1,\ \lambda_2$ are large enough to satisfy

$$1 > \frac{1}{\lambda_1}\left(\frac{1}{\lambda_2}\right)^{\frac{\beta_1}{n(q-1)-\alpha_2}} \|\Omega\|^{\frac{n(q-1)-\alpha_2+\beta_1}{n(q-1)-\alpha_2}} M_1^{\frac{n(q-1)\alpha_1-\alpha_1\alpha_2+\beta_1\beta_2}{n(q-1)-\alpha_2}} M_2^{\frac{n(q-1)\beta_1}{n(q-1)-\alpha_2}},$$
$$1 > \frac{1}{\lambda_2}\left(\frac{1}{\lambda_1}\right)^{\frac{\beta_2}{m(p-1)-\alpha_1}} \|\Omega\|^{\frac{m(p-1)-\alpha_1+\beta_2}{m(p-1)-\alpha_1}} M_2^{\frac{m(p-1)\alpha_2-\alpha_1\alpha_2+\beta_1\beta_2}{m(p-1)-\alpha_1}} M_1^{\frac{m(p-1)\beta_2}{m(p-1)-\alpha_1}}.$$

(iii) When $m(p-1) > \alpha_1$, $n(q-1) > \alpha_2$, $\beta_1\beta_2 > (m(p-1)-\alpha_1)(n(q-1)-\alpha_2)$, we can take Λ_1, Λ_2 small enough such that

$$\Lambda_1^{\frac{(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)-\beta_1\beta_2}{n(q-1)-\alpha_2}}$$
$$> \frac{1}{\lambda_1}\left(\frac{1}{\lambda_2}\right)^{\frac{\beta_1}{n(q-1)-\alpha_2}} \|\Omega\|^{\frac{n(q-1)-\alpha_2+\beta_1}{n(q-1)-\alpha_2}} M_1^{\frac{n(q-1)\alpha_1-\alpha_1\alpha_2+\beta_1\beta_2}{n(q-1)-\alpha_2}} M_2^{\frac{n(q-1)\beta_1}{n(q-1)-\alpha_2}},$$
$$\Lambda_2^{\frac{(m(p-1)-\alpha_1)(n(q-1)-\alpha_2)-\beta_1\beta_2}{m(p-1)-\alpha_1}}$$
$$> \frac{1}{\lambda_2}\left(\frac{1}{\lambda_1}\right)^{\frac{\beta_2}{m(p-1)-\alpha_1}} \|\Omega\|^{\frac{m(p-1)-\alpha_1+\beta_2}{m(p-1)-\alpha_1}} M_2^{\frac{m(p-1)\alpha_2-\alpha_1\alpha_2+\beta_1\beta_2}{m(p-1)-\alpha_1}} M_1^{\frac{m(p-1)\beta_2}{m(p-1)-\alpha_1}},$$

furthermore, if the initial data sufficiently small such that $u_0(x) \le \Lambda_1$ and $v_0(x) \le \Lambda_2$, then (3.1) holds. The proof of Theorem 1.1 is complete. □

4. Blow-up of solution

In this section, we investigate the blow-up property of the solutions to problem (1.1) and prove Theorem 1.2. The main method is constructing a blowing-up lower solution and using comparison principle to achieve our purpose.

Proof of Theorem 1.2. Set

$$\underline{u}(x,t) = (T-t)^{-\gamma_1}V_1(\xi),\ \xi = |x|(T-t)^{-\ell},\ V_1(\xi) = \left(1+\frac{A}{2}-\frac{\xi^2}{2A}\right)_+^{1/m},$$

$$\underline{v}(x,t) = (T-t)^{-\gamma_2}V_2(\xi),\ \xi = |x|(T-t)^{-\ell},\ V_2(\xi) = \left(1+\frac{A}{2}-\frac{\xi^2}{2A}\right)_+^{1/n},$$

where $\ell,\ \gamma_i > 0 (i = 1, 2)$, $A > 1$ and $0 < T < 1$ are parameters to be determined. It is easy to see that $\underline{u}(x,t), \underline{v}(x,t)$ blow up at time T, so it enough to prove $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1). If we choose T small enough such that

$$\text{supp } \underline{u}(\cdot,t) = \overline{B(0, R(T-t)^{\ell})} \subset \overline{B(0, RT^{\ell})} \subset \Omega,$$
$$\text{supp } \underline{v}(\cdot,t) = \overline{B(0, R(T-t)^{\ell})} \subset \overline{B(0, RT^{\ell})} \subset \Omega,$$

where $R = (A(2+A))^{1/2}$, then $\underline{u}(x,t)|_{\partial\Omega} = 0$, $\underline{v}(x,t)|_{\partial\Omega} = 0$. Next if we choose the initial data large enough such that

$$u_0(x) \geq \frac{1}{T^{\gamma_1}} V_1\left(\frac{|x|}{T^\ell}\right), \quad v_0(x) \geq \frac{1}{T^{\gamma_2}} V_2\left(\frac{|x|}{T^\ell}\right),$$

then $(\underline{u}(x,t), \underline{v}(x,t))$ is a lower solution of problem (1.1) if

$$\begin{aligned} &\underline{u}_t - \triangle_{m,p}\underline{u} \leq \underline{u}^{\alpha_1} \int_\Omega \underline{v}^{\beta_1}(x,t)dx, \qquad (x,t) \in \Omega \times (0,T], \\ &\underline{v}_t - \triangle_{n,q}\underline{v} \leq \underline{v}^{\alpha_2} \int_\Omega \underline{u}^{\beta_2}(x,t)dx, \qquad (x,t) \in \Omega \times (0,T]. \end{aligned} \tag{4.1}$$

A direct computation, we obtain

$$\begin{aligned} &\underline{u}_t = \frac{\gamma_1 V_1(\xi) + \ell\xi V_1'(\xi)}{(T-t)^{\gamma_1+1}}, \quad \underline{v}_t = \frac{\gamma_2 V_2(\xi) + \ell\xi V_2'(\xi)}{(T-t)^{\gamma_2+1}}, \\ &\nabla \underline{u}^m = \frac{x}{A(T-t)^{m\gamma_1+2\ell}}, \quad -\triangle\, \underline{u}^m = \frac{N}{A(T-t)^{m\gamma_1+2\ell}}, \\ &\nabla \underline{v}^n = \frac{x}{A(T-t)^{n\gamma_2+2\ell}}, \quad -\triangle\, \underline{v}^n = \frac{N}{A(T-t)^{n\gamma_2+2\ell}}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \triangle_{m,p}\underline{u} &= |\nabla \underline{u}^m|^{p-2} \triangle\, \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} (\nabla \underline{u}^m)^T \cdot (H_x(\underline{u}^m)) \cdot \nabla \underline{u}^m \\ &= |\nabla \underline{u}^m|^{p-2} \triangle\, \underline{u}^m + (p-2)|\nabla \underline{u}^m|^{p-4} \textstyle\sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{u}^m}{\partial x_i} \frac{\partial^2 \underline{u}^m}{\partial x_i x_j} \frac{\partial \underline{u}^m}{\partial x_j}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \triangle_{n,q}\underline{v} &= |\nabla \underline{v}^n|^{q-2} \triangle\, \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} (\nabla \underline{v}^n)^T \cdot (H_x(\underline{v}^n)) \cdot \nabla \underline{v}^n \\ &= |\nabla \underline{v}^n|^{q-2} \triangle\, \underline{v}^n + (q-2)|\nabla \underline{v}^n|^{q-4} \sum_{j=1}^N \sum_{i=1}^N \frac{\partial \underline{v}^n}{\partial x_i} \frac{\partial^2 \underline{v}^n}{\partial x_i x_j} \frac{\partial \underline{v}^n}{\partial x_j}, \end{aligned} \tag{4.4}$$

where $H_x(\underline{u}^m), H_x(\underline{v}^n)$ denote the Hessian matrix of $\underline{u}^m(x,t), \underline{v}^n(x,t)$, respectively.

Denote $d(\Omega) = \mathrm{diam}(\Omega)$, then from (4.2) and (4.3), we get

$$\begin{aligned} |\triangle_{m,p}\, \underline{u}| &\leq \frac{N}{A(T-t)^{m\gamma_1+2\ell}} \left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell}}\right)^{p-2} \\ &\quad + (p-2)\left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell}}\right)^{p-4} \left(\frac{d(\Omega)}{(T-t)^{m\gamma_1+2\ell}}\right)^2 \frac{N}{A(T-t)^{m\gamma_1+2\ell}} \\ &= \frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell)(p-1)}}. \end{aligned} \tag{4.5}$$

Similarly, from (4.2) and (4.4) we obtain

(4.6)
$$\begin{aligned}|\triangle_{n,q}\,\underline{v}| &\le \frac{N}{A(T-t)^{n\gamma_2+2\ell}}\left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell}}\right)^{q-2}\\ &\quad+(q-2)\left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell}}\right)^{q-4}\left(\frac{d(\Omega)}{(T-t)^{n\gamma_2+2\ell}}\right)^{2}\frac{N}{A(T-t)^{n\gamma_2+2\ell}}\\ &=\frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell)(q-1)}}.\end{aligned}$$

Next, we compute the nonlocal term of (4.1)

(4.7)
$$\begin{aligned}&\underline{u}^{\alpha_1}\int_\Omega \underline{v}^{\beta_1}(x,t)dx\\ &=\frac{1}{(T-t)^{\gamma_1\alpha_1+\gamma_2\beta_1}}V_1^{\alpha_1}\left(\frac{|x|}{(T-t)^\ell}\right)\int_{B(0,R(T-t)^\ell)}V_2^{\beta_1}\left(\frac{|x|}{(T-t)^\ell}\right)dx\\ &=\frac{M_1}{(T-t)^{\gamma_1\alpha_1+\gamma_2\beta_1-N\ell}}V_1^{\alpha_1}\left(\frac{|x|}{(T-t)^\ell}\right),\\ &\underline{v}^{\alpha_2}\int_\Omega \underline{u}^{\beta_2}(x,t)dx\\ &=\frac{1}{(T-t)^{\gamma_1\beta_2+\gamma_2\alpha_2}}V_2^{\alpha_2}\left(\frac{|x|}{(T-t)^\ell}\right)\int_{B(0,R(T-t)^\ell)}V_1^{\beta_2}\left(\frac{|x|}{(T-t)^\ell}\right)dx\\ &=\frac{M_2}{(T-t)^{\gamma_1\beta_2+\gamma_2\alpha_2-N\ell}}V_2^{\alpha_2}\left(\frac{|x|}{(T-t)^\ell}\right),\end{aligned}$$

where $M_1=\int_{B(0,R)}V_2^{\beta_1}(|\xi|)d\xi,\ M_2=\int_{B(0,R)}V_1^{\beta_2}(|\xi|)d\xi$.

If $0\le\xi\le A$, then $1\le V_1(\xi)\le\left(1+\frac{A}{2}\right)^{1/m}$, $1\le V_2(\xi)\le\left(1+\frac{A}{2}\right)^{1/n}$, and $V_1'(\xi)\le 0,\ V_2'(\xi)\le 0$. Combining the above inequalities and the definition of M_1 and M_2, we obtain

(4.8)
$$\begin{aligned}M_1V_1^{\alpha_1}\left(\frac{|x|}{(T-t)^\ell}\right)&\ge\int_{B(0,R)}V_2^{\beta_1}(|\xi|)d\xi\ge\|B(0,R)\|,\\ M_2V_2^{\alpha_2}\left(\frac{|x|}{(T-t)^\ell}\right)&\ge\int_{B(0,R)}V_1^{\beta_2}(|\xi|)d\xi\ge\|B(0,R)\|.\end{aligned}$$

Then from (4.2)-(4.8) we get

(4.9)
$$\begin{aligned}&\underline{u}_t-\triangle_{m,p}\underline{u}-\underline{u}^{\alpha_1}\int_\Omega\underline{v}^{\beta_1}(x,t)dx\\ &\le\frac{\gamma_1(1+A/2)^{1/m}}{(T-t)^{\gamma_1+1}}+\frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell)(p-1)}}-\frac{\|B(0,R)\|}{(T-t)^{\gamma_1\alpha_1+\gamma_2\beta_1-N\ell}},\end{aligned}$$

(4.10)

$$\begin{aligned}&\underline{v}_t - \triangle_{n,q}\underline{v} - \underline{v}^{\alpha_2}\int_\Omega \underline{u}^{\beta_2}(x,t)dx\\ &\le \frac{\gamma_2(1+A/2)^{1/n}}{(T-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell)(q-1)}} - \frac{\|B(0,R)\|}{(T-t)^{\gamma_1\beta_2+\gamma_2\alpha_2-N\ell}}.\end{aligned}$$

If $\xi \ge A$, since $m,\ n \ge 1$, we obtain $V_1(\xi) \le 1,\ V_2(\xi) \le 1$ and $V_1'(\xi) \le -\frac{1}{m},\ V_2'(\xi) \le -\frac{1}{n}$. Combining the above inequalities, (4.2)-(4.7) and $M_1 \ge 0,\ M_2 \ge 0$, we get

(4.11)

$$\underline{u}_t - \triangle_{m,p}\underline{u} - \underline{u}^{\alpha_1}\int_\Omega \underline{v}^{\beta_1}(x,t)dx \le \frac{\gamma_1 - \frac{1}{m}\ell A}{(T-t)^{\gamma_1+1}} + \frac{N(p-1)(d(\Omega))^{p-2}}{A(T-t)^{(m\gamma_1+2\ell)(p-1)}},$$

(4.12)

$$\underline{v}_t - \triangle_{n,q}\underline{v} - \underline{v}^{\alpha_2}\int_\Omega \underline{u}^{\beta_2}(x,t)dx \le \frac{\gamma_2 - \frac{1}{n}\ell A}{(T-t)^{\gamma_2+1}} + \frac{N(q-1)(d(\Omega))^{q-2}}{A(T-t)^{(n\gamma_2+2\ell)(q-1)}}.$$

If $0 \le \xi \le A$ and $\xi \ge A$, we have (4.9) and (4.12) hold. If $\xi \ge A$ and $0 \le \xi \le A$, we have (4.11) and (4.10) hold.

So, from the above discussions, (4.1) holds if the right-hand sides of (4.9)-(4.12) are non-positive.

By Lemma 2.2, there exist positive constants $\gamma_1,\ \gamma_2$ to satisfy

$$\begin{aligned}&\gamma_1\alpha_1 + \gamma_2\beta_1 > m(p-1)\gamma_1 + 1, \qquad (m(p-1)-1)\gamma_1 < 1,\\ &\gamma_1\beta_2 + \gamma_2\alpha_2 > n(q-1)\gamma_2 + 1, \qquad (n(q-1)-1)\gamma_2 < 1.\end{aligned}$$

And we can choose ℓ sufficiently small that

$$\ell < \min\left\{\frac{\gamma_1\alpha_1+\gamma_2\beta_1-\gamma_1-1}{N}, \frac{\gamma_1\alpha_1+\gamma_2\beta_1-m(p-1)\gamma_1}{N+2(p-1)}, \frac{\gamma_1+1-m(p-1)\gamma_1}{2(p-1)}\right\},$$

and

$$\ell < \min\left\{\frac{\gamma_1\beta_2+\gamma_2\alpha_2-\gamma_2-1}{N}, \frac{\gamma_1\beta_2+\gamma_2\alpha_2-n(q-1)\gamma_2}{N+2(q-1)}, \frac{\gamma_2+1-n(q-1)\gamma_2}{2(q-1)}\right\}.$$

Thus we have

$$\begin{aligned}(m\gamma_1+2\ell)(p-1) &< \gamma_1+1 < \gamma_1\alpha_1+\gamma_2\beta_1 - N\ell,\\ (n\gamma_2+2\ell)(q-1) &< \gamma_2+1 < \gamma_1\beta_2+\gamma_2\alpha_2 - N\ell.\end{aligned}$$

Furthermore, if we choose $A > \max\{1, m\gamma_1/\ell, n\gamma_2/\ell\}$, then for $T > 0$ sufficiently small, the right-hand sides of (4.9)-(4.12) are non-positive, so (4.1) holds. The proof of Theorem 1.3 is complete. □

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