Commun. Korean Math. Soc. **23** (2008), No. 4, pp. 511–528

SIMPLE VALUATION IDEALS OF ORDER 3 IN TWO-DIMENSIONAL REGULAR LOCAL RINGS

Sunsook Noh

Abstract. Let (R, m) be a 2-dimensional regular local ring with algebraically closed residue field R/m. Let K be the quotient field of R and v be a prime divisor of R, i.e., a valuation of K which is birationally dominating R and residually transcendental over R. Zariski showed that there are finitely many simple v-ideals $m = P_0 \supset P_1 \supset \cdots \supset P_t = P$ and all the other v-ideals are uniquely factored into a product of those simple ones [17]. Lipman further showed that the predecessor of the smallest simple v-ideal P is either simple or the product of two simple v-ideals. The simple integrally closed ideal P is said to be free for the former and satellite for the later.

In this paper we describe the sequence of simple v-ideals when P is satellite of order 3 in terms of the invariant $b_v = |v(x) - v(y)|$, where v is the prime divisor associated to P and $\mathfrak{m} = (x, y)$. Denote b_v by b and let $b = 3k + 1$ for $k = 0, 1, 2$. Let n_i be the number of nonmaximal simple v-ideals of order i for $i = 1, 2, 3$. We show that the numbers $n_v = (n_1, n_2, n_3) = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ and that the rank of P is $\lceil \frac{b+7}{3} \rceil = k+3$. We then describe all the v-ideals from $\mathfrak{m}$ to P as products of those simple v-ideals. In particular, we find the conductor ideal and the v-predecessor of the given ideal P in cases of $b = 1, 2$ and for $b = 3k + 1, 3k + 2, 3k$ for $k \geq 1$. We also find the value semigroup $v(R)$ of a satellite simple valuation ideal P of order 3 in terms of b_v.

1. Backgrounds

Let (R, m) be a 2-dimensional regular local ring with algebraically closed residue field $k = R/m$ and K be the quotient field of R. If v is a valuation of K dominating R whose corresponding valuation ring (V, n) with residue field $k(v) = V/n$, then the residual transcendence degree $\mathtt{tr.deg}_k k(v) \leq 1$. Then v is called a 0-dimensional (1-dimensional, respectively) valuation if $\mathtt{tr.deg}_k k(v) = 0$ (1, respectively). We call v a prime divisor of R if $\mathtt{tr.deg}_k k(v) = 1$.

Received June 21, 2008.

2000 *Mathematics Subject Classification.* Primary 13A18, 13H05, 13B02, 13B22.

Key words and phrases. simple valuation ideal, order of an ideal, prime divisor, proximity of simple integrally closed ideal, regular local ring.

This work was partially supported by KOSEF 97-0701-0201-5.

Let v be a prime divisor of R and (V, n) be the associated valuation ring of v. Such a prime divisor v is a discrete rank 1 valuation with the v-values $v(V) = \mathbf{N}$, the set of nonnegative integers [1, Theorem 1], [15].

For an ideal J of R, $v(J) = \min\{v(a) \mid a \in J\}$ is a nonnegative integer and J is called a v-ideal if $JV \cap R = J$, i.e., if $J = \{r \in R \mid v(r) \geq v(J)\}$. The sequence of contractions of the powers of the maximal ideals of V forms an infinite descending sequence of v-ideals in R

$$n \cap R \supset n^2 \cap R \supset \cdots \supset n^i \cap R \supset \cdots$$

$$m = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_j \supset I_{j+1} \supset \cdots . \tag{1}$$

For each $j \geq 0$, $I_j = \{r \in R \mid v(r) \geq v(I_j)\}$ is the j^{th} largest v-ideal in R. For a consecutive pair $I_j \supset I_{j+1}$ of v-ideals, I_j is called the v-predecessor of I_{j+1} and I_{j+1} is called the v-successor of I_j.

The set of v-values of all the v-ideals in the sequence (1) is called the value semigroup of v on R denoted by $v(R) = \{v(r) \mid r \in R\} = \{v(I_j) \mid \forall j \geq 0\}$:

$$0 < r_0 < r_1 < r_2 < \cdots < r_j < r_{j+1} < \cdots . \tag{2}$$

We denote $v(0) = \infty$. This value semigroup $v(R)$ is known to be symmetric [9, Theorem 1], i.e., there exists some integer z such that $a \in v(R)$ if and only if $z - a \notin v(R)$ for all integer $a \in \mathbf{Z}$. The conductor element of $v(R)$ is the smallest integer $c = r_i$ for some $i \geq 1$ such that $c - 1 \notin v(R)$ but $c + j \in v(R)$ for all $j \geq 0$. The corresponding ideal C with $v(C) = c$ is called the conductor (adjoint) ideal of v.

In [17, Theorem (E), (F), pp. 391–392], Zariski showed that given such a valuation v of K, there is a corresponding simple integrally closed ideal P and a unique quadratic sequence of 2-dimensional regular local rings in the quotient field K:

$$R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_t \tag{3}$$

in which the transform of the simple complete ideal P in R_t is the maximal ideal of R_t and v is the m_t-adic order valuation of K. It was also shown that there exist simple complete ideal P_i whose transforms in R_i is the maximal ideal m_i of R_i for each $i \geq 0$. These are the simple v-ideals

$$P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_t = P, \tag{4}$$

where $m = P_0$ and P is the smallest one. Any other v-ideal I can be uniquely factored into a product $I = \prod_{i=0}^{t} P_i^{a_i}$. The number t of nonmaximal simple v-ideal is said to be the rank of v, or the rank of the smallest simple v-ideal P.

The sequence of v-ideals between $m \supset P$ then can be divided into two parts:

$$m \overset{1}{\supset} P_1 \overset{1}{\supset} I_2 \overset{1}{\supset} \cdots \overset{1}{\supset} C \overset{1}{\supset} \cdots \overset{1}{\supset} P' \overset{1}{\supset} P,$$

and it is also known that this sequence is saturated, i.e., any two consecutive v-ideals are adjacent [10, Lipman, Theorem A.2], and hence P is the s^{th} largest v-ideal I_s, where $s = \lambda(R/P) - 1$ since k is algebraically closed. The v-predecessor of P is denoted by P' in the sequence. The length between any two consecutive v-ideals $I \supset J$ smaller than P can be measured [10, Theorem 3.1] in terms of the largest integer $\nu \in \mathbf{N}$ such that $P^{\nu}|I$.

For a v-ideal $P \supset J$, if J is also a w-ideal for a prime divisor w of R, then the sequence of w-ideals containing J coincides with that of v-ideals [10, Lipman, Theorem A.2].

For two regular local rings $S \supset T$ in K, S is said to be proximate to T (denoted by $S \succ T$) if $V_T \supset S$, where V_T is the $m(T)$-adic order valuation ring [8, (1.3)]. If v_T and v_S are the prime divisors associated to T and S, and hence to the simple integrally closed ideals $P_T \supset P_S$, we also say that $P_S \succ P_T$, i.e., P_S is proximate to P_T.

In the sequence of v-ideals, the v-predecessor of P is the unique integrally closed ideal adjacent to P from above [8, Theorem 4.11], [11, Theorem 3.1], and it was shown that it is either simple P_{t-1} or the product of simple v-ideals $P_{t-1}P_i$ for some $0 \leq i \leq t-2$ since k is algebraically closed. These are the simple v-ideals such that $R_t \succ R_{t-1}, R_i$ by [8, Theorem 4.11]. P is said to be free for the former and satellite for the latter. Note that Lipman showed this result in a general setting, i.e., without the assumption of $k = R/m$ being algebraically closed [8]. We refer [3] for the proximity relations between valuation ideals for 0-dimensional valuation case. Note that the $\mathfrak{m}$-adic order of an ideal I is the integer r such that such that $L \subseteq m^r \backslash m^{r+1}$. We denote it by $o(L)$.

Let us assume that P is a simple complete ideal associated to the prime divisor v. Let us assume that $o(P) = r \geq 1$, $\mathrm{rank}(P) = t \geq 0$ with the simple v-ideals $P_0, P_1, \ldots, P_t = P$. Let n_i be the number of nonmaximal simple v-ideals of order i for $1 \leq i \leq r$. We then may assume that the rank of P is:

$$t = n_1 + n_2 + \cdots + n_{r-1} + n_r,$$

and therefore the sequence of simple v-ideals are:

$$P_0 \supset P_1 \supset \cdots \supset P_{n_1} \supset \cdots \supset P_{n_1+n_2} \supset \cdots \supset P_{n_1+\cdots n_r} = P_t. \tag{5}$$

Let us denote the set of numbers $n_i's$ of v as

$$n_v = (n_1, n_2, \ldots, n_{r-1}, n_r).$$

In this paper, we describe the sequence of v-ideals from $\mathfrak{m}$ to P, find the numbers n_i, the number of simple v-ideals of order i for $1 \leq i \leq r$ in the case when P is a satellite simple complete ideal of order 3. Let $m = (x, y)$, $v(y) = r < v(x) = r + b_v$ for $b_v = v(x) - v(y) > 0$. We denote b_v by b when there is no confusion about v.

Let $o(P) = 1$. If $t = 0$, i.e., $P = P_0$ is the maximal ideal and hence v is the $\mathfrak{m}$-adic order valuation, $m \supset m^2 \supset m^3 \supset m^4 \supset \cdots$ is the sequence of all the v-ideals of R such that $\lambda(m^r/m^{r+1}) = r + 1$ for all $r \geq 1$. If $t > 0$, then

$$t = n_1$$

and the sequence of the v-ideals was then described in detail in [13].

Let $o(P) = 2$ and $\text{rank}(P) = t$. In [2], we showed that

$$t = n_1 + n_2 = \lceil \frac{b+1}{2} \rceil + (t - \lceil \frac{b+1}{2} \rceil),$$

where $b_v = 2k + i$ for $i = 0$ or 1. It was shown that $n_1 = \lceil \frac{b+1}{2} \rceil$ and $o(P_i) = 2$ for $\lceil \frac{b+3}{2} \rceil \leq i \leq t$. We showed that the satellite simple v-ideal of order 2 is $P_{\lceil \frac{b+3}{2} \rceil}$ whose predecessor is $P_{\lceil \frac{b+1}{2} \rceil} P_{\lceil \frac{b-1}{2} \rceil}$ and the conductor ideal $C = P_{\lceil \frac{b-1}{2} \rceil}$ is also simple in [2].

Throughout the paper, we assume $m = (x, y)$, $o(P) = 3$, $\text{rank}(P) = t \geq 3$, $v(y) = 3$, $v(x) = 3 + b_v$ for $b_v \geq 1$. Let n_i denote the number of nonmaximal simple v-ideals of order i for $i = 1, 2, 3$. Then the rank of P is $t = n_1 + n_2 + n_3$, and $n_3 = 1$ if P is proximate simple complete ideal.

In this paper, we describe n_1 and n_2 in terms of b_v (or in terms of k), where $b_v = 0, 1, 2$ or $b_v = 3k + i$ for $k \geq 1$ and $i = 0, 1, 2$. We then describe the sequence of v-ideals from $\mathfrak{m}$ to P using n_1 and n_2.

In Section 2, we show $n_1 = \lceil \frac{b+1}{3} \rceil$ and $n_2 = 1$, i.e., there exists a unique simple v-ideal of order 2. We also showed that the unique simple complete ideal of order 2 is $P_{\lceil \frac{b+4}{3} \rceil}$ and $P = P_{\lceil \frac{b+7}{3} \rceil}$. In particular, the rank of the satellite simple complete ideal P is

$$t = \lceil \frac{b+1}{3} \rceil + 1 + 1 = k + 3.$$

In Section 3, we find the factorizations of v-ideals from $\mathfrak{m}$ to P as products of simple v-ideals $P_i's$ for $0 \leq i \leq k + 3$. We also find factorizations of the v-predecessor of P and the conductor ideal C of v. We also find the value semigroup $v(R)$ of a satellite simple valuation ideal P of order 3 in terms of b_v.

2. The sequence of v-ideals of a satellite simple valuation ideal P of order 3

Throughout this paper we assume that the residue field k is algebraically closed and by an ideal we mean an $\mathfrak{m}$-primary ideal of R. Let v be a prime divisor of R and P be the simple complete ideal associated to v. We also assume that $o(P) = 3$ and that P is also satellite, i.e., v-predecessor P' of P is a product of two simple v-ideals.

Let $\text{rank}(P) = t \geq 3$, $m = (x, y)$, and $v(x) \geq v(y) = v(\mathfrak{m}) = 3$. Let us denote $v(x) - v(y)$ by b_v or often by b. Note that $v(\mathfrak{m}) = o(P) = 3$ by reciprocity by [8, Corollary (4.8)]. The rank of P is then

$$t = n_1 + n_2 + n_3.$$

In the sequence (3) of quadratic sequence along v, consider the first quadratic transformation R_1. Since R_1 has the maximal ideal $\mathfrak{m}_1 = (\frac{x}{y}, y)$ and V dominates R_1 and hence $m(V) \cap R_1 = \mathfrak{m}_1$. Therefore $v(x) > v(y)$ and

$v(x) = 3 + b_v$ for some $b_v > 0$. Let us denote b_v by b and $b = 3k + i$ for $0 \le i \le 2$. Note that then $\lceil \frac{b+1}{3} \rceil = k + 1$, $\lceil \frac{b+4}{3} \rceil = k + 2$, and $\lceil \frac{b+7}{3} \rceil = k + 3$.

Theorem 2.1. *Let (R, m, k) be a 2-dimensional regular local ring with algebraically closed residue field k. Let P be a satellite simple integrally closed ideal of R which is associated to the prime divisor v. Let $o(P) = 3$ and $\operatorname{rank}(P) = t$. Let n_i be the number of nonmaximal simple v-ideals of order i for $1 \le i \le 3$. Then, $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ and $\operatorname{rank}(P) = \lceil \frac{b+7}{3} \rceil$.*

Proof. Denote $b_v = b$. The theorem is true for $b = 1$ case by Proposition 2.2, $b = 2$ case by Proposition 2.3, $b = 3k + 1$ for $k \ge 1$ case by Proposition 2.4, $b = 3k + 2$ for $k \ge 1$ case by Proposition 2.5, and finally $b = 3k$ for $k \ge 1$ case by Proposition 2.6.

It is clear that $n_3 = 1$ since P is satellite. If P is satellite, its v-predecessor $P' = P_{t-1}P_i$ for some $0 \le i < t - 1$. Therefore, $o(P_{t-1}) = 2$ and $o(P_i) = 1$ for $0 \le i \le t - 2$ since $o(P') = 3$. Therefore, $n_2 = n_3 = 1$. $\square$

In [5], Huneke-Sally gave equivalent conditions of an ideal $I = (\mathfrak{m}^n, f)$ to be integrally closed for an element $f \notin \mathfrak{m}^n$. In particular they proved that I is integrally closed if $o(f) = n - 1$ and in this case we may assume that $I = (\mathfrak{m}^n, y^{n-1})$ for $\mathfrak{m} = (x, y)$. It was also shown that such an ideal I is also simple [12]. We then described all the simple v-ideals [12, Lemma 3.6] when v is the prime divisor associated to I. We now describe the sequence of all the v-ideals from $\mathfrak{m}$ to P in the case of $b_v = 1$.

Proposition 2.2. *Let R, $\mathfrak{m}$, P, v, b_v, n_v be as in Theorem 2.1. Let $o(P) = r$ for $r \ge 3$. If $b_v = 1$, then $P = (\mathfrak{m}^{r+1}, x^i)$ and there are r simple v-ideals. Furthermore, $n_v = (\lceil \frac{b+1}{3} \rceil, 1, \dots, 1)$ and $\operatorname{rank}(P) = r$.*

Proof. Note that $v(y) = r$, $v(x) = r+1$, $P = (x^r, x^{r-1}y^2, \dots, xy^r, y^{r+1})$. Then, m^i is a v-ideal for all $0 \le i \le r$ ([10, Theorem 1.2]) since $\lceil \frac{r}{b} \rceil = r$ such that $v(\mathfrak{m}^i) = ri$ for all i. By [12, Lemma 3.6], all the other nonmaximal simple v-ideals are $P_i = (\mathfrak{m}^{i+1}, x^i)$ for each $1 \le i \le r$.

$$\begin{array}{ccccccccccc} P_0 & \supset & P_1 & \supset & P_2 & \supset & \cdots & \supset & P_{r-1} & \supset & P_r \\ || & & \cup & & \cup & & \vdots & & \cup & & || \\ m & \supset & m^2 & \supset & m^3 & \supset & \cdots & \supset & m^r & \supset & P \end{array}$$

These two chains of v-ideals can be relisted as follows:

$$m \supset P_1 \overset{1}{\supset} m^2 \supset P_2 \overset{1}{\supset} m^3 \supset \cdots \supset P_{r-1} \overset{1}{\supset} m^{r-1} \supset P_r = P.$$

We now can fill up this sequence so that we obtain the complete sequence of v-ideals from $\mathfrak{m}$ to P. In general,

$$P_i = (\mathfrak{m}^{i+1}, x^i) = (x^i, x^{i-1}y^2, \dots, xy^i, y^{i+1}) \supset \mathfrak{m}^{i+1}$$

is the i-th simple v-ideal which is adjacent to $\mathfrak{m}^{i+1}$ from above. Since $v(x) = r + 1$ and $v(y) = 1$, we see that

$$v(x^{i-j}y^{j+1}) = (r + 1)i + (r - j)$$

for $1 \le j \le i \le r$ and therefore $v(P_i) = (r+1)i = v(x^i)$ since

$$\begin{array}{lcll} v(x^{i-1}y^2) & = & (r+1)i + (r-1) & > \\ v(x^{i-2}y^3) & = & (r+1)i + (r-2) & > \\ \vdots & & & \\ v(xy^i) & = & (r+1)i + (r-i-1) & > \\ v(y^{i+1}) & = & (r+1)i + (r-i) & \ge \\ v(x^i) & = & (r+1)i & = \quad v(P_i). \end{array}$$

For $1 \le i < r$, $v(y^{i+1}) = ri + r = v(m^{i+1}) > v(x^i) = ir + i = v(P_i)$ since $\mathfrak{m}^i$ is a v-ideal for all $1 \le i \le r$. Note that $v(P_i) = ri + i = v(\mathfrak{m}^{i+1})$ for each $1 \le i \le r$. We can inductively construct all the v-ideals from $\mathfrak{m}^i$ to P_i for each $1 \le i \le r$.

$$\mathfrak{m}^i \supset \mathfrak{m}^{i-1}P_1 \supset \mathfrak{m}^{i-2}P_2 \supset \cdots \supset \mathfrak{m}^2 P_{i-2} \supset \mathfrak{m}P_{i-1} \supset P_i \supset \mathfrak{m}^{i+1},$$

where the v-values of the ideals are

$$ir < ir+1 < ir+2 < \cdots < ir+(i-1) < ir+i < ri+r$$

since $v(\mathfrak{m}^{i-j}P_j) = r(i-j) + jr + j = ri + j$ for $0 \le j \le i$, $1 \le i \le r$. They are $i+2$ distinct saturated ideals since $\lambda(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = i+1$. Since $\mathfrak{m}^i$ is a v-ideal and $\lambda(\mathfrak{m}^i/\mathfrak{m}^{i-1}P_1) = 1$ such that $v(\mathfrak{m}^i) \ne v(\mathfrak{m}^{i-1}P_1)$, therefore we see that $\mathfrak{m}^{i-1}P_1$ is a v-successor of $\mathfrak{m}^i$. Similarly, $\mathfrak{m}^{i-j}P_j$'s are successive v-ideals between $\mathfrak{m}^i \supset \mathfrak{m}^{i+1}$ for all $j = 1, 2, \ldots, i$ and $1 \le i < r$. Therefore, the followings are the complete sequence of all the v-ideals from $\mathfrak{m}$ to P:

$$\begin{array}{lllllllllll} \mathfrak{m} & \supset & P_1 & \supset & & & & & & & \\ \mathfrak{m}^2 & \supset & \mathfrak{m}P_1 & \supset & P_2 & \supset & & & & & \\ \mathfrak{m}^3 & \supset & \mathfrak{m}^2P_1 & \supset & \mathfrak{m}P_2 & \supset & P_3 & \supset & \cdots & & \\ \mathfrak{m}^i & \supset & \mathfrak{m}^{i-1}P_1 & \supset & \mathfrak{m}^{i-2}P_2 & \supset & \mathfrak{m}^{i-3}P_3 & \supset & \cdots & \supset & P_i \supset \quad \cdots \\ \mathfrak{m}^r & \supset & \mathfrak{m}^{r-1}P_1 & \supset & \mathfrak{m}^{r-2}P_2 & \supset & \cdots & \supset & \mathfrak{m}P_{r-1} & \supset & P_r = P. \end{array}$$

The v-values of the v-ideals from $\mathfrak{m}$ to P_{r-1} are in the lower (with the diagonals) triangular matrix of the following $r \times r$ matrix $(v_{ij})_{0 \le i,j \le r-1}$. Then the v-values of the first column are $\mathfrak{m}^i$ for $0 \le i \le r-1$, the main diagonals are v-values of the simple v-ideals, i.e., $v_{ii} = v(P_i)$ for $0 \le i \le r-1$. The last row is the set of v-values of r consecutive valuation ideals with the conductor ideal $C = \mathfrak{m}^{r-1}$ in the first column.

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & r-1 \\ r & r+1 & r+2 & \cdots & \cdots & r+(r-1) \\ 2r & 2r+1 & 2r+2 & \cdots & \cdots & 2r+(r-1) \\ \vdots & \vdots & \vdots & & & \vdots \\ ir & ir+1 & ir+2 & \cdots & \cdots & ir+(r-1) \\ \vdots & \vdots & \vdots & & & \vdots \\ (r-2)r & (r-2)r+1 & (r-2)r+2 & \cdots & \cdots & (r-2)r+(r-1) \\ (r-1)r & (r-1)r+1 & (r-1)r+2 & \cdots & \cdots & (r-1)r+(r-1) \end{pmatrix}$$

The $r+1^{st}$ row of the matrix would start with $v(\mathfrak{m}^r)$, i.e.,

$$\mathfrak{m}^r \supset \mathfrak{m}^{r-1}P_1 \supset \mathfrak{m}^{r-2}P_2 \supset \cdots\cdots \supset \mathfrak{m}P_{r-1} \supset P_r$$

is the saturated sequence of v-values from m^r to P_r. Note that $P_r \supset \mathfrak{m}^{r+1}$ are adjacent, and $P_r = P$ is the smallest simple v-ideal, i.e., P such that $\operatorname{rank}(P) = r$.

Each simple v-ideal P_i transforms to the maximal ideal $\mathfrak{m}_i$ in the i^{th} quadratic transform R_i along v for $1 \leq i \leq r$:

$$R \subset R_1 \subset R_2 \subset \cdots\cdots \subset R_{r-1} \subset R_r = R_v.$$

Note that the v-predecessor of P is $\mathfrak{m}P_{r-1}$ and

$$n_v = (1, 1, 1, \ldots, 1, 1) = (\lceil \frac{b+1}{3} \rceil, 1, 1, \ldots, 1, 1),$$

where $b = 1$ since P_i is the only nonmaximal simple v-ideal of order i for $1 \leq i \leq r$.

The largest v-ideal of order r is $\mathfrak{m}^r$, hence the conductor ideal $C = \mathfrak{m}^r : \mathfrak{m} = \mathfrak{m}^{r-1}$, i.e., this is the largest v-ideal of order $r-1$ and $v(C) = r^2 - r$. The v-predecessor of $\mathfrak{m}^{r-1}$ is P_{r-2}, where $v(P_{r-2}) = v(x^{r-2}) = r^2 - r - 2$, hence $r^2 - r - 1 \notin v(R)$ is the largest number that is not in the value semigroup $v(R)$ of v.

Among v_{ij}'s we see that the elements in the upper triangular matrix, i.e., $v_{ij} \notin v(R)$ for $j > i$. They are exactly the half of the conductor value, i.e., $\frac{r^2-r}{2}$. Hence

$$v(R) = \mathbf{N} \setminus \{ir + j\}_{0 \leq i \leq r-1,\, i+1 \leq j \leq r-1}.$$

This proves the proposition. □

If $r = 3$, then $v(R) = \mathbf{N} \setminus \{1, 2, 5\}$ and $\operatorname{rank}(P) = 3 = \lceil \frac{b+7}{3} \rceil$ since $b = 1$ in the above proposition. From now we assume that $o(P) = 3$ and $t = n_1 + n_2 + n_3$. Therefore, there are t nonmaximal simple v-ideals:

$$\mathfrak{m} \supset P_1 \supset \cdots \supset P_{n_1} \supset P_{n_1+1} \supset \cdots \supset P_{n_1+n_2} \supset \cdots \supset P_{n_1+n_2+n_3}.$$

We further assume that P is satellite, i.e., $n_3 = 1$, $t = n_1 + n_2 + 1$. If $b = 1$, then $n_1 = n_2 = n_3 = 1$ by Proposition 2.2 and therefore $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$.

We often compute the length between two integrally closed ideals by using reciprocity of Lipman [10, Remark 2.2]. When the length between two integrally closed ideals $M \supset N$ are known and another integrally closed ideal L is given, we can compute the length between $ML \supset NL$ as $\lambda(M/L) + (N \cdot L) - (M \cdot L)$, where $(I \cdot J)$ denotes the intersection multiplicity of integrally closed ideals I and J. We also note that if $L \supset M$ complete ideals with M simple, then $u(L) = u(M)$ if and only if M is not a u-ideal for a prime divisor u of R. If J is a simple complete ideal associated to u, then it is equivalent to say that $(L \cdot J) = (M \cdot J)$ [11, Lemma 3.3].

Proposition 2.3. *Let* (R, m, k) *be a* 2*-dimensional regular local ring with algebraically closed residue field* k*. Let* P *be a satellite simple integrally closed ideal of* R *which is associated to the prime divisor* v*. Let* $o(P) = 3$*,* $\text{rank}(P) = t$*,* $b_v = 2$*. Let* n_i *be the number of nonmaximal simple* v*-ideals of order* i *for* $1 \leq i \leq 3$*. Then,* $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ *and* $\text{rank}(P) = \lceil \frac{b+7}{3} \rceil$.

Proof. Assume $b = 2$, i.e., $v(y) = 3$, $v(x) = 5$. In this case, $\mathfrak{m}$ and $\mathfrak{m}^2$ are v-ideals, but $\mathfrak{m}^3$ is not since $\lceil \frac{r}{b} \rceil = 2$ by [10, Theorem 1.2]. Therefore, $P_1 = (x, y^2)$ and

$$m \supset P_1 \supset m^2$$

are consecutive v-ideals of v-values $3, 5, 6$. This implies that $\mathfrak{m}^2 \supset P_2$, i.e., $o(P_2) \geq 2$ and hence P_1 is the only nonmaximal simple v-ideal of order 1 and $n_1 = 1$. Consider the following sequence of ideals:

$$\mathfrak{m} \supset P_1 \supset \mathfrak{m}^2 \supset \mathfrak{m}P_1 \supset I \supset \mathfrak{m}^3,$$

where I is the v-ideal of value $v(\mathfrak{m}^3) = 9$. It is easy to see that this sequence is saturated. Since $\mathfrak{m}^2$ is a v-ideal and $v(\mathfrak{m}P_1) = 8 > v(\mathfrak{m}^2)$, $\mathfrak{m}P_1$ should be the v-successor of $\mathfrak{m}^2$. Since $\mathfrak{m}P_1 \supset I \supset \mathfrak{m}^3$ are also saturated, $v(I) = 9$ and $o(I) = 2$.

If I is not simple, then I is P_1^2 since $o(I) = 2$. However, $v(P_1^2) = 10 > v(I)$, and hence $I \neq P_1^2$. Therefore, $I = P_2$ must simple of value $v(\mathfrak{m}^3) = 9$ of order 2. Consider the following sequence of ideals:

$$\mathfrak{m} \supset P_1 \supset \mathfrak{m}^2 \supset \mathfrak{m}P_1 \supset P_2 \supset J \supseteq P_1^2,$$

where J is the v-ideal of value $v(P_1^2) = 10$. But, $\lambda(mP_1/P_1^2) = 1 + (P_1 \cdot P_1) - (m \cdot P_1) = 2$ by reciprocity. Hence $J = P_1^2$ is the v-ideal adjacent to P_2 from below.

Since we have $7 \notin v(R)$ and $8, 9, 10 \in v(R)$, $8 = v(mP_1)$ is the conductor element of v since $3 \in v(R)$. Let us denote three consecutive v-ideals by

$$C = \mathfrak{m}P_1 \supset D = P_2 \supset E = P_1^2$$

of v-values $8, 9, 10$. Since $o(C) = 2$, $\mathfrak{m}C = \mathfrak{m}^2 P_1$ is the largest v-ideal of order 3 with v-value 11. From calculating the lengths, we have the following sequence of ideals of v-values $11, 12, 13, 14, 15$:

$$\mathfrak{m}C \supset \mathfrak{m}D \supset \mathfrak{m}E = P_1C \supset P_1D \supset P_1E.$$

Note that $\mathfrak{m}E = P_1C = \mathfrak{m}P_1^2$, $v(P_1D) = v(P_1P_2) = 14$ and hence P_1D is a v-ideal which is successive to $\mathfrak{m}E$ since

$$\lambda(\mathfrak{m}D/P_1D) = \lambda(\mathfrak{m}/P_1) + (P_1 \cdot D) - (\mathfrak{m} \cdot D) = 1 + 3 - 2 = 2.$$

Note also that $v(P_1E) = v(P_1^3) = 15$. However,

$$\lambda(\mathfrak{m}E/P_1E) = \lambda(\mathfrak{m}P_1/P_1^2) + 2(P_1 \cdot P_1) - 2(\mathfrak{m} \cdot P_1) = 2 + 4 - 2 = 4$$

implies that P_1^3 is not a v-ideal, i.e., $P_1E = P_1^3$ is not a v-ideal and hence there exists a v-ideal $Q \supset P_1^3$ such that $v(Q) = 15$:

$$\mathfrak{m}P_1^2 = \mathfrak{m}E \supset P_1D = P_1P_2 \supset Q \supset P_1E = P_1^3.$$

Since $o(Q) = 3$, we can factorize $Q = \mathfrak{m}^a P_1^b P_2^c$ for some $a, b, c \geq 0$. Then, $15 = 3a + 5b + 9c$. A possible solution(s) for (a, b, c) are $(5, 0, 0)$, $(2, 0, 1)$, $(0, 3, 0)$. However, $\mathfrak{m}^5$ is not a v-ideal and P_1^3 is not a v-ideal, either. Therefore, $Q = \mathfrak{m}^2 P_2$. But this is not the case since $\lambda(\mathfrak{m}P_2/\mathfrak{m}^2 P_2) = o(\mathfrak{m}P_2) + 1 = 4 \neq 3 = \lambda(mP_2/Q)$. Therefore, Q is the simple v-ideal of order 3, i.e., $Q = P_3$ is the simple v-ideal associated to v with the v-predecessor P_2P_1.

We have shown that $n_v = (1, 1, 1) = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ since $b = 2$. Note that the v-predecessor of P is $P_2P_1 = P_{k+1}P_{k+2}$ since $k = 0$. The following is the complete sequence of v-ideals from $\mathfrak{m}$ to P:

$$\mathfrak{m} \supset P_1 \supset \mathfrak{m}^2 \supset \mathfrak{m}P_1 = C \supset P_2 \supset P_1^2 \supset \mathfrak{m}^2 P_1 \supset \mathfrak{m}P_2 \supset \mathfrak{m}P_1^2 \supset P_1P_2 \supset P_3 = P,$$

where $v(R) = \mathbf{N} \backslash \{1, 2, 4, 7\}$ for $\mathbf{N}$ is the set of nonnegative integers. Furthermore, we have shown that $\operatorname{rank}(P) = 3 = \lceil \frac{b+7}{3} \rceil$ since $b = 2$. □

Now we consider a more general case when $b = 3k + 1$ for $k \geq 1$.

Proposition 2.4. *Let (R, m, k) be a 2-dimensional regular local ring with algebraically closed residue field k. Let P be a satellite simple integrally closed ideal of R which is associated to the prime divisor v. Let $o(P) = 3$, $\operatorname{rank}(P) = t$, $b_v = 3k + 1$ for $k \geq 1$. Then, $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$, $\operatorname{rank}(P) = \lceil \frac{b+7}{3} \rceil$, and P_kP_{k+2} is the v-predecessor of P.*

Proof. We first note that $n_2 > 0$, i.e., we have at least one simple v-ideal of order 2. The v-predecessor P' of P is the product of two simple v-ideals $P' = P_{t-1}P_i$ for $0 \leq i \leq t-2$ since we assume that P is satellite. Therefore, there exists at least one simple v-ideal P_{t-1}, i.e., $n_2 \neq 0$.

Note that $v(y) = 3$, $v(x) = 3 + (3k + 1)$ for $k \geq 1$. Hence, $P_i = (x, y^{i+1})$ is a simple v-ideal such that $v(P_i) = \min\{3k + 4, 3i + 3\}$ for $1 \leq i \leq k + 1$:

$$\mathfrak{m} \supset P_1 = (x, y^2) \supset P_2 = (x, y^3) \supset \cdots \supset P_k = (x, y^{k+1}) \supset P_{k+1} = (x, y^{k+2})$$

is the saturated sequence of v-ideals of value $3, 6, \ldots, 3k, 3k + 3, 3k + 4$, where $b = 3k + 1$ for $k \geq 1$.

Since $\lambda(P_k/\mathfrak{m}P_k) = \mu(P_k) = o(P_k) + 1 = 2$ (cf. [4], [5]) and $v(\mathfrak{m}P_k) = 3k + 6$, $\mathfrak{m}P_k$ is the v-ideal adjacent to P_{k+1}, i.e., $\mathfrak{m}P_k$ is the largest v-ideal of order 2 and hence $o(P_{k+2}) \geq 2$. This implies that $n_1 = k + 1 = \lceil \frac{b+1}{3} \rceil$.

Since $\lambda(P_{k+1}/\mathfrak{m}P_{k+1}) = 2$ and $v(\mathfrak{m}P_{k+1}) = 3k + 7$, $\mathfrak{m}P_{k+1}$ is the v-successor of $\mathfrak{m}P_k$. Therefore,

$$\mathfrak{m} \supset P_1 \supset \cdots \supset P_k \supset P_{k+1} \supset \mathfrak{m}P_k \supset \mathfrak{m}P_{k+1}$$

are all the v-ideals from $\mathfrak{m}$ to $\mathfrak{m}P_{k+1}$ of v-values

$$3 < 6 < \ldots < 3k + 3 < 3k + 4 < 3k + 6 < 3k + 7.$$

By using [2, Corollary 2.2], we can conclude that

$$P_{k+1} \supset mP_k \supset mP_{k+1} \supset P_1P_k \supset P_1P_{k+1} \supset \cdots \supset P_kP_k \supset P_kP_{k+1} \supset P_{k+2}$$

is the saturated sequence of v-ideals from P_{k+1} to P_{k+2}. Note that $o(P_{k+2}) = 2$, $v(P_{k+2}) = 6k + 8$ since $v(P_k^2) = 6k + 6$ and $v(P_kP_{k+1}) = 6k + 7$, $v(P_{k+1}^2) = 6k + 8$. Note that $\lambda(P_kP_{k+1}/P_{k+1}^2) = 2$, hence P_{k+1}^2 is not a v-predecessor of P_kP_{k+1}. Therefore, v-successor of P_kP_{k+1} is a simple v-ideal, P_{k+2}. Since $v(P_{k+1}^2) = 6k + 8 \in v(R)$, $v(P_{k+2}) = v(P_{k+1}^2) = 6k + 8$ and P_{k+1}^2 is not a v-ideal.

Since $P_{k-1}P_{k+1} \supset P_k^2$ are adjacent v-ideals of v-values $6k + 4$ and $v(P_k^2) = 6k + 6$, we have that $6k + 5 \notin v(R)$. Since $6k + 6, 6k + 7, 6k + 8 \in v(R)$, we have the conductor ideal is $C = P_k^2$ such that $v(C) = 6k + 6$. Let

$$C = P_k^2 \supset D = P_kP_{k+1} \supset E = P_{k+2}$$

be three consecutive v-ideals of v-values $6k + 6, 6k + 7, 6k + 8$. Then,

$$\mathfrak{m}C \supset \mathfrak{m}D \supset \mathfrak{m}E \supset P_1C \supset P_1D \supset P_1E \supset \cdots \supset P_kC \supset P_kD \supset P_kE$$

are the consecutive v-ideals of v-values $6k + 9, \ldots, (6k + 9) + (b + 1)$.

Note that $\mathfrak{m}C = \mathfrak{m}P_k^2$ is the largest v-ideal of order 3 and $v(P_kP_{K+2})) = 9k + 11$. Since $v(P_{k+1}P_{k+2}) = 9k + 12$ and $\lambda(P_kP_{k+2}/P_{k+1}P_{k+2}) = 1 + [w(P_{k+1}) - w(P_k)] = 2$, where w is the prime divisor associated to P_{k+2} since then $P_k \supset P_{k+1}$ are both w-ideals whose w-values differ by 1 [14, Theorem 3.3, Theorem 4.1]. Therefore, the v-successor of P_kP_{k+2} has v-value $9k + 12$ and it contains $P_{k+1}P_{k+2}$. Let us call it Q. Since $P_kP_{k+2} \supset Q \supset P_{k+1}P_{k+2}$ are adjacent, Q is either a product of three order 1 simple v-ideals, or $P_{k+2}P_i$ for some $i \leq k$. But the latter cannot be the case for if so, $P_k \supset P_i \supset P_{k+1}$ which is a contradiction since $P_k \supset P_{k+1}$ are adjacent. Let $Q = P_iP_jP_\ell$ for $1 \leq i \leq j \leq \ell \leq k + 1$. Since $\mathfrak{m} \nmid P_k, P_{k+1}, P_{k+2}$, $\mathfrak{m} \nmid Q$ [11, Lemma 1.2]. If $\ell = k + 1$, then $P_{k+1}|Q$ and $v(Q) = 9k + 12 = 3(i + j) + 10$, hence $3|9k + 2$, contradiction. If $\ell = k$, then $P_{k+2} \supset P_iP_j$ are adjacent ideals such that $6k + 8 = 3(i + j) + 6$ which implies that $3|6k + 2$, contradiction. Therefore, P_k does not divide Q, either. Therefore, $Q = P_iP_jP_\ell$ for $1 \leq i \leq j \leq \ell < k$. Since $v(Q) = 3(i + j + \ell) + 9 = 9k + 12$, $i + j + \ell = 3k + 1 < 3k$, a contradiction. Therefore, $Q = P_{k+3}$ is simple of order 3, i.e., $P_{k+3} = P$ is the simple complete ideal associated to v.

Note that the v-predecessor of P is P_kP_{k+2} and P_{k+2} is the only simple v-ideal of order 2. Hence $n_v = (k + 1, 1, 1) = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ since $b = 3k + 1$ for $k \geq 1$. □

Proposition 2.5. *Let (R, m, k) be a 2-dimensional regular local ring with algebraically closed residue field k. Let P be a satellite simple integrally closed ideal of R which is associated to the prime divisor v. Let $o(P) = 3$, $\operatorname{rank}(P) = t$, $b_v = 3k + 2$ for $k \geq 1$. Then, $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ and $\operatorname{rank}(P) = \lceil \frac{b+7}{3} \rceil$.*

Proof. The followings are a saturated sequence of simple v-ideals

$$\mathfrak{m} \supset P_1 = (x, y^2) \supset P_2 = (x, y^3) \supset \cdots \supset P_k = (x, y^{k+1}) \supset P_{k+1} = (x, y^{k+2})$$

whose v-values are $3 < 6 < \cdots < 3k+3 < 3k+5$.

As in the proof of Proposition 2.4, we have $v(\mathfrak{m}P_k)$ is the v-ideal adjacent to P_{k+1}, i.e., mP_k is the largest v-ideal of order 2 and hence $o(P_{k+2}) \geq 2$. This implies that $n_1 = k+1 = \lceil \frac{b+1}{3} \rceil$. It is also true that $\mathfrak{m}P_{k+1}$ is the v-successor of $\mathfrak{m}P_k$ since $v(\mathfrak{m}P_{k+1}) = 3k+8 > v(\mathfrak{m}P_k) = 3k+6$ and $\lambda(\mathfrak{m}P_k/\mathfrak{m}P_{k+1}) = 2$. Therefore,

$$\mathfrak{m} \supset P_1 \supset \cdots \supset P_{k+1} \supset \mathfrak{m}P_k \supset \mathfrak{m}P_{k+1}$$

are all the v-ideals from $\mathfrak{m}$ to $\mathfrak{m}P_{k+1}$ of v-values

$$3 < 6 < \cdots < 3k+3 < 3k+5 < 3k+6 < 3k+8.$$

By using [2, Corollary 2.2], we can also conclude that

$$P_{k+1} \supset \mathfrak{m}P_k \supset \mathfrak{m}P_{k+1} \supset P_1P_k \supset P_1P_{k+1} \supset \cdots \supset P_kP_k \supset P_kP_{k+1} \supset P_{k+2}$$

is the saturated sequence of v-ideals from P_{k+1} to P_{k+2}.

Note that $o(P_{k+2}) = 2$ and $v(P_k^2) = 6k+6$ implies that $6k+9 \in v(R)$. Since $v(P_kP_{k+1}) = 6k+8$ and $v(P_{k+1}^2) = 6k+10$, we conclude that $v(P_{k+2}) = 6k+9$. Since $\lambda(P_kP_{k+1}/P_{k+1}^2) = 2$ with $v(P_{k+1}^2) = 6k+10$, hence P_{k+1}^2 is a v-ideal adjacent to P_{k+2}. Therefore,

$$P_kP_{k+1} \supset P_{k+2} \supset P_{k+1}^2$$

are consecutive v-ideals of v-values $6k+8, 6k+9, 6k+10$. Since $6k+7 \notin v(R)$, $C = P_kP_{k+1}$ is the conductor ideal. Let

$$C = P_kP_{k+1} \supset D = P_{k+2} \supset E = P_{k+1}^2$$

be three consecutive v-ideals of v-values $6k+8, 6k+9, 6k+10$. Then,

$$\mathfrak{m}C \supset \mathfrak{m}D \supset \mathfrak{m}E \supset P_1C \supset P_1D \supset P_1E \supset \cdots \supset P_kC \supset P_kD \supset P_kE$$

are the consecutive v-ideals.

Note that $\mathfrak{m}C$ is the largest v-ideal of order 3 with v-value $6k+11$. Note also that $v(P_kP_{k+2}) = 9k+13$, $v(P_{k+1}P_{k+2}) = 9k+15$, and $\lambda(P_kP_{k+2}/P_{k+1}P_{k+2}) = 2$. Therefore, there exist a v-ideal Q such that

$$P_kP_{k+2} \supset Q \supset P_{k+1}P_{k+2}$$

are consecutive v-ideals of v-values $9k+13 < 9k+14 < 9k+15$.

As in the proof of $b = 3k+1$ case, we can show that $P_{k+2} \nmid Q$. Suppose $P_{k+1}|Q$. Then, $Q = P_{k+1}Q' \supset P_{k+1}P_{k+2}$ are adjacent, and hence $Q' = P_kP_{k+1}$ is the adjacent ideal to P_{k+2} from above. Note that

$$\begin{aligned}\lambda(P_kP_{k+1}^2/P_{k+1}P_{k+2}) &= \lambda(P_kP_{k+1}/P_{k+2}) + [(P_{k+1} \cdot P_{k+2}) - (P_{k+1} \cdot P_kP_{k+1})] \\ &= 1 + [w(P_{k+2}) - w(P_kP_{k+1})] \\ &= 1\end{aligned}$$

since $P_kP_{k+1} \supset P_{k+2}$ are adjacent and P_{k+2} is not a w-ideal, where w is the prime divisor associated to P_{k+1} [11, Lemma 3.3]. However, $v(P_kP_{k+1}^2) = 9k+13 = v(P_kP_{k+2})$ implies that $Q \neq P_kP_{k+1}^2$, i.e., $P_{k+1} \nmid Q$. This leaves the case to $Q = P_iP_jP_\ell$ for $i, j, \ell \leq k$. Since then $v(Q) = 3(i+j+k)+9 = 9k+14$ implies that $3|3k+5$, a contradiction. Therefore, $Q = P_{k+3}$ is the simple v-ideal which is P.

We showed that $n_v = (k+1, 1, 1) = (\lceil \frac{b+1}{3} \rceil, 1, 1)$ and the rank of P is $k+3 = \lceil \frac{b+7}{3} \rceil$ since $b = 3k+2$ for $k \geq 1$. □

Our proof does heavily depend on the reciprocity formula of Lipman which may be stated as $w(I) = v(J)$ for prime divisors v and w associated to simple $\mathfrak{m}$-primary complete ideals I and J. We often use this formula to compute the intersection multiplicity $(L \cdot M)$ of two complete $\mathfrak{m}$-primary ideals L and M (cf. [6, Corollary (3.7)], [4, Corollary 4.4]).

Proposition 2.6. *Let (R, m, k) be a 2-dimensional regular local ring with algebraically closed residue field k. Let P be a satellite simple integrally closed ideal of R which is associated to the prime divisor v. Let $o(P) = 3$, $b_v = 3k$ for $k \geq 1$. Then, $n_v = (\lceil \frac{b+1}{3} \rceil, 1, 1)$, $\text{rank}(P) = \lceil \frac{b+7}{3} \rceil$, and P_kP_{k+2} is the v-predecessor of P.*

Proof. The followings are simple v-ideals

$$P_1 = (x, y^2) \supset P_2 = (x, y^3) \supset \cdots \supset P_k = (x, y^{k+1})$$

whose v-values are $6 < 9 < \cdots < 3k < 3k+3$ for $k \geq 1$. Since $\lambda(P_k/\mathfrak{m}P_k) = 2$, $v(\mathfrak{m}P_k) = 3k+6$, and $P_k \supset I \supset \mathfrak{m}P_k$, where I is the v-successor of P_k containing $\mathfrak{m}P_k$.

We then have that $I = (x - \alpha y^{k+1}, y^{k+2})$ for a unit α of R. Such an ideal I is simple, and hence $I = P_{k+1}$ is the $k+1^{st}$ simple v-ideal. Note that the v-successor of P_{k+1} is $\mathfrak{m}P_k$. Since $v(P_k) = 3k+3$ and $v(\mathfrak{m}P_k) = 3k+6$, we have either $v(P_{k+1}) = 3k+4$ or $3k+5$. Therefore, $\mathfrak{m}P_k$ is the largest v-ideal of order 2 and hence $n_1 = k+1$.

Claim 1: $P_{i-1}P_k \supset P_{i-1}P_{k+1} \supset P_iP_k \supset P_iP_{k+1}$ are successive, adjacent v-ideals for $1 \leq i \leq k$.

Since $o(P_{k+1}) = 1$, we have $\lambda(P_{k+1}/\mathfrak{m}P_{k+1}) = 2$. Therefore, $\mathfrak{m}P_{k+1}$ is the successor of $\mathfrak{m}P_k$ since $v(\mathfrak{m}P_{k+1}) > v(\mathfrak{m}P_k)$. Let w be the prime divisor associated to the simple integrally closed ideal $P_1 = (x, y^2)$. Hence $w(y) = 1$ and $w(x) = 2$. Since $k+1 \geq 2$, we also have $w(P_k) = w(x, y^{k+1}) = 2$. Then

$$\lambda(\mathfrak{m}P_k/P_1P_k) = \lambda(\mathfrak{m}/P_1) + [(P_1 \cdot P_k) - (\mathfrak{m} \cdot P_k)] = 1 + w(P_k) - o(P_k) = 2,$$

we have $\mathfrak{m}P_k \supset \mathfrak{m}P_{k+1}$ are adjacent v-ideals of v-value $3k+6 < 3k+7$ or $3k+8$. Since $v(P_1P_k) = 3k+9$, we have that $\mathfrak{m}P_{k+1} \supset P_1P_k$ are adjacent, i.e., therefore

$$\mathfrak{m}P_k \supset \mathfrak{m}P_{k+1} \supset P_1P_k$$

are consecutive v-ideals. In general, we have by reciprocity

$$\begin{aligned}\lambda(P_{i-1}P_k/P_iP_k) &= \lambda(P_{i-1}/P_i)+[(P_i\cdot P_k)-(P_{i-1}\cdot P_k)]\\ &= 1+[w_i(P_k)-w_{i-1}(P_k)]\\ &= 1+[w_i(P_i)-w_{i-1}(P_{i-1})]\\ &= 1+[e(P_i)-e(P_{i-1})]\\ &= 1+[(i+1)-i]=2,\end{aligned}$$

where w_i is the prime divisor associated to the simple v-ideal P_i for $1\le i\le k$ and $e(\cdot)$ denotes the multiplicity of the ideal. Therefore,

$$P_{i-1}P_k\supset P_{i-1}P_{k+1}\supset P_iP_k$$

are the adjacent v-ideals since their v-values are $3(i+k)+3<3(i+k)+4,5<3(i+k)+6$. Inductively, we can show that these are v-ideals.

Similarly, we prove that $\lambda(P_iP_{k+1}/P_{i+1}P_{k+1})=2$ and hence that

$$P_{i-1}P_{k+1}\supset P_iP_k\supset P_iP_{k+1}$$

are adjacent v-ideals since

$$\begin{aligned}v(P_{i-1}P_{k+1}) &= 3(i+k)+4(\text{or }\ 3(i+k)+5)\\ &< v(P_iP_k)=3(i+k)+6\\ &< v(P_iP_{k+1})=3(i+k)+7(\text{or }\ 3(i+k)+8)\end{aligned}$$

for all $1\le i\le k$. Therefore, the following is the complete sequence of v-ideals from $\mathfrak{m}P_k$ to P_kP_{k+1}:

$$mP_k\supset mP_{k+1}\supset P_1P_k\supset\cdots\supset P_kP_k\supset P_kP_{k+1}.$$

This proves Claim 1.

Let Q be the v-successor of P_kP_{k+1}. Then $o(Q)=2$ and hence $Q=P_iP_j$ for some $0\le i,j\le k+1$. If $P_{k+1}|Q$, since if so $Q=P_{k+1}^2$ since $P_kP_{k+1}\supset Q$ are adjacent. However, the length between $P_kP_{k+1}\supset Q$

$$\lambda(P_kP_{k+1}/P_{k+1}^2)=1+[e(P_{k+1})-e(P_k)]=1+[(k+2)-(k+1)]=2$$

gives a contradiction. Therefore, $P_{k+1}\nmid Q$. If $P_k|Q$, then $Q=P_kQ'$ for some simple v-ideal of order 1 which is smaller than P_{k+1}, contradiction. Therefore, $P_k\nmid Q$. Suppose now that $Q=P_iP_j$ for $i,j<k$. Then, $v(Q)=3(i+j)+6<6k+7$ or $6k+8$ which is $v(P_kP_{k+1})$, contradiction to $P_kP_{k+1}\supset Q$ are v-ideals. Therefore, Q is simple, i.e., $Q=P_{k+2}$ is the largest simple v-ideal of order 2. Now we further claim the following:

Claim 2: The conductor ideal is $C=P_k^2$ and $P_{k+2}\supset\mathfrak{m}P_kP_k\supset\mathfrak{m}P_kP_{k+1}\supset\mathfrak{m}P_{k+2}$ are successive, adjacent v-ideals.

Note that the v-values of those three ideals are

$$v(P_{k+2})<6k+9<6k+10,6k+11<v(P_{k+2})+3.$$

Therefore, $v(P_{k+1})=3k+4$, $v(P_kP_{k+1})=6k+7$, and $v(P_{k+2})=6k+8$. Hence $\mathfrak{m}P_k^2\supset\mathfrak{m}P_kP_{k+1}\supset\mathfrak{m}P_{k+2}$ are another three successive v-ideals of v-value $6k+9,6k+10,6k+11$ due to the length computations. Since $v(P_{k-1}P_{k+1})=$

$6k+4$ and hence $6k+5 \notin v(R)$. Since $6k+6, 6k+7, 6k+8 \in v(R)$ and $3 \in v(R)$, we see that $6k+6$ is the conductor element and P_k^2 is the conductor ideal. This proves Claim 2.

Let $C = P_k^2 \supset D = P_k P_{k+1} \supset E = P_{k+2}$ be three consecutive v-ideals of v-values $6k+6, 6k+7, 6k+8$. Then, we construct the v-ideals further as follows:

Claim 3: $\lambda(P_{i-1}C/P_iC) = \lambda(P_{i-1}D/P_iD) = \lambda(P_{i-1}D/P_iD) = 3$ for $1 \le i \le k$.

Let w_i be the prime divisor associated to P_i for each $1 \le i \le k$. We multiply $P_{i-1} \supset P_i$ by $C = P_k^2$ to calculate the lengths:

$$\begin{aligned}
\lambda(P_{i-1}C/P_iC) &= \lambda(P_{i-1}/P_i) + [(P_i \cdot C) - (P_{i-1} \cdot C)] \\
&= \lambda(P_{i-1}/P_i) + [(P_i \cdot P_k^2) - (P_{i-1} \cdot P_k^2)] \\
&= 1 + 2[w_i(P_k) - w_{i-1}(P_k)] \\
&= 1 + 2[w_i(P_i) - w_{i-1}(P_{i-1})] \\
&= 1 + 2[e(P_i) - e(P_{i-1})] \\
&= 3
\end{aligned}$$

since P_k is not a w_i-, w_{i-1}-ideal [11, Lemma 3.3], where $e(\cdot)$ denotes the multiplicity of the ideal. Similarly, we multiply $P_{i-1} \supset P_i$ by $D = P_k P_{k+1}$ and compute the length:

$$\begin{aligned}
\lambda(P_{i-1}D/P_iD) &= \lambda(P_{i-1}/P_i) + [(P_i \cdot D) - (P_{i-1} \cdot D)] \\
&= 1 + [w_i(P_k P_{k+1}) - w_{i-1}(P_k P_{k+1})] \\
&= 1 + [w_i(P_k) + w_i(P_{k+1})] - [w_{i-1}(P_k) + w_{i-1}(P_{k+1})] \\
&= 1 + 2w_i(P_i) - 2w_{i-1}(P_{i-1}) \\
&= 1 + 2[e(x, y^{i+1}) - e(x, y^i)] \\
&= 3
\end{aligned}$$

since $P_i = (x, y^{i+1})$ for $1 \le i \le k$. Finally, we multiply $P_{i-1} \supset P_i$ by $E = P_{k+2}$. Let w be the prime divisor associated to P_{k+2}. Then by reciprocity, $w_i(P_{k+2}) = w(P_i) = w(x, y^{i+1})$ for all $1 \le i \le k$:

$$\begin{aligned}
\lambda(P_{i-1}E/P_iE) &= \lambda(P_{i-1}/P_i) + [(P_i \cdot E) - (P_{i-1} \cdot E)] \\
&= \lambda(P_{i-1}/P_i) + [(P_i \cdot P_{k+2}) - (P_{i-1} \cdot P_{k+2})] \\
&= 1 + [w_i(P_{k+2}) - w_{i-1}(P_{k+2})] \\
&= 1 + [w(P_i) - w(P_{i-1})] \\
&= 1 + [(2i+2) - (2i)] \\
&= 3
\end{aligned}$$

as in the proof of [2, Theorem 2.1] since $w(y) = w(m) = o(P_{k+2}) = 2$, $w(P_i) = w(x, y^{i+1}) = 2(i+1)$ for $1 \le i \le k$. This proves Claim 3.

Let us denote $C = P_k^2$, $D = P_k P_{k+1}$, and $E = P_{k+2}$. We have constructed all the successive v-ideals from m to $P_k E$ using Claim 1, Claim 2, Claim 3 as follows:

$$\begin{array}{rl} m & \supset P_1 \supset P_2 \supset \cdots \supset P_k \supset P_{k+1} \\ & \supset mP_k \supset mP_{k+1} \supset \cdots \supset C = P_kP_k \supset D = P_kP_{k+1} \supset E = P_{k+2} \\ & \supset mC \supset mD \supset mE \supset P_1C \supset P_1D \supset P_1E \supset \cdots \\ & \supset P_kC = P_k^3 \supset P_kD = P_k^2P_{k+1} \supset P_kE = P_kP_{k+2}. \end{array}$$

The v-values of ideals in the last row are $9k+9, 9k+10, 9k+11$ since P_k^2 is the conductor ideal by Claim 2. Let M be the v-successor of P_kP_{k+2}, i.e., $v(M) = 9k+12$. Then, M is either simple or a product of $P_i's$ for $0 \leq i \leq k+2$. Since $\mathfrak{m}C = \mathfrak{m}P_k^2 \supset M$, the order of M is 3, too. Therefore, M can be factored into $P_{k+2}P_i$ for $i \leq k+1$, or it is a product of three P_i's for $i \leq k+1$.

If the former, i.e., $P_{k+2}|M$, then $P_kP_{k+2} \supset M = P_iP_{k+2}$ for some $P_k \supset P_i$, hence $M = P_{k+2}P_{k+1}$. However,

$$\lambda(P_kP_{k+2}/P_{k+1}P_{k+2}) = 1 + w(P_{k+1}) - w(P_k) > 1$$

since $P_k \supset P_{k+1}$ are w-ideals, where w is the prime divisor associated to P_{k+2} of order 2. Therefore, $P_{k+2} \nmid M$.

For the latter, let us assume that $M = P_{k+1}P_iP_j$ for some $i, j \leq k$. Then, $v(M) = (3k+4)+(3i+3)+(3j+3) = 9k+12$ which implies that $6k+2 = 3(i+j)$, this is also a contradiction. Therefore, $P_{k+1} \nmid M$ either. Finally, suppose that $P_k|M$, i.e., $M = P_kL$ for some integrally closed ideal L. Since $P_kP_{k+2} \supset M$ are adjacent, $P_{k+2} \supset L$ are also adjacent v-ideals. Therefore, we show that $L = \mathfrak{m}P_k^2$ and $M = P_k(\mathfrak{m}P_k^2) = \mathfrak{m}P_k^3$ has order 4, contradiction. Therefore, $M = P_iP_jP_\ell$ for $i, j, \ell < k$. Therefore, $v(M) = 3(i+j+\ell)+9 = 9k+12$ implies that $3k+1 = i+j+\ell < 3k$, a contradiction.

Therefore, we conclude that M is simple of order 3 with $v(M) = 9k+12$, i.e., $M = P_{k+3}$ is the simple complete ideal associated to v adjacent to $P_{k+2}P_k$. The above sequence of v-ideals are the complete sequence of v-ideals from $\mathfrak{m}$ to $P = P_{k+3}$. We also have $\text{rank}(P) = k+3 = \lceil \frac{b+7}{3} \rceil$ since $b = 3k$. $\square$

3. The conductor ideal and the value semigroup $v(R)$ of a satellite simple valuation ideal P of order 3

In the previous section, we described the complete sequence of v-ideals associated to a simple integrally closed ideal P of order 3. In particular, we measure n_i, the number of nonmaximal simple v-ideals of order i for $i = 1, 2, 3$ in the case of when P is a satellite simple complete ideal of order 3.

In describing the v-ideals from $\mathfrak{m}$ to the smallest simple v-ideal P, we also found the factorization of the v-predecessor of P in terms of larger simple v-ideals.

Theorem 3.1. *Let P, v be as in Theorem 2.1. Let $v(y) = 3$, $v(x) = 3 + b_v$, where $\mathfrak{m} = (x, y)$. Let $b_v = 3k + i$ for $i = 0, 1, 2$. Then, the v-predecessor of P is*

(i) $P_{k+2}P_{k+1}$, *if* $b_v = 2$

(ii) $P_{k+2}P_k$, *otherwise.*

Proof. We denote b_v by b.

(i) If $b = 1$, then the v-predecessor of P is $\mathfrak{m}P_2$ since $k = 0$ and $P = P_3$ by Proposition 2.2.

(ii) If $b = 2$, then the v-predecessor of $P = P_{k+3} = P_3$ is $P_2P_1 = P_{k+2}P_{k+1}$ since $k = 0$ by Proposition 2.3.

(iii) If $b = 3k + 1$ for $k \geq 1$, we refer to the proof of Proposition 2.4.

(iv) If $b = 3k + 2$ for $k \geq 1$, then $P_{k+2}P_k$ is the v-predecessor of $P = P_{k+3}$ by Proposition 2.5.

(v) If $b = 3k$ for $k \geq 1$, then the v-predecessor of $P = P_{k+3}$ is $P_{k+2}P_k$ by Proposition 2.6. □

We also obtained the unique factorization of the conductor ideal C in the previous section. The conductor ideal of P (or v) is the v-ideal C such that for any successive v-ideals $J \supset J'$ smaller than C have $v(J') = v(J) + 1$. It is known that $C = L : m$ for the largest v-ideal L of order $o(P)$ [7, Theorem 2.2]. Using this we obtained the conductor ideal of v in Section 2.

Corollary 3.2. *Let P, v be as in Theorem 2.1. Let $v(y) = 3$, $v(x) = 3 + b_v$, where $\mathfrak{m} = (x, y)$. Let $b_v = 3k + i$ for $i = 0, 1, 2$. Then, the conductor ideal of P is as follows*:

(i) $C = P_kP_{k+1}, v(C) = 6k + 8$ *if* $b_v = 3k + 2$, $k \geq 0$

(ii) $C = P_kP_k, v(C) = 6k + 6$ *otherwise.*

Proof. Let $b_v = b$ and $b = 3k + i$ for $i = 0, 1, 2$. If $b = 3k$, we assume $k \geq 1$.

(i) If $b = 2$, then the conductor ideal is $\mathfrak{m}P_1 = P_kP_{k+1}$ of v-value $8 = 6k + 8$. It was shown that $\text{rank}(P) = 3$ in by Proposition 2.3. If $k \geq 1$, then it was shown that $C = P_kP_{k+1}$ of v-value $6k + 8$ by Proposition 2.5.

(ii) If $b = 1$, it was shown that the conductor ideal is $C = \mathfrak{m}^2 = P_kP_k$ with $v(C) = 6 = 6k + 6$ since $k = 0$ in Proposition 2.2. It was also shown that the conductor ideal of $P = P_{k+3}$ is P_kP_k with $v(C) = 6k + 6$ in Proposition 2.4, Proposition 2.6 for $k \geq 0$. □

We showed that there exists a unique simple v-ideal P_{k+2} of order 2, i.e., $n_2 = 1$ for a satellite simple complete ideal P in Theorem 2.1. In Corollary 3.2, we showed the factorization of the conductor ideal. Note that $\mathfrak{m}C$ is the largest v-ideal of order 3, we have $\lambda(C/\mathfrak{m}C) = 4$. Hence, we showed that the simple v-ideal P_{k+2} of order 2 is in between C and $\mathfrak{m}C$ as follows.

Corollary 3.3. *Let P, v be as in Theorem 2.1. Let $v(y) = 3$, $v(x) = 3 + b_v$, where $\mathfrak{m} = (x, y)$. Let $b_v = 3k + i$ for $i = 0, 1, 2$. Then, the successive v-ideals from C to $\mathfrak{m}C$ are as follows*:

(i) $C = P_kP_{k+1} \supset P_{k+2} \supset P_{k+1}^2 \supset \mathfrak{m}C$ *if* $b_v = 3k + 2$, $k \geq 0$

(ii) $C = P_kP_k \supset P_kP_{k+1} \supset P_{k+2} \supset \mathfrak{m}C$ *otherwise.*

Proof. Let us denote b_v by b. (i) If $b = 2$, then $C = \mathfrak{m}P_1 \supset P_2 \supset P_1^2 \supset \mathfrak{m}^2P_1$ are consecutive v-ideals of v-values $8, 9, 10, 11$ as in Proposition 2.2. We also showed that

$$C = P_kP_{k+1} \supset P_{k+2} \supset P_{k+1}^2 \supset \mathfrak{m}C$$

are the consecutive v-ideals of v-values $6k+8, 6k+9, 6k+10, 6k+11$ in Proposition 2.5 for $k \geq 1$ case as well.

(ii) If $b = 1$, then $k = 0$ and these are

$$C = \mathfrak{m}^2 \supset \mathfrak{m}P_1 \supset P_2 \supset \mathfrak{m}^3$$

are the such ideals of v-values $6k + 6, 6k + 7, 6k + 8, 6k + 9$ in Proposition 2.2. If $b = 3k + 1$ for $k \geq 1$, then

$$C = P_kP_k \supset P_kP_{k+1} \supset P_{k+2} \supset \mathfrak{m}C$$

are consecutive v-ideals from C whose v-values are $6k+6, 6k+7, 6k+8, 6k+9$ in Proposition 2.4. The same is true if $b = 3k$ for $k \geq 1$ as in Proposition 2.6. □

Corollary 3.4. *Let P, v be as in Theorem 2.1. Let $v(y) = 3$, $v(x) = 3 + b_v$, where $\mathfrak{m} = (x, y)$. Let $b_v = 3k + i$ for $i = 0, 1, 2$ and $k \geq 0$. Then, the value semigroup $v(R)$ is as follows*:

(i) $v(R) = \mathbf{N}\backslash\{3i+1, 3j+2 \mid 0 \leq i \leq k, 0 \leq j \leq 2k+1\}$ *if* $b_v = 3k+1$, $k \geq 0$,
(ii) $v(R) = \mathbf{N}\backslash\{3i+2, 3j+1 \mid 0 \leq i \leq k,\ 0 \leq j \leq 2k+2\}$ *if* $b_v = 3k+2$, $k \geq 0$,
(iii) $v(R) = \mathbf{N}\backslash\{3i + 1, 3j + 2 \mid 0 \leq i \leq k,\ 0 \leq j \leq 2k + 1\}$ *or*
$v(R) = \mathbf{N}\backslash\{3i + 2, 3j + 1 \mid 0 \leq i \leq k,\ 0 \leq j \leq 2k + 1\}$ *if* $b_v = 3k$, $k \geq 1$.

Proof. Let us denote b_v by b. We can prove (i) by Proposition 2.2 and Proposition 2.4. Note that $v(R) = \mathbf{N}\backslash\{1, 2, 5\}$ if $b = 1$. We can prove (ii) by Proposition 2.3 and Proposition 2.5. Note that $v(R) = \mathbf{N}\backslash\{1, 2, 4, 7\}$ if $b = 2$. We prove (iii) the proof of Proposition 2.6. In case of (iii), $v(R) = \mathbf{N}\backslash\{3i + 1\}_{0\leq i\leq k} \cup \{3j + 2\}_{0\leq j\leq 2k+1}$ if $v(P_{k+1}) = 3k + 4$. If $v(P_{k+1}) = 3k + 5$, then $v(R) = \mathbf{N}\backslash\{3i + 2\}_{0\leq i\leq k} \cup \{3j + 1\}_{0\leq j\leq 2k+1}$. □

References

[1] S. S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. **78** (1956), 70–99.

[2] J. Hong, H. Lee, and S. Noh, *Simple valuation ideals of order two in 2-dimensional regular local rings*, Commun. Korean Math. Soc. **20** (2005), no. 3, 427–436.

[3] M. A. Hoskin, *Zero-dimensional valuation ideals associated with plane curve branches*, Proc. London Math. Soc. **6** (1956), no. 3, 70–99.

[4] C. Huneke, *Integrally closed ideals in two–dimensional regular local rings*, Proc. Microprogram, in: Commutative Algebra, June 1987, MSRI Publication Series, Vol. 15, Springer-Verlag, New York, 1989, 325–337.

[5] C. Huneke and J. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra **115** (1988), 481–500.

[6] J. Lipman, *On complete ideals in regular local rings*, in: Algebraic Geometry and Commutative Algebra, Collected Papers in Honor of Massayoshi Nagata, Academic Press, New York, 1988, pp. 203–231.

[7] ———, *Adjoints and polars of simple complete ideals in two-dimensional regular local rings*, Bull. Soc. Math. Belgique **45** (1993), 223–244.

[8] ———, *Proximity inequalities for complete ideals in two-dimensional regular local rings*, Contemporary Math. **159** (1994), 293–306.

[9] S. Noh, *The value semigroups of prime divisors of the second kind on 2-dimensional regular local rings*, Trans. Amer. Math. Soc. **336** (1993), 607–619.

[10] ———, *Sequence of valuation ideals of prime divisors of the second kind in 2-dimensional regular local rings*, J. Algebra **158** (1993), 31–49.

[11] ———, *Adjacent integrally closed ideals in dimension two*, J. Pure and Applied Algebra **85** (1993), 163–184.

[12] ———, *Powers of simple complete ideals in two-dimensional reular local rings*, Comm. Algebra **23** (1995), no. 8, 3127–3143.

[13] ———, *Valuation ideals of order one in two-dimensional regular local rings*, Comm. Algebra **28** (2000), no. 2, 613–624.

[14] ———, *Valuation ideals of order two in 2-dimensional regular local rings*, Math. Nachr. **261-262** (2003) 123–140.

[15] P. Ribenboim, *The Theory of Classical Valuations*, Sringer-Verlag, New York, 1999.

[16] O. Zariski, *Polynomial ideals defined by infinitely near base points*, Amer. J. Math. **60** (1938) 151–204.

[17] O. Zariski and P. Samuel, *Commutative Algebra, Vol. 2*, D. Van Nostrand, Princeton, 1960.

Department of Mathematics Education
Ewha Womans University
Seoul 120-750, Korea
E-mail address: noh@ewha.ac.kr