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ON THE STRUCTURE OF THE GRADE THREE PERFECT IDEALS OF TYPE THREE

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Abstract. Buchsbaum and Eisenbud showed that every Gorenstein ideal of grade 3 is generated by the submaximal order pfaffians of an alternating matrix. In this paper, we describe a method for constructing a class of type 3, grade 3, perfect ideals which are not Gorenstein. We also prove that they are algebraically linked to an even type grade 3 almost complete intersection.

1. Introduction

An ideal I in a Noetherian local ring R with maximal ideal $\mathfrak{m}$ is perfect if the length of a maximal R-sequence contained in I, the grade of I, is the same as the projective dimension of R/I. When I is a perfect ideal of grade g, the type of I, denoted by type I, is the dimension of $R/\mathfrak{m}$-vector space $\operatorname{Ext}_R^g(R/\mathfrak{m}, R/I)$. Equivalently, if

$$\mathbb{F} : 0 \longrightarrow F_g \longrightarrow F_{g-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R$$

is the minimal free resolution of R/I, then $\operatorname{type} I = \operatorname{rank} F_g$. An ideal I of grade g is a complete intersection if I can be generated by a regular sequence $x_1, x_2, \ldots, x_g$. It is well-known that such an ideal is a perfect ideal of type 1. More generally, a perfect ideal is said to be Gorenstein if it has type 1. A perfect ideal of grade g is an almost complete intersection if it can be minimally generated by $g + 1$ elements.

For long years, many people have been studying the structure of classes of perfect ideals. Burch [4] proved a structure theorem for grade 2 perfect ideals. The Hilbert-Burch theorem asserts that every perfect ideal of grade 2 is generated by maximal minors of a certain matrix. Buchsbaum and Eisenbud [3] proved a structure theorem for Gorenstein ideals of grade 3 which says that every Gorenstein ideal of grade 3 is generated by the submaximal order pfaffians of a certain alternating matrix. They also showed a structure theorem for almost complete intersections of grade 3. The self duality and commutativity

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of the algebra structure on the minimal free resolutions of R/I, where I is a Gorenstein ideal, was exploited to understand the structure of the perfect ideal I. Brown [2] gave a structure theorem for a class of type 2, grade 3 perfect ideals and $\lambda(I) = \dim_k \Lambda_1^2 > 0$, where $\lambda(I)$ is the numerical invariant. The number $\lambda(I)$ was used by Kustin and Miller [8] to distinguish classes of grade 4 Gorenstein ideals I in terms of free resolutions of R/I. Sanchez [10] proved a structure theorem for a class of type 3, grade 3 perfect ideals and $\lambda(I) \geq 2$. In [6] we introduced a generalized alternating matrix to be a skew-symmetrizable matrix and gave a structure theorem for grade 4 complete intersection ideals. A generalized alternating matrix has been further elaborated to describe the structure theorem for a class of type 2, grade 3 perfect ideals I minimally generated by an odd number $n \geq 5$ elements [7]. The main purpose of this paper is to describe and produce a new class of type 3, grade 3 perfect ideals which is not Gorenstein.

In Section 2, we introduce useful properties of pfaffians, and review the theory of linkage. In Section 3, we give a new class of non-Gorenstein type 3, grade 3 perfect ideals minimally generated by the quotients of the submaximal order pfaffians of the alternating matrix induced by a skew-symmetrizable matrix. We also show that the ideals in this class are geometrically linked to an even type, grade 3 almost complete intersection.

2. Preliminaries

To describe the structure theorems mentioned in the introduction, we need some properties of an alternating matrix. An alternating matrix is a square skew-symmetric matrix whose diagonal entries are zero. Let $T = (t_{ij})$ be an $n \times n$ alternating matrix with entries in a commutative ring R. Then it turns out that the determinant of an alternating matrix T is a square of a homogeneous polynomial of degree $\frac{n}{2}$ in R and is zero when n is odd. The pfaffian of T is defined as the uniquely determined square root of the determinant of T and is denoted by $\mathrm{Pf}(T)$. We define $\mathrm{Pf}_s(T)$ to be an ideal generated by the sth order pfaffians of T. If $s < n$, we let $T(i_1, i_2, \ldots, i_s)$ denote the alternating submatrix of T obtained by deleting rows and columns $i_1, i_2, \ldots, i_s$ from T. Let $(i) = i_1, i_2, \ldots, i_s$ denote the index of integers. Let $\theta(i)$ denote the sign of permutation that rearranges (i) in increasing order. If (i) has a repeated index, then we set $\theta(i) = 0$. Let $\tau(i)$ be the sum of the entries of (i). Define

$$T_{(i)} = (-1)^{\tau(i)+1}\theta(i)\mathrm{Pf}\,(T(i_1, i_2, \ldots, i_s)). \tag{2.1}$$

If $s = n$, we let $T_{(i)} = (-1)^{\tau(i)+1}\theta(i)$ and if $s > n$, we let $T_{(i)} = 0$. Let $\mathbf{t} = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix}$ be the row vector of the pfaffians of T of order $n-1$ signed appropriately according to the conventions described above. Pfaffians can be developed along a row just like the determinants. There is a "Laplace expansion" for developing pfaffians in terms of ones of lower order.

Lemma 2.1 ([8]). *Let T be an $n \times n$ alternating matrix and j a fixed integer, $1 \leq j \leq n$. Then*

(1) $\mathrm{Pf}(T) = \sum_{i=1}^{n} t_{ij}T_{ij}$, *and*

(2) $\mathbf{t}T = 0$.

The following lemma follows from Lemma 2.1.

Lemma 2.2 ([10]). *Let T be an $n \times n$ alternating matrix. Let a, b, c, d, and e be distinct integers between 1 and n. Then*

(1) $\sum_{i=1}^{n} t_{ik}T_{iab} = -\delta_{ka}T_b + \delta_{kb}T_a$,

(2) $\sum_{i=1}^{n} t_{ik}T_{iabc} = \delta_{ka}T_{bc} - \delta_{kb}T_{ac} + \delta_{kc}T_{ab}$,

(3) $\sum_{i=1}^{n} t_{ik}T_{iabcd} = -\delta_{ka}T_{bcd} + \delta_{kb}T_{acd} - \delta_{kc}T_{abd} + \delta_{kd}T_{abc}$,

(4) $\sum_{i=1}^{n} t_{ik}T_{iabcde} = \delta_{ka}T_{bcde} - \delta_{kb}T_{acde} + \delta_{kc}T_{abde} - \delta_{kd}T_{abce} + \delta_{ke}T_{abcd}$,

where δ_{ij} is the Kronecker's delta.

The following lemma which is a direct consequence from Lemma 2.1 and Lemma 2.2 will be used in the sequel.

Lemma 2.3. *Let n be a positive integer and T an $n \times n$ alternating matrix. Assume that i, j, k, and l are elements of a set $\{1, 2, \ldots, n\}$. Then*

$$T_iT_{jkl} + T_jT_{ikl} + T_kT_{ijl} + T_lT_{ijk} = 0.$$

The Buchsbaum-Eisenbud structure theorem identifies every grade 3 Gorenstein ideal as the ideal $\mathrm{Pf}_{n-1}(T) = (T_1, T_2, \ldots, T_n)$ of a certain $n \times n$ alternating matrix T.

Theorem 2.4 ([3]). *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $n \geqslant 3$ be an odd integer. Let F be a free R-module with* rank $F = n$. *Let $f : F^* \to F$ be an alternating map whose image is contained in $\mathfrak{m}F$. Suppose that* $\mathrm{Pf}_{n-1}(f)$ *has grade* 3. *Then* $\mathrm{Pf}_{n-1}(f)$ *is a Gorenstein ideal minimally generated by n elements.*

(2) *Every grade* 3 *Gorenstein ideal arises as in this way.*

We notice that as in [3] or [9], in most cases, linkage is used in the case of perfect ideals in Gorenstein or Cohen-Macaulay local rings. However the result that we use here is true for perfect ideals in any commutative ring, as shown by Golod [5].

Definition 2.5. Let I and J be perfect ideals of grade g. An ideal I is linked to J, $I \sim J$ if there exists a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g \in I \cap J$ such that $J = (\mathbf{x}) : I$ and $I = (\mathbf{x}) : J$, and *geometrically* linked to J if $I \sim J$ and $I \cap J = (\mathbf{x})$.

A fundamental result is that linkage is a symmetric relation on the set of perfect ideals in a Noetherian ring R.

Theorem 2.6 ([9]). *Let R be a Noetherian ring. If I is a grade g perfect ideal and $\mathbf{x} = x_1, x_2, \ldots, x_g$ is a regular sequence in I, then $J = (\mathbf{x}) : I$ is a grade g perfect ideal and $I = (\mathbf{x}) : J$.*

An almost complete intersection of grade g is linked to a grade g Gorenstein ideal by a regular sequence $\mathbf{x}$.

Proposition 2.7 ([3]). *Let I and J be perfect ideals of the same grade g in a Noetherian local ring R and suppose that I is linked to J by a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$. Then*

(1) *If I is Gorenstein, then $J = (\mathbf{x}, w)$ for some $w \in R$.*
(2) *If J is minimally generated by $\mathbf{x}$ and w, then I is Gorenstein.*

Now we review the structure theorems for a class of $\lambda(I) > 0$, type 2, grade 3 perfect ideals I and for a class of $\lambda(I) \geq 2$, type 3, grade 3 perfect ideals I given by Brown [2] and Sanchez [10], respectively.

Let I be any ideal in a Noetherian local ring R. Let $(\mathbb{F}, d)$ be a minimal free resolution of R/I. Let C be the image of d_2 and K the submodule of C which is generated by the Koszul relations on the entries of d_1. We note that if I is minimally generated by $r_1, r_2, \ldots, r_n$, and $\{e_1, e_2, \ldots, e_n\}$ is a basis of F_1, then K is generated by the set $\{r_j e_i - r_i e_j \mid 1 \leq i < j \leq n\}$. Define

$$\lambda(I) = \dim_k (K + \mathfrak{m}C)/\mathfrak{m}C,$$

where $\mathfrak{m}$ is the maximal ideal of R and $k = R/\mathfrak{m}$. Since $\lambda(I)$ is the maximum number of minimal generators of K which can be chosen to be the part of a minimal basis for C, we see that $\lambda(I)$ is also the maximum number of Koszul relations which can appear as rows of a matrix for d_2. Brown gave a structure theorem for a class of $\lambda(I) > 0$, type 2, grade 3 perfect ideals I. The minimal free resolution $\mathbb{F}$ of R/I is described in [2].

Theorem 2.8 ([2]). *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $n > 4$ be an integer. Let I be a type 2, grade 3 perfect ideal minimally generated by n elements. If $\lambda(I) > 0$, then there is an $n \times n$ alternating matrix $T = (t_{ij})$ with $t_{12} = 0$ and $t_{ij} \in \mathfrak{m}$ such that*
(1) *if n is odd, then $I = (T_1, T_2, z_1 T_{12j} + z_2 T_j : 3 \leq j \leq n)$ for some $z_1, z_2 \in \mathfrak{m}$,*
(2) *if n is even, then $I = (\mathrm{Pf}(T), T_{12}, z_1 T_{1j} + z_2 T_{2j} : 3 \leq j \leq n)$ for some $z_1, z_2 \in \mathfrak{m}$.*

Sanchez gave a structure theorem for a class of $\lambda(I) \geq 2$, type 3, grade 3 perfect ideals I. The minimal free resolution $\mathbb{F}$ of R/I is described in [10].

Theorem 2.9 ([10]). *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let I be a type 3, grade 3 perfect ideal minimally generated by $n > 4$ elements. If $\lambda(I) \geq 2$, then there exists an $n \times n$ alternating matrix $T = (t_{ij})$ and a 2×3 matrix $X = (x_{ij})$ with $t_{ij}, x_{ij} \in \mathfrak{m}$ such that*

(1) *If $n > 3$ is odd, then either*

$$\begin{aligned} I =&(T_1, x_{11}T_2 + x_{12}T_3 + x_{13}T_{123}, x_{21}T_2 + x_{22}T_3 + x_{23}T_{123}, \\ &\Delta_3 T_j + \Delta_2 T_{12j} + \Delta_1 T_{13j} | 4 \leq j \leq n) \end{aligned}$$

or

$$\begin{aligned} I =&(T_{123}, x_{11}T_1 + x_{12}T_2 + x_{13}T_3, x_{21}T_1 + x_{22}T_2 + x_{23}T_3, \\ &\Delta_3 T_{12j} + \Delta_2 T_{13j} + \Delta_1 T_{23j} | 4 \leq j \leq n), \end{aligned}$$

where Δ_i is the determinant of the 2×2 submatrix of X obtained by deleting the ith column.

(2) *If $n > 3$ is even, then either*

$$\begin{aligned} I =&(\mathrm{Pf}(T), x_{11}T_{12} + x_{12}T_{13} + x_{13}T_{23}, x_{21}T_{12} + x_{22}T_{13} + x_{23}T_{23}, \\ &\Delta_3 T_{1j} + \Delta_2 T_{2j} + \Delta_1 T_{123j} | 4 \leq j \leq n) \end{aligned}$$

or

$$\begin{aligned} I =&(T_{12}, x_{11}\mathrm{Pf}(T) + x_{12}T_{13} + x_{13}T_{23}, x_{21}\mathrm{Pf}(T) + x_{22}T_{13} + x_{23}T_{23}, \\ &\Delta_3 T_{1j} + \Delta_2 T_{2j} + \Delta_1 T_{123j} | 4 \leq j \leq n). \end{aligned}$$

3. Perfect ideals of grade 3

In this section we construct a new class of non-Gorenstein type 3, grade 3 perfect ideals generated by certain quotients of the submaximal order pfaffians of the alternating matrix induced by some skew-symmetrizable matrix.

To this end, we begin this section with the definition of a generalized alternating matrix.

Definition 3.1. Let R be a commutative ring with identity. An $n \times n$ matrix X over R is said to be *generalized alternating* or *skew-symmetrizable* if there exist nonzero diagonal matrices $D' = \mathrm{diag}\{u_1, u_2, \ldots, u_n\}$ and $D = \mathrm{diag}\{w_1, w_2, \ldots, w_n\}$ with entries in R such that $D'XD$ is an alternating matrix.

We denote by GA_n the set of all $n \times n$ skew-symmetrizable matrices over R. Let X be an $n \times n$ skew-symmetrizable matrix. We denote by $\mathcal{A}(X)$ the alternating matrix induced by X as follows:

$$\mathcal{A}(X) = \begin{cases} X & \text{if } X \text{ is alternating,} \\ D'XD & \text{if } X \text{ is not alternating.} \end{cases} \tag{3.1}$$

The following lemma gives us relations between the determinants of the skew-symmetrizable matrices and the pfaffians of the alternating matrices induced by the skew-symmetrizable matrices.

Lemma 3.2. *Let X be an $n \times n$ skew-symmetrizable matrix and $\mathcal{A}(X)$ the alternating matrix $D'XD$ induced by X defined in* (3.1). *Let $\hat{u}_i = \prod_{j=1}^{n} u_j/u_i$ and $\hat{w}_i = \prod_{j=1}^{n} w_j/w_i$ for $i = 1, 2, \ldots, n$. Let $X(i)$ be the $(n-1) \times (n-1)$ submatrix of X obtained by deleting the ith row and the corresponding column of it. Then*

$$\hat{u}_i \hat{w}_i \det X(i) = \mathcal{A}(X)_i^2.$$

Proof. It follows from the basic properties of determinants of square matrices and pfaffians of alternating matrices. □

Example 3.3. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let n be an odd integer with $n > 3$. Let $Y = (y_{ij})$ be an $n \times n$ alternating matrix with $y_{12} = 0$ and entries in $\mathfrak{m}$ and A the submatrix of Y obtained by deleting the first two columns and the last $(n-2)$ rows of Y from Y. For two elements $v, w \in \mathfrak{m}$, we define the $n \times n$ skew-symmetrizable matrix G_1 by

$$G_1 = \left[\begin{array}{c|c} B & vA \\ \hline -A^t & Y(1,2) \end{array}\right], \quad \text{where} \quad B = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}, \tag{3.2}$$

and $Y(1,2)$ is the $(n-2) \times (n-2)$ alternating submatrix of Y obtained by removing the first, second rows and columns from Y. The alternating matrix $\mathcal{A}(G_1)$ is obtained by multiplying the first two columns of G_1 by v.

Let x_i be an element defined by

$$x_i = \mathcal{A}(G_1)_i/v \quad \text{for } i = 1, 2, 3, \ldots, n. \tag{3.3}$$

We define $\overline{\mathrm{Pf}_{n-1}(G_1)}$ to be the ideal generated by n elements x_i. The next theorem says that $\overline{\mathrm{Pf}_{n-1}(G_1)}$ characterizes a perfect ideal I of grade 3 satisfying the following properties : (1) I has type 2, (2) the number of generators for I is odd, and (3) $\lambda(I) > 0$.

Theorem 3.4 ([7]). *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let n be an odd integer with $n > 3$ and v, w elements in $\mathfrak{m}$. Let G_1 be the $n \times n$ skew-symmetrizable matrix defined in* (3.2). *Then*

(1) *if $I = \overline{\mathrm{Pf}_{n-1}(G_1)}$ is an ideal of grade* 3 *with $\lambda(I) > 0$, then I is a perfect ideal of type* 2.

(2) *Every perfect ideal of grade* 3, *type* 2, $\lambda(I) > 0$ *minimally generated by n elements arises as in the way of* (1).

Proof. See the proof of Theorem 4.3 [7]. □

Now we construct a skew-symmetrizable matrix which defines a type 3, grade 3 perfect ideal. Let $A = (a_{ij})$ and $Y = (y_{ij})$ be an $r \times 3$ matrix and an $r \times r$ alternating matrix with entries in $\mathfrak{m}$, respectively. Set F to be the $3 \times r$ matrix defined by

$$F = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r\,1} \\ -a_{12} & -a_{22} & \cdots & -a_{r\,2} \\ a_{13} & a_{23} & \cdots & a_{r\,3} \end{bmatrix}$$

with an even integer $r \geq 4$. Define an $(r+3) \times (r+3)$ skew-symmetrizable matrix G_2 by

$$G_2 = \left[\begin{array}{c|c} \mathbf{0} & \bar{F} \\ \hline -F^t & Y \end{array}\right], \quad \text{where } \bar{F} = \begin{bmatrix} va_{11} & va_{21} & \cdots & va_{r\,1} \\ -ua_{12} & -ua_{22} & \cdots & -ua_{r\,2} \\ uva_{13} & uva_{23} & \cdots & uva_{r\,3} \end{bmatrix}$$

with $u, v \in \mathfrak{m}$. The alternating matrix $\mathcal{A}(G_2)$ is obtained by multiplying the first column of G_2 by v, the second column of it by u, and the third column of it by uv. To describe non-Gorenstein perfect ideal of grade 3, we need the ideal $\overline{\mathrm{Pf}_{r+2}(G_2)}$ induced from the submaximal order pfaffians of $\mathcal{A}(G_2)$ as follows.

Definition 3.5. With notations as above, we let

$$\mathcal{A}(G_2) = \left[\begin{array}{c|c} \mathbf{0} & \bar{F} \\ \hline -\bar{F}^t & Y \end{array}\right]$$

be the alternating matrix induced by G_2. We define $\overline{\mathrm{Pf}_{r+2}(G_2)}$ to be the ideal generated by the quotients of the submaximal order pfaffians of $\mathcal{A}(G_2)$ by uv,

$$\overline{\mathrm{Pf}_{r+2}(G_2)} = (\mathcal{A}(G_2)_1/uv, \mathcal{A}(G_2)_2/uv, \ldots, \mathcal{A}(G_2)_{r+3}/uv).$$

Next, we prove our main theorem which gives a class of type 3, grade 3 perfect ideals algebraically linked to a type r, grade 3 almost complete intersection.

Theorem 3.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. With notations as above, let $x_1 = \mathcal{A}(G_2)_1/uv, x_2 = \mathcal{A}(G_2)_2/uv, x_3 = \mathcal{A}(G_2)_3/uv$. If $\mathbf{x} = x_1, x_2, x_3$ is a regular sequence in $\overline{\mathrm{Pf}_{r+2}(G_2)}$, then* (1) $(\mathbf{x}) : \overline{\mathrm{Pf}_{r+2}(G_2)}$ *is a type even r, grade* 3 *almost complete intersection and* (2) $\overline{\mathrm{Pf}_{r+2}(G_2)}$ *is a type* 3*, grade* 3 *perfect ideal.*

Proof. (1) Define an $(r+3) \times (r+3)$ alternating matrix T by

$$T = \widetilde{G}_2 = \left[\begin{array}{c|c} \mathbf{0} & F \\ \hline -F^t & Y \end{array}\right].$$

Then we observe that

$$\begin{aligned} &x_1 = uT_1, \;\; x_2 = vT_2, \;\; x_3 = T_3, \;\; x_k = \mathcal{A}(G_2)_k/uv = uvT_k \\ &\quad \text{for } \; k = 4, 5, \ldots, r+3. \end{aligned} \tag{3.4}$$

To see that $\mathrm{Pf}_{r+2}(T)$ is a Gorenstein ideal under the hypothesis of the theorem, we first note that by Lemma 2.3, $\overline{\mathrm{Pf}_{r+2}(G_2)}$ has grade 3. Next, since $\overline{\mathrm{Pf}_{r+2}(G_2)} \subset \mathrm{Pf}_{r+2}(T)$, $\mathrm{Pf}_{r+2}(T)$ has grade 3 by Lemma 2.3. Finally, since T is an alternating matrix, $\mathrm{Pf}_{r+2}(T)$ is a Gorenstein ideal by Theorem 2.4. Let $\mathbf{y} = y_1, y_2, y_3$ be a sequence of elements in R, where $y_1 = T_1, y_2 = T_2, y_3 = T_3$. Since $\mathbf{x}$ is a regular sequence, $\mathbf{y}$ is also regular sequence. By Theorem 2.6 and Proposition 2.7, $J = (\mathbf{y}) : \mathrm{Pf}_{r+2}(T)$ is a grade 3 perfect ideal, and J is an almost complete intersection. We claim that J has type even r and $J = (\mathbf{x}) : \overline{\mathrm{Pf}_{r+2}(G_2)}$. Direct computation gives us

$$\mathrm{Pf}(Y)T_k \in (\mathbf{y}) \quad \text{for } k = 4, 5, 6, \ldots, r+3. \tag{3.5}$$

Hence we have

$$J = (\mathbf{y}, \mathrm{Pf}(Y)).$$

Since u and v are contained in $\mathfrak{m}$, the fact that J has type r follows from the Bass result ([1] Proposition 2.9), which says that if J is a grade 3 perfect ideal algebraically linked to a perfect ideal $\mathrm{Pf}_{r+2}(T)$ by a regular sequence $\mathbf{y}$ in J, then the type of J is equal to the minimal number of generators for the canonical module $\mathrm{Ext}^3_R(R/J, R)$ and

$$\mathrm{Ext}^3_R(R/J, R) \cong (\mathbf{y}) : J/(\mathbf{y}) \cong \mathrm{Pf}_{r+2}(T)/(\mathbf{y}).$$

From (3.4) and (3.5), it is easy to see that

$$J = (\mathbf{x}) : \overline{\mathrm{Pf}_{r+2}(G_2)}.$$

(2) We will prove that $\overline{\mathrm{Pf}_{r+2}(G_2)}$ is a type 3, grade 3 perfect ideal. Since we have already shown that $\overline{\mathrm{Pf}_{r+2}(G_2)}$ is a grade 3 perfect ideal, it suffices to prove that the type of $\overline{\mathrm{Pf}_{r+2}(G_2)}$ is 3. From the Bass result and

$$\mathrm{Ext}^3_R(R/\overline{\mathrm{Pf}_{r+2}(G_2)}, R) \cong (\mathbf{x}) : \overline{\mathrm{Pf}_{r+2}(G_2)}/(\mathbf{x}) \cong J/(\mathbf{x}),$$

we have type $\overline{\mathrm{Pf}_{r+2}(G_2)} = 3$. □

The minimal free resolution $\mathbb{F}$ of $R/\overline{\mathrm{Pf}_{r+2}(G_2)}$ is given by

$$\mathbb{F} : 0 \longrightarrow R^3 \xrightarrow{f_3} R^{r+5} \xrightarrow{f_2} R^{r+3} \xrightarrow{f_1} R\ ,$$

where

$$f_1 = \begin{bmatrix} x_1 & x_2 & x_3 \cdots & x_{r+3} \end{bmatrix},\ f_2 = \begin{bmatrix} \mathbf{0} & \bar{F} & B \\ -F^t & Y & \mathbf{0} \end{bmatrix},\ f_3 = \begin{bmatrix} C \\ Q \\ N \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & x_3 \\ x_3 & 0 \\ -x_2 & -x_1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -\mathrm{Pf}(Y) & T_1 \\ \mathrm{Pf}(Y) & 0 & T_2 \\ 0 & 0 & T_3 \end{bmatrix},$$

$$Q = \begin{bmatrix} -q_{21} & q_{11} & T_4 \\ -q_{22} & q_{12} & T_5 \\ \vdots & \vdots & \vdots \\ -q_{2r} & q_{1r} & T_{r+3} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & u & 0 \\ v & 0 & 0 \end{bmatrix},$$

$$q_{ij} = (-1)^{i+1} \sum_{1\le k\le r} Y_{jk} a_{ki} \text{ for } i = 1, 2.$$

The following example illustrates Theorem 3.6.

Example 3.7. Let $R = \mathbb{C}[[x, y, z]]$ be the formal power series ring over the field $\mathbb{C}$ of the complex numbers with indeterminates x, y, z. Let A and Y be a 4×3 matrix and a 4×4 alternating matrix, respectively, given by

$$A = \begin{bmatrix} 0 & z & -y \\ -y & z & 0 \\ z & -x & 0 \\ 0 & 0 & -x \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & x & 0 & z \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ -z & 0 & -y & 0 \end{bmatrix}.$$

Then F is the 3×4 matrix given by

$$F = \begin{bmatrix} 0 & -y & z & 0 \\ -z & -z & x & 0 \\ -y & 0 & 0 & -x \end{bmatrix}.$$

We set $u = x$ and $v = y$ and define G_2 to be a 7×7 skew-symmetrizable matrix given by

$$G_2 = \left[\begin{array}{c|c} \mathbf{0} & \bar{F} \\ \hline -F^t & Y \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & -y^2 & yz & 0 \\ 0 & 0 & 0 & -xz & -xz & x^2 & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & 0 & -x^2y \\ 0 & z & y & 0 & x & 0 & z \\ y & z & 0 & -x & 0 & 0 & 0 \\ -z & -x & 0 & 0 & 0 & 0 & y \\ 0 & 0 & x & -z & 0 & -y & 0 \end{bmatrix}.$$

Define the 7×7 alternating matrix $T = \widetilde{G}_2$ by

$$T = \widetilde{G}_2 = \left[\begin{array}{c|c} \mathbf{0} & F \\ \hline -F^t & Y \end{array}\right].$$

Then we will show that $\overline{\mathrm{Pf}_6(G_2)} = (x^4 + xy^2z, -y^4 - x^2yz, xyz + y^2z - z^3, x^3y^2 - x^2yz^2, x^2yz^2, x^2y^2z, -x^2y^3 + xy^2z^2)$ is a type 3, grade 3 perfect ideal and linked to a type 4, grade 3 almost complete intersection by a regular sequence $\mathbf{x} = x^4+xy^2z, -y^4-x^2yz, -z^3+xyz+y^2z$. The minimal free resolution $\mathbb{F}$ of $R/\overline{\mathrm{Pf}_6(G_2)}$ has the form:

$$\mathbb{F} : 0 \longrightarrow R^3 \xrightarrow{f_3} R^9 \xrightarrow{f_2} R^7 \xrightarrow{f_1} R\ ,$$

where the map f_2 has the following matrix:

$$f_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -y^2 & yz & 0 & 0 & x_3 \\ 0 & 0 & 0 & -xz & -xz & x^2 & 0 & x_3 & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & 0 & -x^2y & -x_2 & -x_1 \\ 0 & z & y & 0 & x & 0 & z & 0 & 0 \\ y & z & 0 & -x & 0 & 0 & 0 & 0 & 0 \\ -z & -x & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & x & -z & 0 & -y & 0 & 0 & 0 \end{bmatrix}.$$

To describe f_1 and f_3 in concise form, we set

$$\begin{aligned} &x_1 = x^4 + xy^2z,\ \ x_2 = -y^4 - x^2yz,\ \ x_3 = xyz + y^2z - z^3, \\ &x_4 = x^3y^2 - x^2yz^2,\ \ x_5 = x^2yz^2,\ \ x_6 = x^2y^2z,\ \ x_7 = -x^2y^3 + xy^2z^2, \\ &T_1 = x^3 + y^2z,\ \ T_2 = -y^3 - x^2z,\ \ T_3 = xyz + y^2z - z^3,\ \ T_4 = x^2y - xz^2, \\ &T_5 = xz^2,\ \ T_6 = xyz,\ \ T_7 = -xy^2 + yz^2. \end{aligned}$$

Then we have

$$f_1 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix},$$

$$f_3 = \begin{bmatrix} 0 & -xy & 0 & -yz & xz+yz & z^2 & -x^2 & 0 & y \\ xy & 0 & 0 & y^2 & -z^2 & -yz & xz & x & 0 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & 0 & 0 \end{bmatrix}^t.$$

If K is the ideal generated by x_1, x_2, x_3, then the radical of K is equal to the maximal ideal $\mathfrak{m} = (x, y, z)$. Since $\mathfrak{m}$ has grade 3, $\mathbf{x} = x_1, x_2, x_3$ is a regular sequence and so $\overline{\mathrm{Pf}_6(G_2)}$ has grade 3 by Lemma 2.3. Now we will show that $(\mathbf{x}) : \overline{\mathrm{Pf}_6(G_2)}$ is a type 4, grade 3 almost complete intersection. Let $y_1 = T_1, y_2 = T_2, y_3 = T_3$. Since $\mathbf{x}$ is a regular sequence, $\mathbf{y} = y_1, y_2, y_3$ is a regular sequence. By Theorem 2.4 and Proposition 2.7, $I = (T_1, T_2, \ldots, T_7)$ is a grade 3 Gorenstein ideal, and $J = (\mathbf{y}) : I$ is a grade 3 almost complete intersection. Since $w = \mathrm{Pf}(Y) = -xy$, $J = (\mathbf{y}, w) = (-x^3 - y^2z, -y^3 - x^2z, -y^2z + z^3, -xy)$. As seen in the proof of Theorem 3.6, we have $J = (\mathbf{x}) : \overline{\mathrm{Pf}_6(G_2)}$. From the Bass result and

$$\begin{aligned} \mathrm{Ext}^3_R(R/J, R) &\cong (\mathbf{x}) : J/(\mathbf{x}) \cong \overline{\mathrm{Pf}_6(G_2)}/(\mathbf{x}) \\ &= (\mathbf{x}, xyT_4, xyT_5, xyT_6, xyT_7)/(\mathbf{x}), \end{aligned}$$

we have type $J = 4$. By the same method, $\overline{\mathrm{Pf}_6(G_2)}$ has type 3.

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