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WEIERSTRASS POINTS ON $\Gamma_0(p)$ AND ITS APPLICATION

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Abstract. In this note, we study arithmetic properties for the exponents of modular forms on $\Gamma_0(p)$ for primes p. Our aim is to refine the result of [4] by using the geometric property of the modular curve of $\Gamma_0(p)$.

1. Introduction and results

Let $\theta := \frac{1}{2\pi i}\frac{d}{dz}$. This operator is called as the Ramanujan theta operator and plays important roles in number theory. Let N be a positive integer. Suppose $f(z)$ are modular forms on $\Gamma_0(N)$. When $N = 1$, Bruinier, Kohnen, and Ono studied in [3] the images of modular forms under the Ramanujan theta operator. They also provided a relation between the infinite product expansion of a modular form and the values of a certain meromorphic modular function at points in the divisor of f. These results were extended to modular forms on the genus zero congruence subgroups in [1] and [5]. The author obtained in [4] analogues of these results for modular forms on $\Gamma_0(p)$, where p is a prime. In this note, our aim is to refine the result of [4] by using the geometric property of the modular curve of $\Gamma_0(p)$.

Let p be a prime and g be the genus of $\Gamma_0(p)$. Let $\mathbb{H}$ denote the complex upper half plane. A modular curve $X_0(p)$ is defined by

$$X_0(p) := \Gamma_0(p)\backslash\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}.$$

Note that ∞ is not a Weierstrass point on $X_0(p)$ whose genus is larger than 1. This implies that for each integer $m \geq g+1$ there exists a unique modular function $j_{p,m}(z) = q^{-m} + O\left(q^{-g}\right)$ that has its only pole at ∞ and a zero at the cusp 0 (see Section 2 for details). Let l_τ be the order of isotropic subgroup of $\Gamma_0(p)$ at $\tau \in \mathbb{H}$. The order of zero or pole of f at $\tau \in \mathbb{H}$ is denoted by $\nu_\tau^{(p)}(f)$ and has the form

$$\nu_\tau^{(p)}(f) = \frac{1}{l_\tau}\mathrm{ord}_\tau(f),$$

where $\mathrm{ord}_\tau(f)$ denotes the order of zero or pole of f at τ as a complex function on $\mathbb{H}$. With these notations, we state our first theorem.

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Theorem 1. *Suppose that* $f(z) := q^h \prod_{n=1}^{\infty}(1-q^n)^{c(n)}$ *is a normalized modular form of weight* k *on* $\Gamma_0(p)$ *and that the genus of* $\Gamma_0(p)$ *is larger than 1. Then*

$$\sum_{m\geq g+1} \sum_{\tau\in\mathcal{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m$$

is a meromorphic modular form of weight 2. Moreover,

$$f_\theta(z) := \frac{\theta f(z)}{f(z)} - \frac{k}{12}E_2(z) + \sum_{m\geq g+1} \sum_{\tau\in\mathcal{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m + \left(\frac{k}{12} - h\right)\frac{1}{p-1}(pE_2(pz) - E_2(z))$$

is a cusp form of weight 2 on $\Gamma_0(p)$.

Remark 1.1. The main difference of our result from [4] is the definition of $j_{p,m}$. In [4], $j_{p,m}$ is defined by the sum of eta-quotients. Following the definition of $j_{p,m}$ in [4], we have that

$$\sum_{m\geq g+1} \sum_{\tau\in\mathcal{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) q^m$$

is not a modular form in general.

Let K be the number field and $\mathcal{O}_K$ denote the ring of integers in K. Using Theorem 1, we have the following congruence for the exponents of modular forms.

Theorem 2. *Let* $f(z) := q^h \prod_{n=1}^{\infty}(1-q^n)^{c(n)} \in \mathcal{O}_K[[q]]$ *be a normalized modular form of weight* k *on* $\Gamma_0(p)$ *and* $\beta \nmid (p-1)$ *denote a prime ideal of* $\mathcal{O}_K$. *Suppose that* f_θ *is* β*-integral, and that* s *is a positive integer, and that the genus of* $\Gamma_0(p)$ *is larger than 1. Then for almost all* m *coprime to* p

$$\sum_{d|m} d\cdot c(d) \equiv \sum_{\tau\in\mathcal{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) + \frac{2pk-24h}{p-1}\sigma_1(m) + \frac{24h-2k}{p-1}p\sigma_1(m/p) \pmod{\beta^s},$$

where $\sigma_k(n) := \sum_{d|n} d^k$.

Remark 1.2. In Theorem 2, we mean "almost all" in the sense of density (i.e.,

$$x \sim \sharp\{0\leq m\leq x \mid \sum_{d|m} d\cdot c(d) \equiv \sum_{\tau\in\mathcal{H}_p} \nu_\tau^{(p)}(f(z)) j_{p,m}(\tau) + \frac{2pk-24h}{p-1}\sigma_1(m) + \frac{24h-2k}{p-1}p\sigma_1(m/p) \pmod{\beta^s}\}).$$

Remark 1.3. Our method gives no information on the first g coefficients of $\frac{\theta f}{f}$. Thus, from the argument of this note we can not obtain an analogue for the recursive relations of the Fourier coefficients of modular forms in [3], [1] and [5].

2. Prerequisites

Suppose that p is a prime. The group $\Gamma_0(p)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ defined as

$$\Gamma_0(p) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \ (\mathrm{mod}\ p) \right\}.$$

Let Γ denote $SL_2(\mathbb{Z})$ and $\mathcal{F}_p$ be a fundamental domain for the action of $\Gamma_0(p)$ on $\mathbb{H}$. We denote the set of distinct cusps as S_p,

$$S_p = \{0, \infty\}.$$

From now on, we suppose that if t is a cusp point, then t is in S_p. The period of q-expansion at t is denoted by p_t, where p_t is given by the following way:

$$p_t = 1 \text{ if } t = \infty \text{ and } p_t = p \text{ if } t = 0.$$

Adjoining the cusps to $\Gamma_0(p)\backslash\mathbb{H}$, we obtain a compact Riemann surface $X_0(p)$. For $\tau \in \mathbb{H} \cup S_p$, let Q_τ be the image of τ by the canonical map from $\mathbb{H} \cup S_p$ to $X_0(p)$.

Suppose G is a meromorphic modular form of weight 2 on $\Gamma_0(p)$. The residue of G at Q_τ on $X_0(p)$, denoted by $\mathrm{Res}_{Q_\tau} G dz$, is well defined since we have the canonical correspondence between a meromorphic modular form of weight 2 on $\Gamma_0(p)$ and a meromorphic 1-form of $X_0(p)$. If $\mathrm{Res}_\tau G$ denotes the residue of G at τ on $\mathbb{H}$, then for $\tau \in \mathbb{H}$ we obtain

$$\mathrm{Res}_{Q_\tau} G dz = \frac{1}{l_\tau} \mathrm{Res}_\tau G.$$

Here, l_τ is the order of isotropy group at τ. Especially, if f is a meromorphic modular form of weight k on $\Gamma_0(p)$ and $G = \frac{\theta f}{f}$, then the residue of G at Q_τ on $\tau \in \mathbb{H}$ is computed with the order of zero or pole of f at $\tau \in \mathbb{H}$. The order of zero or pole of f at $\tau \in \mathbb{H}$ is denoted by $\nu_\tau^{(p)}(f)$ and has the form

$$\nu_\tau^{(p)}(f) = \frac{1}{l_\tau} \mathrm{ord}_\tau(f),$$

where $\mathrm{ord}_\tau(f)$ denotes the order of zero or pole of f at τ as a complex function on $\mathbb{H}$. Then we have

$$2\pi i \cdot \mathrm{Res}_{Q_\tau} \frac{\theta f}{f} = \nu_\tau^{(p)}(f). \tag{2.1}$$

We introduce some notations to formulate $\mathrm{Res}_{Q_t} G dz$ at every cusp t. First, recall the usual slash operator $f(z)|_k\, \gamma$ given as

$$f(z)|_k\, \gamma = \det(\gamma)^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$ and γz denotes $\frac{az+b}{cz+d}$. From now on, q denotes $e^{2\pi i z}$. We define a matrix $\gamma_t^{(p)}$ as the following way;

$$\gamma_t^{(p)} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \qquad \text{if } t = 0,$$

$$\gamma_t^{(p)} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \qquad \text{if } t = \infty.$$

If G has the Fourier expansion of the form at each cusps

$$G(z) \mid_2 \gamma_t^{(p)} = \sum_{n=m_t}^{\infty} a_t(n) q^n \text{ at } \infty,$$

then we have

$$\operatorname{Res}_{Q_t} G dz = \frac{a_t(0)}{2\pi i} \text{ for } t \in S_p \,. \tag{2.2}$$

Now, we recall the definition of Weierstrass point. Let X be a compact Riemann surface with the genus g. At a given point P of a Riemann surface X, genus g, we say that m is a gap if no function exists with a pole of order m at P and regular elsewhere on X. It is known that there are just g gaps at P, and that these satisfy $1 \leq m \leq 2g-1$; moreover except for finitely many P, the gaps are just the integers 1 to g. Those exceptional P for which this is not so are called Weierstrass points of X. It is known that the point ∞ on $X_0(p)$ is not a Weierstrass point (see [8]). So, for each integer $m \geq g+1$ there exists a modular function on $\Gamma_0(p)$ such that $\operatorname{ord}_\infty(F_m(z)) = -m$ and that $F_m(z)$ is holomorphic elsewhere on $X_0(p)$. Using $F_j(z)$ for $g+1 \leq j \leq m$, we can construct a modular function $G_m(z)$ on $X_0(p)$ satisfying the followings:

- $G_m(z) = q^{-m} + O(q^{-g})$,
- $\operatorname{ord}_0(G_m(z)) \geq 1$,
- $G_m(z)$ is holomorphic on $X_0(p)$ except ∞.

Moreover, $G_m(z)$ is uniquely determined by its properties.

3. Proofs

We begin by stating a lemma which was proved by Eholzer and Skoruppa in [6].

Lemma 3.1. *Suppose that $f = \sum_{n=h}^{\infty} a(n) q^n$ is a meromorphic modular function in a neighborhood of $q = 0$ and that $a(h) = 1$. Then there are uniquely determined complex number $c(n)$ such that*

$$f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. Moreover, the following identity is true

$$\frac{\theta f}{f} = h - \sum_{n \geq 1} \sum_{d \mid n} c(d) d q^n.$$

Proof of Theorem 1. Let

$$F(z) = \frac{\theta f(z)}{f(z)} - \frac{k}{12} E_2(z) + \left(\frac{k}{12} - h \right) \frac{1}{p-1} (pE_2(pz) - E_2(z)).$$

Here, $E_2(z)$ is the usual normalized Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n.$$

The function $F(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(p)$ and has the q-expansion of the form

$$F(z) = \sum_{n=1}^{\infty} a_F(n) q^n.$$

Since ∞ is not a Weierstrass point on $X_0(p)$ for each integer v, $1 \leq v \leq g$, there exits a cusp form $w(z)$ of weight 2 such that $\mathrm{ord}_\infty(w) = v$ (see [7] or [2]). So, we can choose a cusp form $g(z) := \sum_{n=1}^{\infty} a_g(n)$ of weight 2 such that $a_g(n) = a_F(n)$ for $1 \leq n \leq g$.

Let $F'(z) = F(z) - g(z)$. Its Fourier expansion at $t \in S_N$ is given by

$$\begin{aligned} F'(z) \mid_2 \gamma_t^{(p)} &= \left(\frac{\theta f(z)}{f(z)}\right)\bigg|_2 \gamma_t^{(p)} + \frac{p(k-12h)}{12(p-1)} \, E_2(pz)|_2 \, \gamma_t^{(p)} \\ &\qquad - \frac{pk-12h}{12(p-1)} \, E_2(z)|_2 \, \gamma_t^{(p)} - g(z)|_2 \gamma_t^{(p)} \\ &= \frac{\theta(f|_k \gamma_t^{(p)})}{f|_k \gamma_t^{(p)}} + \frac{p(k-12h)}{12(p-1)} \, E_2(pz)|_2 \, \gamma_t^{(p)} \\ &\qquad - \frac{pk-12h}{12(p-1)} \, E_2(z)|_2 \, \gamma_t^{(p)} - g(z)|_2 \gamma_t^{(p)}. \end{aligned}$$

Since $F'(z) j_{p,m}(z) dz$ is a meromorphic 1-form on $X_0(p)$, we obtain from (2.2) that

$$\begin{aligned} &2\pi i \mathrm{Res}_{Q_\infty} F'(z) j_{p,m}(z) dz \\ &= \, - a_g(m) - \left(\sum_{d|m} c_t(d) d\right) + \frac{2pk - 24h}{p-1} \sigma_1(m) + \frac{24h - 2k}{p-1} p \sigma_1(m/p), \end{aligned}$$

and that $2\pi i \mathrm{Res}_{Q_0} F'(z) j_{p,m}(z) dz = 0$ since $\mathrm{ord}_0(j_{p,m}(z)) \geq 1$ and $F'(z)$ is holomorphic at 0. Next we compute $\mathrm{Res}_{Q_\tau} F'(z) j_{p,m}(z) dz$ for $\tau \in \mathbb{H}$. For each $\tau \in \mathbb{H}$, we obtain that from (2.1)

$$2\pi i \mathrm{Res}_{Q_\tau} F'(z) j_{p,m}(z) dz = 2\pi i \frac{1}{l_\tau} \mathrm{Res}_\tau \frac{\theta f(z)}{f(z)} j_{p,m}(z) = \nu_\tau^{(N)}(f) j_{p,m}(z)$$

since $E_2(z)$and $j_{p,m}(z)$ are holomorphic on $\mathbb{H}$.

The residue theorem implies that

$$2\pi i \sum_{Q_\tau \in X_0(N)} \mathrm{Res}_{Q_\tau} F'(z) j_{p,m}(z) dz = 0$$

since $X_0(N)$ is a compact Riemann surface. Thus, we have

$$\begin{aligned}
&-\sum_{m\geq g+1}\sum_{\tau\in\mathcal{H}_p}\nu_\tau^{(p)}(f(z))j_{p,m}(\tau)q^m\\
&=F'(z)=F(z)-g(z)\\
&=\frac{\theta f(z)}{f(z)}-g(z)+\frac{-kp+12h}{12(p-1)}E_2(z)+\frac{-12h+k}{12(p-1)}pE_2(pz).
\end{aligned}$$

Therefore, this completes the proof. □

To prove Theorem 2 we need the following proposition.

Proposition 3.2 (Serre [9], Corollaire du Théorème 1)**.** *Let*

$$f(z)=\sum_{n=0}^{\infty}c_f(n)q^{n/M},\ M\geq 1$$

be a modular form of integral weight $k\geq 1$ on a congruence subgroup of $SL_2(\mathbb{Z})$, and suppose that the coefficients $c_f(n)$ lie in the ring of integers of an algebraic number field K. Then for any integer $m\geq 1$,

$$c_f(n)\equiv 0 \pmod m$$

for almost all n.

Proof of Theorem 2. Theorem 1 implies that

$$\begin{aligned}
g(z):=&\sum_{m\geq g+1}\sum_{\tau\in\mathcal{H}_p}\nu_\tau^{(p)}(f(z))j_{p,m}(\tau)q^m+\frac{\theta f(z)}{f(z)}\\
&+\frac{-kp+12h}{12(p-1)}E_2(z)+\frac{-12h+k}{12(p-1)}pE_2(pz)
\end{aligned}$$

is a cusp form. From the assumption the coefficients of $g(z)$ are β-integral. Applying Proposition 3.2 to $g(z)$, we complete the proof. □

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