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ϵ-FUZZY CONGRUENCES ON SEMIGROUPS

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Abstract. We define an ϵ-fuzzy congruence, which is a weakened fuzzy congruence, find the ϵ-fuzzy congruence generated by the union of two ϵ-fuzzy congruences on a semigroup, and characterize the ϵ-fuzzy congruences generated by fuzzy relations on semigroups. We also show that the collection of all ϵ-fuzzy congruences on a semigroup is a complete lattice and that the collection of ϵ-fuzzy congruences under some conditions is a modular lattice.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([8]). Subsequently, Goguen ([1]) and Sanchez ([6]) studied fuzzy relations in various contexts. In [4] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([3]) developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. Samhan ([5]) characterized the fuzzy congruences generated by fuzzy relations on a semigroup and studied the lattice of fuzzy congruences on a semigroup. Also Gupta et al. ([2]) proposed a generalized definition of a fuzzy equivalence relation on a set, which is called a G-fuzzy equivalence relation, and developed some properties of that relation. The standard definition of a reflexive fuzzy relation μ on a set X, which Murali ([3]), Nemitz ([4]), and Samhan ([5]) used in their papers, is $\mu(x,x) = 1$ for all $x \in X$. Yeh ([7]) weakened the standard reflexive fuzzy relation to $\mu(x,x) \geq \epsilon > 0$ for all $x \in X$, which is called an ϵ-reflexive fuzzy relation.

We define an ϵ-fuzzy congruence, which is a weakened fuzzy congruence based on the ϵ-refexive fuzzy relation, and characterize the generated ϵ-fuzzy congruences on semigroups and some lattice properties of ϵ-fuzzy congruences. In Section 2 we review some basic definitions and properties of fuzzy relations and ϵ-fuzzy congruences. In Section 3 we find the ϵ-fuzzy congruence generated

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by the union of two ϵ-fuzzy congruences on a semigroup, find the ϵ-fuzzy congruence generated by a fuzzy relation which is fuzzy left and right compatible on a semigroup, and find the ϵ-fuzzy congruence generated by a fuzzy relation on a semigroup. In Section 4 we show that the set $C(S)$ of all ϵ-fuzzy congruences on a semigroup is a complete lattice and every sublattice H of the lattice $\{\mu \in C(S) : \mu(x,y) \leq \epsilon \text{ for } x,y \in S \text{ and } x \neq y\}$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$ is modular.

2. Preliminaries

In this section we recall some basic definitions and properties of fuzzy relations and ϵ-fuzzy congruences which will be used in the next section.

Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a reflexive fuzzy relation μ in a set X demands $\mu(x,x) = 1$ for all $x \in X$. Yeh ([7]) weakened this definition as follows.

Definition 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *ϵ-reflexive* in X if $\mu(x,x) \geq \epsilon > 0$ for all $x \in X$. μ is *symmetric* in X if $\mu(x,y) = \mu(y,x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x,y) = \underset{z \in X}{\rightarrow} \sup \min(\lambda(x,z), \mu(z,y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called *ϵ-fuzzy equivalence relation* if μ is ϵ-reflexive, symmetric, and transitive.

Example. Let $X = \{x, y, z\}$ be a set. Let ν be a fuzzy relation in X such that $\nu(x,x) = \nu(y,y) = \nu(z,z) = 1,\ \nu(x,y) = \nu(y,x) = 0.2,\ \nu(x,z) = \nu(z,x) = 0.1$, and $\nu(y,z) = \nu(z,y) = 0.1$. Then ν is a fuzzy equivalence relation in X. Let μ be a fuzzy relation in X such that $\mu(x,x) = 0.5,\ \mu(y,y) = 0.3,\ \mu(z,z) = 0.7,\ \mu(x,y) = \mu(y,x) = 0.2,\ \mu(x,z) = \mu(z,x) = 0.1$, and $\mu(y,z) = \mu(z,y) = 0.1$. Then μ is an ϵ-fuzzy equivalence relation. That is, μ need not to be $\mu(x,x) = \mu(y,y) = \mu(z,z) = 1$.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then it is easy to see that the composition $\circ$ is associative and $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$.

Definition 2.3. Let μ be a fuzzy relation in a set X. μ is called *fuzzy left* (*right*) *compatible* if $\mu(x,y) \leq \mu(zx, zy)$ ($\mu(x,y) \leq \mu(xz, yz)$) for all $x, y, z \in X$. An ϵ-fuzzy equivalence relation on X is called an *ϵ-fuzzy left congruence* (*right congruence*) if it is fuzzy left compatible (right compatible). An ϵ-fuzzy equivalence relation on X is an *ϵ-fuzzy congruence* if it is an ϵ-fuzzy left and right congruence.

Definition 2.4. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y)=\mu(y,x)$.

It is easy to see that $(\mu\circ\nu)^{-1}=\nu^{-1}\circ\mu^{-1}$ for fuzzy relations μ and ν.

Proposition 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty}\mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n=\mu\circ\mu\circ\cdots\circ\mu$.*

Proof. See Proposition 2.3 of [5]. □

Proposition 2.6. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty}\mu^n$, where $\mu^n=\mu\circ\mu\circ\cdots\circ\mu$.*

Proof. See Proposition 2.4 of [5]. □

Proposition 2.7. *If μ is a fuzzy relation on a semigroup S that is fuzzy left and right compatible, then so is $\cup_{n=1}^{\infty}\mu^n$, where $\mu^n=\mu\circ\mu\circ\cdots\circ\mu$.*

Proof. See Proposition 3.6 of [5]. □

Proposition 2.8. *Let μ be a fuzzy relation on a set S. If μ is ϵ-reflexive, then so is $\cup_{n=1}^{\infty}\mu^n$, where $\mu^n=\mu\circ\mu\circ\cdots\circ\mu$.*

Proof. Clearly μ is ϵ-reflexive. Suppose μ^k is ϵ-reflexive. Then $\mu^{k+1}(x,x)=(\mu^k\circ\mu)(x,x)=\underset{z\in X}{\rightarrow}\sup\min[\mu^k(x,z),\ \mu(z,x)]\geq\min[\mu^k(x,x),\ \mu(x,x)]\geq\epsilon>0$. By the mathematical induction, μ^n is ϵ-reflexive for all natural numbers n. Thus $[\cup_{n=1}^{\infty}\mu^n](x,x)=\sup[\mu(x,x),\ (\mu\circ\mu)(x,x),\dots]\geq\epsilon>0$. Hence $\cup_{n=1}^{\infty}\mu^n$ is ϵ-reflexive. □

Proposition 2.9. *Let μ and each ν_i be fuzzy relations in a set X for all $i\in I$. Then $\mu\circ(\underset{i\in I}{\rightarrow}\cap\nu_i)\subseteq\underset{i\in I}{\rightarrow}\cap(\mu\circ\nu_i)$ and $(\underset{i\in I}{\rightarrow}\cap\nu_i)\circ\mu\subseteq\underset{i\in I}{\rightarrow}\cap(\nu_i\circ\mu)$.*

Proof. Straightforward. □

3. ϵ-fuzzy congruences on semigroups

In this section we characterize the generated ϵ-fuzzy congruences on semigroups.

Proposition 3.1. *Let μ and ν be ϵ-fuzzy congruences in a set X. Then $\mu\cap\nu$ is an ϵ-fuzzy congruence.*

Proof. It is clear that $\mu\cap\nu$ is ϵ-reflexive and symmetric. By Proposition 2.9, $[(\mu\cap\nu)\circ(\mu\cap\nu)]\subseteq[\mu\circ(\mu\cap\nu)]\cap[\nu\circ(\mu\cap\nu)]\subseteq[(\mu\circ\mu)\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap(\nu\circ\nu)]\subseteq[\mu\cap(\mu\circ\nu)]\cap[(\nu\circ\mu)\cap\nu]\subseteq\mu\cap\nu$. That is, $\mu\cap\nu$ is transitive. Clearly $\mu\cap\nu$ is fuzzy left and right compatible. Thus $\mu\cap\nu$ is an ϵ-fuzzy congruence. □

Example. Let $X=\{x,y,z\}$ be a semigroup such that $xy=y,\ yx=x,\ xz=z,\ zx=x,\ yz=z,\ zy=y$. Let μ and ν be fuzzy relations in X such that $\mu(x,x)=\mu(y,y)=\mu(z,z)=0.8,\ \mu(x,y)=\mu(y,x)=0.4,\ \mu(x,z)=\mu(z,x)=0.2,\ \mu(y,z)=\mu(z,y)=0.2,\ \nu(x,x)=\nu(y,y)=\nu(z,z)=0.6,\ \nu(x,y)=$

$\nu(y,x) = 0.3$, $\nu(x,z) = \nu(z,x) = 0.3$, $\nu(y,z) = \nu(z,y) = 0.5$. Then $(\mu \cup \nu)(x,x) = (\mu \cup \nu)(y,y) = (\mu \cup \nu)(z,z) = 0.8$, $(\mu \cup \nu)(x,y) = (\mu \cup \nu)(y,x) = 0.4$, $(\mu \cup \nu)(x,z) = (\mu \cup \nu)(z,x) = 0.3$, and $(\mu \cup \nu)(y,z) = (\mu \cup \nu)(z,y) = 0.5$. It is easily checked that μ and ν are ϵ-fuzzy congruences on X, but $\mu \cup \nu$ is not an ϵ-fuzzy congruence on X.

Even though μ and ν are ϵ-fuzzy congruences, $\mu \cup \nu$ is not necessarily an ϵ-fuzzy congruence as shown in the above example. We find the ϵ-fuzzy congruence generated by $\mu \cup \nu$ on a semigroup in the following proposition.

Proposition 3.2. *Let μ and ν be ϵ-fuzzy congruences on a semigroup S. Then the ϵ-fuzzy congruence generated by $\mu \cup \nu$ in S is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \cdots$.*

Proof. Clearly $(\mu \cup \nu)(x,x) \geq \epsilon > 0$. That is, $\mu \cup \nu$ is ϵ-reflexive. By Proposition 2.8, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is ϵ-reflexive. Clearly $\mu \cup \nu$ is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is transitive. Hence $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is an ϵ-fuzzy equivalence relation containing $\mu \cup \nu$. It is straightforward to see that $\mu \cup \nu$ is fuzzy left and right compatible. By Proposition 2.7, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is an ϵ-fuzzy congruence containing $\mu \cup \nu$. Let λ be an ϵ-fuzzy congruence in S containing $\mu \cup \nu$. Then $\cup_{n=1}^{\infty}(\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty}\lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \cdots \subseteq \lambda \cup \lambda \cup \cdots = \lambda$. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is the ϵ-fuzzy congruence generated by $\mu \cup \nu$. $\square$

We now turn to the characterization of the ϵ-fuzzy congruences generated by fuzzy relations on semigroups.

Theorem 3.3. *Let μ be a fuzzy relation on a semigroup S such that μ is fuzzy left and right compatible. Then the ϵ-fuzzy congruence generated by μ in S is $\cup_{n=1}^{\infty}(\mu \cup \mu^{-1} \cup \theta)^n$, where θ is a fuzzy relation in S such that $\theta(a,a) = \epsilon$ for all $a \in S$ and $\theta(x,y) = \theta(y,x) \leq \min[\mu(x,y), \mu(y,x)]$ for all $x, y \in S$ with $x \neq y$.*

Proof. $(\mu \cup \mu^{-1} \cup \theta)(a,a) \geq \theta(a,a) = \epsilon > 0$ for all $a \in S$. Thus $\mu \cup \mu^{-1} \cup \theta$ is ϵ-reflexive. Let $\mu_1 = \mu \cup \mu^{-1} \cup \theta$. By Proposition 2.8, $\cup_{n=1}^{\infty}\mu_1^n$ is ϵ-reflexive. $\mu_1(x,y) = (\mu \cup \mu^{-1} \cup \theta)(x,y) = \max[\mu(x,y), \mu^{-1}(x,y), \theta(x,y)] = \max[\mu^{-1}(y,x), \mu(y,x), \theta(y,x)] = (\mu \cup \mu^{-1} \cup \theta)(y,x) = \mu_1(y,x)$. Thus μ_1 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty}\mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}\mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty}\mu_1^n$ is an ϵ-fuzzy equivalence relation containing μ. Since $\theta(x,y) \leq \mu(x,y) \leq \mu(zx,zy)$, $\mu_1(x,y) = (\mu \cup \mu^{-1} \cup \theta)(x,y) = \max[\mu(x,y), \mu^{-1}(x,y), \theta(x,y)] = \max[\mu(x,y), \mu(y,x), \theta(x,y)] \leq \max[\mu(zx,zy), \mu(zy,zx), \theta(zx,zy)] = \max[\mu(zx,zy), \mu^{-1}(zx,zy), \theta(zx,zy)] = (\mu \cup \mu^{-1} \cup \theta)(zx,zy) = \mu_1(zx,zy)$ for all $x, y, z \in S$ such that $x \neq y$. Since $\theta(a,a) = \epsilon$ for all $a \in S$, $\mu_1(x,x) = (\mu \cup \mu^{-1} \cup \theta)(x,x) = \max[\mu(x,x), \theta(x,x)] \leq \max[\mu(zx,zx), \theta(zx,zx)] = (\mu \cup \mu^{-1} \cup \theta)(zx,zx) = \mu_1(zx,zx)$ for all $x, z \in S$. Thus μ_1 is fuzzy left compatible. Similarly we may show μ_1 is fuzzy right compatible. By

Proposition 2.7, $\cup_{n=1}^{\infty}\ \mu_1^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty}\ \mu_1^n$ is an ϵ-fuzzy congruence containing μ. Let ν be an ϵ-fuzzy congruence containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$ for all $x,y \in S$ such that $x \neq y$. That is, $\nu(x,y) \geq (\mu \cup \mu^{-1} \cup \theta)(x,y)$ for all $x,y \in S$ such that $x \neq y$. $\nu(a,a) \geq \mu(a,a) = \mu^{-1}(a,a)$ for all $a \in S$. Since $\theta(a,a) = \epsilon$ and $\nu(a,a) \geq \epsilon$ for all $a \in S$, $\theta(a,a) \leq \nu(a,a)$. That is, $\mu_1(a,a) \leq \nu(a,a)$ for all $a \in S$. Thus $\mu_1 = (\mu \cup \mu^{-1} \cup \theta) \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(x,y) = (\mu_1^k \circ \mu_1)(x,y) = \underset{z\in S}{\rightarrow} \sup\min[\mu_1^k(x,z), \mu_1(z,y)] \leq \underset{z\in S}{\rightarrow} \sup\min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu \circ \nu \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for $n = 1, 2, \ldots$. Hence $\cup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \cdots \subseteq \nu$. □

We need the following definition and proposition to find the ϵ-fuzzy congruence generated by a fuzzy relation on a semigroup.

Definition 3.4. Let μ be a fuzzy relation on a semigroup S and let $S^1 = S \cup \{e\}$, where e is the identity of S. We define the fuzzy relation μ^* on S as

$$\mu^*(c,d) = \bigcup_{\substack{x,y\in S^1,\\ xay=c,\\ xby=d}} \mu(a,b) \quad \text{for all } c,d \in S.$$

Proposition 3.5. *Let μ and ν be two fuzzy relations on a semigroup S. Then*

(1) $\mu \subseteq \mu^*$
(2) $(\mu^*)^{-1} = (\mu^{-1})^*$
(3) *If* $\mu \subseteq \nu$, *then* $\mu^* \subseteq \nu^*$
(4) $(\mu \cup \nu)^* = \mu^* \cup \nu^*$
(5) $\mu = \mu^*$ *if and only if* μ *is fuzzy left and right compatible*
(6) $(\mu^*)^* = \mu^*$

Proof. See Proposition 3.5 of [5]. □

Theorem 3.6. *Let μ be a fuzzy relation on a semigroup S. Then the ϵ-fuzzy congruence generated by μ in S is $\cup_{n=1}^{\infty}\ [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n$, where θ is a fuzzy relation in S such that $\theta(a,a) = \epsilon$ for all $a \in S$ and $\theta(x,y) = \theta(y,x) \leq \min\ [\mu(x,y), \mu(y,x)]$ for all $x,y \in S$ with $x \neq y$, and μ^* and θ^* are fuzzy relations defined in Definition 3.4.*

Proof. Since $\theta(a,a) = \epsilon$, $\theta^*(a,a) \geq \epsilon > 0$ for all $a \in S$ by Proposition 3.5(1). Let $\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^*$. Then $\mu_1(a,a) \geq \epsilon > 0$. That is, μ_1 is ϵ-reflexive. By Proposition 2.8, $\cup_{n=1}^{\infty}\mu_1^n$ is ϵ-reflexive. Let $x,y \in S$ with $x \neq y$. Since $\theta = \theta^{-1}$, $\theta^* = (\theta^{-1})^* = (\theta^*)^{-1}$ by Proposition 3.5(2). $\mu_1(x,y) = \max[\mu^*(x,y), (\mu^*)^{-1}(x,y), \theta^*(x,y)] = \max[(\mu^*)^{-1}(y,x), \mu^*(y,x), (\theta^*)^{-1}(x,y)] = \max[(\mu^*)^{-1}(y,x), \mu^*(y,x), \theta^*(y,x)] = (\mu^* \cup (\mu^*)^{-1} \cup \theta^*)(y,x) = \mu_1(y,x)$. Thus μ_1 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty}\ \mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}\ \mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty}\ \mu_1^n$ is an ϵ-fuzzy equivalence relation containing μ. By Proposition 3.5(2), (4), and (6), $\mu_1^* = (\mu^* \cup (\mu^*)^{-1} \cup \theta^*)^* = (\mu^* \cup (\mu^{-1})^* \cup \theta^*)^* = (\mu^*)^* \cup ((\mu^{-1})^*)^* \cup (\theta^*)^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu_1$. Thus μ_1 is fuzzy left and right compatible by Proposition 3.5(5). By Proposition 2.7, $\cup_{n=1}^{\infty} \mu_1^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty} \mu_1^n$ is an ϵ-fuzzy congruence containing μ. Let ν be an ϵ-fuzzy congruence containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$. That is, $(\mu \cup \mu^{-1} \cup \theta)(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. $\nu(a,a) \geq \mu(a,a) = \mu^{-1}(a,a)$ for all $a \in S$. Since $\theta(a,a) = \epsilon$ and $\nu(a,a) \geq \epsilon$ for all $a \in S$, $\theta(a,a) \leq \nu(a,a)$. That is, $(\mu \cup \mu^{-1} \cup \theta)(a,a) \leq \nu(a,a)$ for all $a \in S$. Thus $\mu \cup \mu^{-1} \cup \theta \subseteq \nu$. By Proposition 3.5(2), (4), and (3), $\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = (\mu \cup \mu^{-1} \cup \theta)^* \subseteq \nu^*$. Since $\nu = \nu^*$ by Proposition 3.5(5), $\mu_1 \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(x,y) = (\mu_1^k \circ \mu_1)(x,y) = \underset{z \in X}{\rightarrow} \sup \min[\mu_1^k(x,z), \mu_1(z,y)] \leq \underset{z \in X}{\rightarrow} \sup \min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu \circ \nu \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for $n = 1, 2, \ldots$. Thus $\cup_{n=1}^{\infty} [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n = \cup_{n=1}^{\infty} {\mu_1}^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$. □

4. Lattices of ϵ-fuzzy congruences

In this section we discuss some lattice theoretic properties of ϵ-fuzzy congruences.

Theorem 4.1. *Let $C(S)$ be the collection of all ϵ-fuzzy congruences on a semigroup S. Then $(C(S), \leq)$ is a complete lattice, where $\leq$ is a relation on the set of all ϵ-fuzzy congruences on S defined by $\mu \leq \nu$ if and only if $\mu(x,y) \leq \nu(x,y)$ for all $x, y \in S$.*

Proof. Clearly $\leq$ is a partial order relation. It is easy to check that the equality relation σ defined by $\sigma(x,y) = 1$ for all $x, y \in S$ is in $C(S)$ and the relation λ defined by $\lambda(x,y) = \epsilon$ for $x = y$ and $\lambda(x,y) = 0$ for $x \neq y$ is in $C(S)$. Also σ is the greatest element and λ is the least element of $C(S)$ with respect to the ordering $\leq$. Let $\{\mu_j\}_{j \in J}$ be a non-empty collection of ϵ-fuzzy congruences in $C(S)$. Let $\mu(x,y) = \underset{j \in J}{\rightarrow} \inf \mu_j(x,y)$ for all $x, y \in S$. It is easy to see that $\mu(x,x) \geq \epsilon$ for all $x \in S$, $\mu = \mu^{-1}$, $\mu(x,y) \leq \mu(zx, zy)$, and $\mu(x,y) \leq \mu(xz, yz)$ for all $x, y, z \in S$. $\mu \circ \mu(x,y) = \underset{z \in X}{\rightarrow} \sup \min[\underset{j \in J}{\rightarrow} \inf \mu_j(x,z), \underset{j \in J}{\rightarrow} \inf \mu_j(z,y)] = \underset{z \in X}{\rightarrow} \sup \underset{j \in J}{\rightarrow} \inf \underset{i \in J}{\rightarrow} \inf \min[\mu_j(x,z), \mu_i(z,y)] \leq \underset{z \in X}{\rightarrow} \sup \underset{j \in J}{\rightarrow} \inf \min[\mu_j(x,z), \mu_j(z,y)] \leq \underset{j \in J}{\rightarrow} \inf \mu_j \circ \mu_j(x,y) \leq \underset{j \in J}{\rightarrow} \inf \mu_j(x,y) = \mu(x,y)$. That is, $\mu \in C(S)$. Since μ is the greatest lower bound of $\{\mu_j\}_{j \in J}$, $(C(S), \leq)$ is a complete lattice. □

Let $MC(S) = \{\mu \in C(S) : \mu(x,y) \leq \epsilon$ for all $x, y \in S$ such that $x \neq y\}$. Then it is easy to see that $(MC(S), \leq)$ is a sublattice of $(C(S), \leq)$. We define addition and multiplication on $MC(S)$ by $\mu + \nu = \langle \mu \cup \nu \rangle_c$ and $\mu \cdot \nu = \mu \cap \nu$, where $\langle \mu \cup \nu \rangle_c$ is the ϵ-fuzzy congruence generated by $\mu \cup \nu$.

Definition 4.2. A lattice $(L, +, \cdot)$ is called *modular* if $(x + y) \cdot z \leq x + (y \cdot z)$ for all $x, y, z \in L$ with $x \leq z$.

Lemma 4.3. *Let μ and ν be ϵ-fuzzy congruences on a semigroup S such that $\mu(x, x) \geq \nu(x, y)$ and $\nu(y, y) \geq \mu(x, y)$ for all $x, y \in S$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the ϵ-fuzzy congruence on S generated by $\mu \cup \nu$.*

Proof. $(\mu \circ \nu)(x, x) = \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \nu(z, x)] \geq \min(\mu(x, x), \nu(x, x)) \geq \epsilon > 0$ for all $x \in S$. That is, $\mu \circ \nu$ is ϵ-reflexive. Since μ and ν are symmetric, $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \nu \circ \mu = \mu \circ \nu$. Thus $\mu \circ \nu$ is symmetric. Since μ and ν are transitive and the operation $\circ$ is associative, $(\mu \circ \nu) \circ (\mu \circ \nu) = \mu \circ (\nu \circ \mu) \circ \nu = \mu \circ (\mu \circ \nu) \circ \nu = (\mu \circ \mu) \circ (\nu \circ \nu) \subseteq \mu \circ \nu$. Hence $\mu \circ \nu$ is an ϵ-fuzzy equivalence relation. Since S is a semigroup, $(\mu \circ \nu)(x, y) = \underset{a\in S}{\rightarrow} \sup \min[\mu(x, a), \nu(a, y)] \leq \underset{za\in S}{\rightarrow} \sup \min[\mu(zx, za), \nu(za, zy)] \leq \underset{t\in S}{\rightarrow} \sup \min[\mu(zx, t), \nu(t, zy)] = (\mu \circ \nu)(zx, zy)$ for all $x, y, z \in S$. Thus $\mu \circ \nu$ is fuzzy left compatible. Similarly we may show $\mu \circ \nu$ is fuzzy right compatible. Hence $\mu \circ \nu$ is an ϵ-fuzzy congruence in S. Since $\nu(y, y) \geq \mu(x, y)$, $(\mu \circ \nu)(x, y) = \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \nu(z, y)] \geq \min(\mu(x, y), \nu(y, y)) = \mu(x, y)$. Since $\mu(x, x) \geq \nu(x, y)$, $(\mu \circ \nu)(x, y) = \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \nu(z, y)] \geq \min(\mu(x, x), \nu(x, y)) = \nu(x, y)$. Thus $(\mu \circ \nu)(x, y) \geq \max(\mu(x, y), \nu(x, y)) = (\mu \cup \nu)(x, y)$ for all $x, y \in S$. Thus $\mu \cup \nu \subseteq \mu \circ \nu$. Let λ be an ϵ-fuzzy congruence in S containing $\mu \cup \nu$. Since λ is transitive, $\mu \circ \nu \subseteq (\mu \cup \nu) \circ (\mu \cup \nu) \subseteq \lambda \circ \lambda \subseteq \lambda$. Thus $\mu \circ \nu$ is the ϵ-fuzzy congruence generated by $\mu \cup \nu$. □

Lemma 4.3 also gives sufficient conditions for the composition $\mu \circ \nu$ of two ϵ-fuzzy congruences μ and ν on a semigroup to be the ϵ-fuzzy congruence generated by $\mu \cup \nu$.

Theorem 4.4. *Let S be a semigroup and H be a sublattice of $(MC(S), +, \cdot)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$. Then H is a modular lattice.*

Proof. Let $\mu, \nu, \rho \in H$ with $\mu \leq \rho$. Let $x, y \in S$. $\min[(\mu \circ \nu)(x, y), \rho(x, y)] = \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \nu(z, y), \rho(x, y)] \leq \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \rho(x, z), \nu(z, y), \rho(x, y)] \leq \underset{z\in S}{\rightarrow} \sup \min[\mu(x, z), \nu(z, y), \rho(z, y)] = [\mu \circ \min(\nu, \rho)](x, y)$. Thus $(\mu \circ \nu) \cdot \rho \leq \mu \circ (\nu \cdot \rho)$. Since $\mu, \nu \in MC(S)$, $\mu(x, x) \geq \nu(x, y)$ and $\nu(y, y) \geq \mu(x, y)$ for all $x, y \in S$. By Lemma 4.3, $\mu \circ \nu$ is the ϵ-fuzzy congruence generated by $\mu \cup \nu$. That is, $\mu + \nu = \mu \circ \nu$. Similarly we may show $\mu + (\nu \cdot \rho) = \mu \circ (\nu \cdot \rho)$. Thus $(\mu + \nu) \cdot \rho \leq \mu + (\nu \cdot \rho)$. Hence H is modular. □

Proposition 4.5. *If S is a group, then $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in C(S)$.*

Proof. Straightforward. □

Corollary 4.6. *If S is a group, then $(MC(S), +, \cdot)$ is modular.*

Proof. By Theorem 4.4 and Proposition 4.5, $(MC(S), +, \cdot)$ is modular. □

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