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OUTER AUTOMORPHISM GROUPS OF POLYGONAL PRODUCTS OF CERTAIN CONJUGACY SEPARABLE GROUPS

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Abstract. Grossman [7] showed that certain cyclically pinched 1-relator groups have residually finite outer automorphism groups. In this paper we prove that tree products of finitely generated free groups amalgamating maximal cyclic subgroups have residually finite outer automorphism groups. We also prove that polygonal products of finitely generated central subgroup separable groups amalgamating trivial intersecting central subgroups have residually finite outer automorphism groups.

1. Introduction

Outer automorphism groups have been widely studied for various purposes. The residual finiteness of outer automorphism groups of certain cyclically pinched 1-relator groups was first studied by E. Grossman [7] in which she proved that outer automorphism groups of the fundamental groups of orientable surfaces are residually finite ($\mathcal{RF}$). It follows that the mapping class groups of orientable surfaces are $\mathcal{RF}$. Wehrfritz [13] proved that outer automorphism groups of finitely generated nilpotent groups are isomorphic to subgroups of $GL(n, Z)$, whence such groups are $\mathcal{RF}$. D. Wise [14] gave an example of a finitely generated $\mathcal{RF}$ group whose outer automorphism group is not $\mathcal{RF}$, contrasting Baumslag's result [5] that automorphism groups of finitely generated residually finite groups are $\mathcal{RF}$. Thus it is of interest to find out which finitely generated $\mathcal{RF}$ groups have $\mathcal{RF}$ outer automorphism groups. In [3], Allenby, Kim, and Tang showed that cyclically pinched 1-relator groups have $\mathcal{RF}$ outer automorphism groups. From this it follows that mapping class groups of orientable and non-orientable surface groups are $\mathcal{RF}$. Applying Fine and Rosenberger's result [6] that Fuchsian groups are conjugacy separable, Allenby, Kim, and Tang [4] showed that finitely generated Fuchsian groups have $\mathcal{RF}$ outer automorphism groups. Further Allenby, Kim, and Tang [1], [2] showed that

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most of the Seifert groups have $\mathcal{RF}$ outer automorphism groups. In this paper we prove that tree products of finitely generated free groups amalgamating maximal cyclic subgroups and polygonal products of finitely generated central subgroup separable groups amalgamating trivial intersecting central subgroups have $\mathcal{RF}$ outer automorphism groups. Thus, in particular, polygonal products of finitely generated abelian groups amalgamating trivial intersecting subgroups have $\mathcal{RF}$ outer automorphism groups.

In Section 2, we give a brief summary of notation and results we need. In Section 3, we prove that tree products of finitely generated free groups amalgamating maximal cyclic subgroups have $\mathcal{RF}$ outer automorphism groups. In Section 4, we prove a criterion for generalized free products amalgamating retracts of certain split extensions to be have Property E. Applying this result we prove, in Section 5, our main result Theorem 5.6.

2. Preliminaries

Throughout this paper we use standard notation and terminology.

If A, B are groups, $G = A *_H B$ denotes the generalized free product of A and B amalgamating the subgroup H.

If $x \in G = A *_H B$, then $\|x\|$ denotes the free product length of x in G.

If $g \in G$, Inn g denotes the inner automorphism of G induced by g.

Out G denotes the outer automorphism group of G.

$x \sim_G y$ means that x and y are conjugate in G.

$Z(G)$ is the center of G.

$\mathcal{RF}$ is an abbreviation for "residually finite".

We use the somewhat unusual notation $1 \neq a \in A$ to mean $a \neq 1$ and $a \in A$.

Definition 2.1. By a *conjugating endomorphism/automorphism* of a group G we mean an endomorphism/automorphism α which is such that, for each $g \in G$, there exists $k_g \in G$, depending on g, so that $\alpha(g) = k_g^{-1} g k_g$.

Definition 2.2 (Grossman [7])**.** A group G has *Property A* if, for each conjugating automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$, i.e., $\alpha = \text{Inn } k$.

We extend Grossman's Property A to include endomorphisms.

Definition 2.3 ([4])**.** A group G has *Property E* if, for each conjugating endomorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1}gk$ for all $g \in G$, i.e., $\alpha = \text{Inn } k$.

Clearly, every abelian group has Property E and every group having Property E has Property A. We will make use of the following result of Grossman [7]:

Theorem 2.4 (Grossman [7])**.** *Let B be a finitely generated, conjugacy separable group with Property A. Then* Out B *is* $\mathcal{RF}$.

Theorem 2.5 ([3, Corollary 3.3])**.** *Free groups and free products of cycles have Property E.*

Theorem 2.6 ([11, Theorem 4.6])**.** *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $||x|| = 0$, then $||y|| \leq 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that*

$$y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x.$$

(2) *If $||x|| = 1$, then $||y|| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*

(3) *If $||x|| \geq 2$, then $||x|| = ||y||$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

Theorem 2.7 ([8])**.** *Let $G = A *_H B$, where $A \neq H \neq B$ and $H \subset Z(B)$. Suppose A has Property E and the following conditions hold*:

(C1) *If $u \in A$ and $u^{-1}hu = h$ for all $h \in H$, then $u \in H$.*

(C2) *There exists an element $a \in A$ such that $\{a\}^A \cap H = \emptyset$ and if $u^{-1}au = h'ah$, where $u \in A$ and $h', h \in H$, then $h' = h^{-1}$.*

Then G has Property E.

3. On tree products of free groups

A subgroup H of a group G is *malnormal* if $u^{-1}Hu \cap H = 1$ for each $u \in G \backslash H$. We shall use the following result (Theorem 3.1 in [4]).

Theorem 3.1 ([4])**.** *Let $G = A *_H B$, where $A \neq H \neq B$. Suppose the following conditions hold*:

(A1) *H is malnormal in A and B.*

(A2) *There exists an element $a \in A$ such that* (i) *$\{a\}^A \cap H = \emptyset$ and* (ii) *if $u^{-1}au = h'ah$, where $u \in A$ and $h', h \in H$, then $h' = h^{-1}$.*

Then G has Property E.

We shall use $\{K\}^B = \{b^{-1}kb \mid k \in K, b \in B\}$.

Lemma 3.2. *Let $G = A *_H B$, where $A \neq H \neq B$. Suppose H is malnormal in A and B. If K is a malnormal subgroup in B, then K is a malnormal subgroup in G.*

Proof. Let $g^{-1}k_1g = k_2$, where $g \in G$ and $k_1, k_2 \in K$ such that $k_1 \neq 1 \neq k_2$. We shall show that $g \in K$. Let $g = u_1u_2 \cdots u_r$ be the alternating product of g in $G = A *_H B$.

Case 1. $H \cap \{K\}^B = 1$.

In this case $k_1 \in B \backslash H$. So if $u_1 \in A \backslash H$, then $r \geq 1$ and the length of

$$g^{-1}k_1g = u_r^{-1} \cdots u_1^{-1}k_1u_1 \cdots u_r \tag{3.1}$$

is $2r+1 \geq 3$. Thus $g^{-1}k_1g \neq k_2$. Therefore we may assume $u_1 \in B$. Since $H \cap \{K\}^B = 1$, $u_1^{-1}k_1u_1 \in B \backslash H$. Thus, if $r \geq 2$, then the length of (3.1) is $2r-1 \geq 3$. Therefore, for $g^{-1}k_1g = k_2$, we must have $r = 1$, that is, $g = u_1 \in B$. Since K is malnormal in B and $g^{-1}k_1g = k_2$, we have $g \in K$.

Case 2. $H \cap \{K\}^B \neq 1$.

(1) Suppose $u_1 \in A \backslash H$. Let $r \geq 1$. If $k_1 \notin H$, then the length of (3.1) is $2r+1 \geq 3$. Hence $g^{-1}k_1g \neq k_2$. Therefore $k_1 \in H$. Then, since H is malnormal in A and $u_1 \in A \backslash H$, we have $u_1^{-1}k_1u_1 \notin H$. Then the length of (3.1) is $2r-1$. Since $g^{-1}k_1g = k_2 \in B$, we have $r = 1$ and $g = u_1$. Thus $k_2 = g^{-1}k_1g \in A$. Hence $k_2 \in A \cap B = H$. Since H is malnormal in A, $g = u_1 \in H$ which is impossible by the assumption $u_1 \in A \backslash H$.

(2) $u_1 \in B$. Suppose $r \geq 2$. Then $u_2 \in A \backslash H$. If $u_1^{-1}k_1u_1 \in H$, then $u_2^{-1}(u_1^{-1}k_1u_1)u_2 \notin H$ (H is malnormal in A). Hence $g^{-1}k_1g \neq k_2 \in B$. On the other hand, if $u_1^{-1}k_1u_1 \notin H$, then the length of (3.1) is $2r-1 \geq 3$. Hence $g^{-1}k_1g \neq k_2$. Therefore we must have $r \leq 1$. Then $g = u_1 \in B$ and $g^{-1}k_1g = k_2$. Since K is malnormal in B, we have $g \in K$.

This completes the proof that K is malnormal in G. □

Theorem 3.3. *Let G be a tree product of finite number of free groups A_i ($1 \leq i \leq n$) amalgamating maximal cyclic subgroups. Then G has Property E.*

Proof. Without loss of generality, let A_n be an extremal vertex group of G and let T be the subgroup of G generated by the rest of the vertices. Then T is a tree product of A_i ($1 \leq i \leq n-1$) amalgamating maximal cyclic subgroups. Thus we can consider $G = T *_H A_n$ where H is a maximal cyclic subgroup in A_n and in some vertex group of T. Hence we may assume A_n is a free group of rank ≥ 2. Since A_n is free, as in Theorem 3.6 [4], there exists an element $a \in A_n$ such that (i) $\{a\}^{A_n} \cap H = \emptyset$ and (ii) if $u^{-1}au = h'ah$, where $u \in A_n$ and $h', h \in H$, then $h' = h^{-1}$.

Since maximal cyclic subgroups in free groups are malnormal, H is malnormal in T by Lemma 3.2. Thus, by Theorem 3.1, $G = T *_H A_n$ has Property E. □

Since tree products of free groups, amalgamating maximal cyclic subgroups, are conjugacy separable [12], we have the following by Theorem 2.4:

Theorem 3.4. *Let G be a tree product of finite number of finitely generated free groups A_i ($1 \leq i \leq n$) amalgamating maximal cyclic subgroups. Then* Out G *is residually finite.*

4. A criterion

If there is a normal subgroup N of a group G such that $G = N \cdot H$ and $N \cap H = 1$, then G is called a *split extension* of N by a *retract* H.

Lemma 4.1. *Let A_0, B_0 be split extensions of A, B, respectively, by a retract H. Let $G = A_0 *_H B_0$, where $A \subset Z(A_0)$. Suppose α is a conjugating endomorphism of G such that $\alpha(g) = k_g^{-1} g k_g$ for $g \in G$. If there exist $1 \neq a \in A$ and $1 \neq b \in B$ such that $\alpha(a) = a$ and $\alpha(b) = b$, then we can choose $k_h \in H$, $k_x \in H$ and $k_y \in H$ for all $h \in H$, $x \in A$ and $y \in B$.*

Proof. Suppose there exist $1 \neq a \in A$, $1 \neq b \in B$ such that $\alpha(a) = a$ and $\alpha(b) = b$.

We first show that we can choose $k_x \in H$ for all $x \in A$. Let $1 \neq x \in A$ and $k_x = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in G such that $\alpha(x) = k_x^{-1} x k_x$. Since $x \in A \subset Z(A_0)$, we can assume $u_1 \in B_0$. Then $k_{xa}^{-1}(xa)k_{xa} = \alpha(xa) = \alpha(x)\alpha(a) = k_x^{-1} x k_x \cdot a = u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1} x u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r \cdot a$. Thus,

$$xa \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1} x u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r a u_r^{-1}. \tag{4.1}$$

Since $1 \neq x \in A$, if $u_1 \in B_0 \backslash H$, then $\|u_1^{-1} x u_1\| = 3$. Hence if $u_1 \in B_0 \backslash H$, then the length of the R.H.S. of (4.1) is greater than 1, whereas the length of the L.H.S. of (4.1) is at most 1, which is not possible by Theorem 2.6. Thus we may assume $k_x = u_1 \in H$. Since $x \in A \subset Z(A_0)$, $\alpha(x) = u_1^{-1} x u_1 = x$. Therefore $\alpha(x) = x$ and $k_x = 1 \in H$ for all $x \in A$.

We shall show that $k_e \in H$ for all $e \in H$. Let $1 \neq e \in H$ and $k_e = u_1 u_2 \cdots u_r$ be an alternating product of the shortest length in G such that $\alpha(e) = k_e^{-1} e k_e$. Then $k_{ea}^{-1}(ea)k_{ea} = \alpha(ea) = \alpha(e)\alpha(a) = k_e^{-1} e k_e \cdot a = u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1} e u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r \cdot a$. Thus,

$$ea \sim_G u_{r-1}^{-1} \cdots u_2^{-1} \cdot u_1^{-1} e u_1 \cdot u_2 \cdots u_{r-1} \cdot u_r a u_r^{-1}. \tag{4.2}$$

(1) Suppose $u_1 \in B_0 \backslash H$.

(i) Suppose $u_1^{-1} e u_1 \in H$. Let $u_1^{-1} e u_1 = c \in H$ and $u_1 = bh$ where $b \in B$ and $h \in H$. Then $h^{-1} b^{-1} e b h = c \in H$. Thus $b^{-1} e b = h c h^{-1} \in H$. Since $B \lhd B_0$, $e b e^{-1} = b' \in B$. This implies $b^{-1} b' = h c h^{-1} e^{-1} \in H \cap B = 1$. Hence $e = h c h^{-1}$. Then $u_1^{-1} e u_1 = c = h^{-1} e h$, which would reduce the length of k_e.

(ii) Suppose $u_1^{-1} e u_1 \notin H$. Then $u_1^{-1} e u_1 \in B_0 \backslash H$. Then the R.H.S. of (4.2) is greater than 1. Since the length of the L.H.S. of (4.2) is at most 1, this would contradict Theorem 2.6.

(2) Suppose $u_1 \in A_0$.

If $u_1^{-1} e u_1 = c \in H$, then the length of k_e would be reduced as in (i) above.

If $u_1^{-1} e u_1 \notin H$, then $u_1^{-1} e u_1 \in A_0 \backslash H$. Since the length of the L.H.S. of (4.2) is at most 1, we must have $r \leq 1$. Then $k_e = u_1 \in A_0$. By assumption, there exists $1 \neq b \in B$ such that $\alpha(b) = b$. Now $eb \sim \alpha(eb) = u_1^{-1} e u_1 b$. Since $eb \in B_0$ and $u_1 \in A_0$, $u_1^{-1} e u_1 \in H$ by Theorem 2.6. Then, as (i) above, $u_1^{-1} e u_1 = h^{-1} e h$ for some $h \in H$. Hence $\alpha(e) = u_1^{-1} e u_1 = h^{-1} e h$, that is, $k_e = h \in H$.

Finally we show that $k_y \in H$ for all $y \in B$. Let $1 \neq y \in B$ and $k_y = u_1u_2\cdots u_r$ be an alternating product of the shortest length in G such that $\alpha(y) = k_y^{-1}yk_y$. Then $k_{ya}^{-1}(ya)k_{ya} = \alpha(ya) = \alpha(y)\alpha(a) = k_y^{-1}yk_y \cdot a = u_r^{-1}\cdots u_2^{-1}\cdot u_1^{-1}yu_1\cdot u_2\cdots u_{r-1}\cdot u_r\cdot a$. Thus,

$$ya \sim_G u_{r-1}^{-1}\cdots u_2^{-1}\cdot u_1^{-1}yu_1\cdot u_2\cdots u_{r-1}\cdot u_rau_r^{-1}. \tag{4.3}$$

Clearly $u_1 \notin A\backslash H$, so we may suppose $u_1 \in B_0$. Since $y \in B \lhd B_0$ and $B\cap H = 1$, we have $u_1^{-1}yu_1 \notin H$. By considering the length of (4.3), we must have $r \leq 2$ and $k_y = u_1u_2 \in B_0A_0$. Thus $ya \sim_G u_1^{-1}yu_1u_2au_2^{-1}$. By Theorem 2.6, $ya \sim_H u_1^{-1}yu_1u_2au_2^{-1}$. Hence

$$y = h_1^{-1}u_1^{-1}yu_1h_2 \text{ and } a = h_2^{-1}u_2au_2^{-1}h_1$$

for some $h_1, h_2 \in H$. Let $u_1 = b_1h$ where $b_1 \in B$ and $h \in H$. Then $y = h_1^{-1}h^{-1}b_1^{-1}yb_1hh_2 = c \in H$. Since $B \lhd B_0$, we have $h^{-1}b_1^{-1}yb_1h = b_2 \in B$. Then $y = h_1^{-1}b_2h_2 = h_1^{-1}h_2b_3$ where $b_3 = h_2^{-1}b_2h_2 \in B$. This implies $yb_3^{-1} = h_1^{-1}h_2 \in H\cap B = 1$. Hence $h_1 = h_2$. Thus $u_1^{-1}yu_1 = h_1yh_1^{-1}$. It follows that we can choose $k_y = h_1^{-1}u_2 \in A_0$. Since $y, b \in B_0$, $h^{-1}u_2 \in A_0$ and

$$yb \sim_G \alpha(yb) = k_y^{-1}yk_yb = (h_1^{-1}u_2)^{-1}y(h_1^{-1}u_2)b,$$

by Theorem 2.6 we must have $h_1^{-1}u_2 \in H$. Hence $k_y = h_1^{-1}u_2 \in H$ for each $y \in B$. □

Lemma 4.2. *Let A_0, B_0 be split extensions of A, B, respectively, by a retract H. Let $G = A_0 *_H B_0$, where $A \subset Z(A_0)$. Suppose A_0 has property E. If α is a conjugating endomorphism of G, then there exists $g \in G$ such that $\alpha' = \text{Inn } g\circ\alpha$, where $\alpha'(e) = e$ for all $e \in A_0$ and $\alpha'(y) = h_y^{-1}yh_y$, where $h_y \in H$ for all $y \in B$.*

Proof. Let α be a conjugating endomorphism of G and $\alpha(g) = k_g^{-1}gk_g$ for $g \in G$. Without loss of generality, we can assume $\alpha(b) = b$ for a fixed element $1 \neq b \in B$. Let $1 \neq a \in A$ be a fixed element. Let $k_a = u_1u_2\cdots u_r$ be an alternating product of the shortest length from G such that $\alpha(a) = k_a^{-1}ak_a$. Since $a \in A \subset Z(A_0)$, we can assume $u_1 \in B_0$. In fact, we shall show that $k_a \in B_0$. Since $k_{ab}^{-1}(ab)k_{ab} = \alpha(ab) = k_a^{-1}ak_a\cdot b = u_r^{-1}\cdots u_1^{-1}au_1\cdots u_r\cdot b$. Hence

$$ab \sim_G u_{r-1}^{-1}\cdots u_2^{-1}\cdot u_1^{-1}au_1\cdot u_2\cdots u_{r-1}\cdot u_rbu_r^{-1}. \tag{4.4}$$

Suppose $r \geq 2$. If $u_r \in A_0\backslash H$, then the R.H.S. of (4.4) is cyclically reduced of length $2r+2$. If $u_r \in B_0\backslash H$, then $u_rbu_r^{-1} \in B_0\backslash H$ ($b \in B \lhd B_0$). Hence the R.H.S. of (4.4) is cyclically reduced of length $2r$. Since the L.H.S. of (4.4) is cyclically reduced of length 2, both cases are impossible. Therefore, $r \leq 1$, that is, $k_a = u_1 \in B_0$. So $\alpha(a) = u_1^{-1}au_1$.

By (4.4), we have $ab \sim_G au_1bu_1^{-1}$. From Theorem 2.6 there exist $h, h_1 \in H$ such that $a = h^{-1}ah_1$ and $b = h_1^{-1}u_1bu_1^{-1}h$. Since $a \in Z(A_0)$, we have $h = h_1$.

Thus

$$b = h_1^{-1}u_1bu_1^{-1}h_1. \tag{4.5}$$

Let $\overline{\alpha} = \text{Inn } u_1^{-1}h_1 \circ \alpha$. Then $\overline{\alpha}(a) = (u_1^{-1}h_1)^{-1}\alpha(a)u_1^{-1}h_1 = h_1^{-1}ah_1 = a$, since $\alpha(a) = u_1^{-1}au_1$ and $a \in Z(A_0)$. Moreover, $\overline{\alpha}(b) = (u_1^{-1}h_1)^{-1}\alpha(b)u_1^{-1}h_1 = (u_1^{-1}h_1)^{-1}bu_1^{-1}h_1 = b$ by (4.5).

For convenience, we again use α instead of $\overline{\alpha}$. Then α is a conjugating endomorphism of G and $\alpha(g) = k_g^{-1}gk_g$ for $g \in G$, where $\alpha(a) = a$ and $\alpha(b) = b$ for fixed elements $1 \neq a \in A$ and $1 \neq b \in B$. By Lemma 4.1, we have $k_h \in H$ for all $h \in H$, $k_x \in H$ for all $x \in A$ and $k_y \in H$ for all $y \in B$.

Since $A \subset Z(A_0)$, $\alpha(xh) = \alpha(x)\alpha(h) = k_x^{-1}xk_x \cdot k_h^{-1}hk_h = xk_h^{-1}hk_h = k_h^{-1}xhk_h$ for all $x \in A$ and $h \in H$. Therefore, the restriction of α to A_0 is a conjugating endomorphism of A_0. Since A_0 has property E, there exists $c \in A_0$ such that $\alpha(e) = c^{-1}ec$ for all $e \in A_0$. Let $c = c_1h_1$, where $c_1 \in A$ and $h_1 \in H$. Since $A \subset Z(A_0)$, we have $\alpha(e) = h_1^{-1}eh_1$ for all $e \in A_0$. Let $\alpha' = \text{Inn } h_1^{-1} \circ \alpha$. Then $\alpha'(e) = e$ for all $e \in A_0$ and $\alpha'(y) = h_1k_y^{-1}yk_yh_1^{-1}$ for all $y \in B$, where $h_1, k_y \in H$. □

Theorem 4.3. *Let A_0, B_0 be split extensions of A, B, respectively, by a retract H. Let $G = A_0 *_H B_0$, where $A \subset Z(A_0)$. Suppose A_0, B_0 have property E and B_0 satisfies the following*:

(D) *For $u \in B_0$, if $u^{-1}hu = h$ for all $h \in H$, then $u \in Z(B_0)$.*

Then G has Property E.

Proof. Let α be a conjugating endomorphism of G. By Lemma 4.2, we can assume that $\alpha(e) = e$ for all $e \in A_0$ and $\alpha(y) = k_y^{-1}yk_y$, where $k_y \in H$ for all $y \in B$.

We shall show that $k_f \in B_0$ for all $f \in B_0$. Consider $1 \neq y \in B$ and $h \in H$. Let $k_{yh} = u_1u_2 \cdots u_r$ be an alternating product of the shortest length from G such that $\overline{\alpha}(yh) = k_{yh}^{-1}yhk_{yh}$. Since $\alpha(yh) = \alpha(y)\alpha(h) = k_y^{-1}yk_yh$, we have

$$u_r^{-1} \cdots u_2^{-1} \cdot u_1^{-1}yhu_1 \cdot u_2 \cdots u_r = k_y^{-1}yk_yh. \tag{4.6}$$

Note that the R.H.S. of (4.6) is of length at most 1.

(1) Since the L.H.S. of (4.6) is of length at least 3, we must have $u_1 \notin A_0\backslash H$.

(2) Suppose $u_1 \in B_0\backslash H$ and $r \geq 2$. If $u_1^{-1}yhu_1 \notin H$, then the L.H.S. of (4.6) is of length at least 3. Hence, suppose $u_1^{-1}yhu_1 \in H$. Since $u_2 \in A_0$, let $u_2 = rt$ for $r \in A$ and $t \in H$. Then $u_2^{-1}(u_1^{-1}yhu_1)u_2 = t^{-1}(u_1^{-1}yhu_1)t = (u_1t)^{-1}yh(u_1t)$, which reduces the length of k_{yh}.

Therefore, $k_{yh} = u_1 \in B_0$ for all $1 \neq y \in B$ and $h \in H$. Since $\alpha(e) = e$ for each $e \in A_0$, $\alpha(h) = h$ for all $h \in H$. Consequently, we have $k_f \in B_0$ for all $f \in B_0$.

Hence the restriction of α to B_0 is a conjugating endomorphism of B_0. Since B_0 has Property E, there exists $s \in B_0$ such that $\alpha(f) = s^{-1}fs$ for all $f \in B_0$. In particular, $\alpha(h) = s^{-1}hs$ for all $h \in H$. Since $\alpha(h) = h$, we have $s^{-1}hs = h$ for all $h \in H$. It follows from assumption (D) that $s \in Z(B_0)$. Therefore,

$\alpha(f) = s^{-1}fs = f$ for all $f \in B_0$. Consequently α is an identity on G. This proves that G has Property E. □

5. On polygonal products

In this section we shall show that outer automorphism groups of polygonal products of certain conjugacy separable groups, amalgamating central subgroups with trivial intersections, are residually finite. For convenience, we use the following notation:

Define $A_1 = H_0 \times H_1$ and $H_iH_{i+1} = H_i \times H_{i+1}$. For $r \geq 2$, define

$$A_r = H_0H_1 *_{H_1} H_1H_2 *_{H_2} \cdots *_{H_{r-1}} H_{r-1}H_r, \tag{5.1}$$

$$B_{r-1} = \langle H_0, H_1, \ldots, H_{r-1}\rangle^{A_r}, \tag{5.2}$$

where the H_i are finitely generated abelian groups and $H_i \cap H_{i+1} = 1$. Then A_r is a split extension of B_{r-1} by the retract H_r. Note that $A_{r+1} = A_r *_{H_r} H_rH_{r+1}$.

Theorem 5.1. *The group A_{r+1} has Property E for $r \geq 0$.*

Proof. Clearly the abelian group $A_1 = H_0 \times H_1$ has Property E. Since $A_2 = H_0H_1 *_{H_1} H_1H_2 = (H_0 * H_2) \times H_1$, A_2 has Property E. This is because by Theorem 2.2 [4] free products of groups with Property E have Property E and it is easy to check direct products of groups with Property E also have Property E.

We shall show that $A_3 = A_2 *_{H_2} H_2H_3$ has Property E. Let α be a conjugating endomorphism of A_3 such that $\alpha(g) = k_g^{-1}gk_g$ for $g \in A_3$. By Lemma 4.2, we can assume that $\alpha(e) = e$ for all $e \in H_2H_3$ and $\alpha(y) = k_y^{-1}yk_y$, where $k_y \in H_2$ for all $y \in B_1$. Since H_2 is abelian we have, for $h \in H_2$, $y \in B_1$, that $\alpha(yh) = \alpha(y)\alpha(h) = k_y^{-1}yk_yh = k_y^{-1}yhk_y$, where $k_y \in H_2$. Hence the restriction of α to A_2 is a conjugating endomorphism of A_2. Since A_2 has Property E, there exists $s \in A_2$ such that $\alpha(f) = s^{-1}fs$ for all $f \in A_2$. In particular, $\alpha(h) = s^{-1}hs$ for all $h \in H_2$. Since $\alpha(h) = h$ for $h \in H_2$, we have $h = s^{-1}hs$. Since $s \in A_2 = (H_0 * H_2) \times H_1$, let $s = wh_1$, where $w \in H_0 * H_2$ and $h_1 \in H_1$. Then $h = s^{-1}hs = w^{-1}hw$. Hence $w \in H_2$ and $s = wh_1 \in H_2H_1$. Then $\alpha(f) = s^{-1}fs = w^{-1}fw$ for all $f \in A_2$ and $\alpha(e) = e = w^{-1}ew$ for all $e \in H_2H_3$. Therefore $\alpha = \mathrm{Inn}\, w$. This proves that A_3 has Property E.

Inductively we assume A_r has Property E for $r \geq 3$. Let α be a conjugating endomorphism of A_{r+1}. Consider $A_{r+1} = A_r *_{H_r} H_rH_{r+1}$, where $A_r = B_{r-1}H_r$ (see (5.2)). By Lemma 4.2, we can assume that $\alpha(e) = e$ for all $e \in H_rH_{r+1}$ and $\alpha(y) = k_y^{-1}yk_y$, where $k_y \in H_r$, for all $y \in B_{r-1}$. Since H_r is abelian for all $h \in H_r$ and $y \in B_{r-1}$, we have $\alpha(yh) = \alpha(y)\alpha(h) = k_y^{-1}yk_yh = k_y^{-1}yhk_y$. Hence the restriction of α to A_r is a conjugating endomorphism of A_r. Since A_r has Property E by induction, there exists $s \in A_r$ such that $\alpha(f) = s^{-1}fs$ for all $f \in A_r$. In particular, for $h \in H_r$ we have $h = \alpha(h) = s^{-1}hs$. Since $h \in H_r$ and $s \in A_r = A_{r-1} *_{H_{r-1}} H_{r-1}H_r$, by considering the lengths of h and $s^{-1}hs$, we have $s \in H_{r-1}H_r$. Let $s = h_1h_2$ where $h_1 \in H_{r-1}$ and $h_2 \in H_r$. Hence, for $f \in H_0$, we have $\alpha(f) = s^{-1}fs = h_2^{-1}(h_1^{-1}fh_1)h_2$. Since $\alpha(f) = k_f^{-1}fk_f$ for $k_f \in H_r$, we have $h_2^{-1}(h_1^{-1}fh_1)h_2 = k_f^{-1}fk_f$, where $h_1 \in H_{r-1}$ and $h_2 \in H_r$. Thus we have

$$h_2^{-1} = k_f^{-1}h_3, \tag{5.3}$$

$$h_1^{-1}fh_1 = h_3^{-1}fh_4, \tag{5.4}$$

$$h_2 = h_4^{-1}k_f \tag{5.5}$$

for some $h_3, h_4 \in H_{r-1}$. Since $H_{r-1} \cap H_r = 1$, from (5.3) we have $h_3 = 1$ and $h_2 = k_f$. Similarly, from (5.5) we have $h_4 = 1$. Then we must have $h_1 = 1$ from (5.4), since $f \in H_0$ and $h_1 \in H_{r-1}$ where $r \geq 3$. Thus $\alpha(f) = s^{-1}fs = h_2^{-1}fh_2$ for all $f \in A_r$ and $\alpha(e) = e = h_2^{-1}eh_2$ for all $e \in H_rH_{r+1}$. Therefore $\alpha = \text{Inn } h_2$. This proves that A_{r+1} has Property E. □

Since tree products of finitely generated abelian groups are conjugacy separable [10], we have the following by Theorem 2.4:

Theorem 5.2. *The group* Out A_r *is residually finite.*

For $r \geq 2$, we can consider A_r as a split extension of $D_{r-1} = \langle H_1, \ldots, H_{r-1}\rangle^{A_r}$ by a retract $H = H_0 * H_r$.

Lemma 5.3. *Let $A_r = D_{r-1}H$ as above, where $r \geq 2$. If $u \in A_r$ and $u^{-1}hu = h$ for all $h \in H$, then $u \in Z(A_r)$.*

Proof. Since $Z(A_2) = H_1$ and $Z(A_r) = 1$ for $r \geq 3$, we consider two cases separately.

Case 1. $r = 2$. Note $A_2 = H_0H_1 *_{H_1} H_1H_2 = (H_0 * H_2) \times H_1$. Suppose $u \in A_2$ and $u^{-1}hu = h$ for all $h \in H = H_0 * H_2$. For $1 \neq a \in H_0$, since $u^{-1}au = a$, by considering the lengths of $u^{-1}au = a$, u can not be of the form $u = f_1e_1\cdots$, where $f_1 \in H_1H_2\backslash H_1$ and $e_1 \in H_0H_1\backslash H_1$. Also considering $1 \neq b \in H_2$ and $u^{-1}bu = b$, u can not be of the form $u = e_1f_1\cdots$, where $e_1 \in H_0H_1\backslash H_1$ and $f_1 \in H_1H_2\backslash H_1$. Therefore $u \in H_1 = Z(A_2)$.

Case 2. $r \geq 3$. Note $A_r = A_{r-1} *_{H_{r-1}} H_{r-1}H_r$ and $H = H_0 * H_r$. As before, by considering $1 \neq a \in H_0$, since $u^{-1}au = a$, u can not be of the form $u = f_1e_1\cdots$, where $f_1 \in H_{r-1}H_r\backslash H_{r-1}$ and $e_1 \in A_{r-1}\backslash H_{r-1}$. Also considering $1 \neq b \in H_r$ and $u^{-1}bu = b$, u can not be of the form $u = e_1f_1\cdots$, where $e_1 \in A_r\backslash H_{r-1}$ and $f_1 \in H_{r-1}H_r\backslash H_{r-1}$. Thus $u \in H_{r-1}$. Again, for $1 \neq a \in H_0$, we have $u^{-1}au = a$, where $u \in H_{r-1}$. Since $r \geq 3$, $\langle H_0, H_{r-1}\rangle = H_0 * H_{r-1}$. Hence $u = 1 \in Z(A_r)$. □

Now for our main theorem, let G be a polygonal product of groups $S_1, S_2, \ldots, S_n$ ($n > 3$), amalgamating central subgroups $H_1, H_2, \ldots, H_0$, with trivial intersections. Hence $H_i \subset Z(S_i) \cap Z(S_{i+1})$ and $H_i \cap H_{i+1} = 1$ where $1 \leq i \leq n$ and the subscripts are taken modulo n. Then the subgroup of G generated by $H_1, H_2, \ldots, H_0$ is called a *reduced polygonal product* P_0 of G.

Thus P_0 is the polygonal product of $H_0H_1, H_1H_2, \ldots, H_{n-1}H_0$, amalgamating $H_1, H_2, \ldots, H_0$, and

$$\begin{aligned} P_0 &= B_0 *_H A_0, \text{ where} \\ B_0 &= H_0H_1 *_{H_1} H_1H_2 *_{H_2} \cdots *_{H_{n-3}} H_{n-3}H_{n-2}, \\ A_0 &= H_0H_{n-1} *_{H_{n-1}} H_{n-1}H_{n-2} \text{ and} \\ H &= H_0 * H_{n-2}. \end{aligned}$$

We first show that P_0 has Property E.

Theorem 5.4. *The reduced polygonal product P_0 has Property E.*

Proof. Clearly B_0 is a split extensions of $\langle H_1, \ldots, H_{n-3}\rangle^{B_0}$ with a retract H and A_0 is a split extensions of H_{n-1} with retract H, where $H_{n-1} \subset Z(A_0)$. By Lemma 5.1, A_0, B_0 have Property E. By Lemma 5.3, B_0 satisfies (D) in Theorem 4.3. Hence, by Theorem 4.3, P_0 has Property E. $\square$

Theorem 5.5. *Let G be a polygonal product of groups $S_1, S_2, \ldots, S_n$ $(n > 3)$, amalgamating central subgroups $H_1, H_2, \ldots, H_0$, with trivial intersections. Then G has Property E.*

Proof. Let P_0 be the reduced polygonal product of G as before. For $i = 0, 1, \ldots, n-1$, let

$$P_{i+1} = (\cdots(P_0 *_{H_0H_1} S_1) *_{H_1H_2} \cdots) *_{H_iH_{i+1}} S_{i+1}.$$

Then $G = P_n$. By Theorem 5.4, P has Property E. Thus, by induction, we can assume that $P_0, P_1, \ldots, P_i$ have Property E and we shall show that $P_{i+1} = P_i *_{H_iH_{i+1}} S_{i+1}$ has Property E.

To prove (C1) in Theorem 2.7, let $c \in P_i$ and $ch = hc$ for all $h \in H_iH_{i+1}$. Let

$$\begin{aligned} E &= S_i *_{H_i} H_iH_{i+1}, \\ F &= S_{i-1} *_{H_{i-2}} \cdots *_{H_1} S_1 *_{H_0} H_0H_{n-1} *_{H_{n-1}} \cdots *_{H_{i+2}} H_{i+2}H_{i+1} \text{ and} \\ H &= H_{i-1} * H_{i+1}, \end{aligned}$$

where $S_0 = H_0H_{n-1}$, $S_{-1} = H_{n-1}H_{n-2}$, $S_{-2} = H_{n-2}H_{n-3}$ and the subscripts of H_i are taken modulo n. Then $P_i = E *_H F$. Since $c^{-1}hc = h$ for all $1 \neq h \in H_i$, by considering the length of $c^{-1}hc = h$, c can not be of the form $f_1e_1\cdots$, of length ≥ 1, where $f_1 \in F\backslash H$ and $e_1 \in E\backslash H$. If $c = e_1f_1\cdots$ of length ≥ 2 where $e_1 \in E\backslash H$ and $f_1 \in F\backslash H$, then $c^{-1}hc = \cdots f_1^{-1}e_1^{-1}he_1f_1\cdots = \cdots f_1^{-1}hf_1\cdots$ is of length ≥ 3. Hence $c^{-1}hc \neq h$. Therefore $c = e_1 \in E = S_i *_{H_i} H_iH_{i+1}$.

On the other hand, we consider

$$\begin{aligned} E' &= H_iH_{i+1} *_{H_{i+1}} H_{i+1}H_{i+2}, \\ F' &= S_i *_{H_{i-1}} \cdots *_{H_1} S_1 *_{H_0} H_0H_{n-1} *_{H_{n-1}} \cdots *_{H_{i+3}} H_{i+3}H_{i+2} \text{ and} \\ H' &= H_i * H_{i+2}, \end{aligned}$$

where $S_0 = H_0H_{n-1}$ and the subscripts of H_i are taken modulo n. Then $P_i = E' *_{H'} F'$. Since $c^{-1}h'c = h'$ for all $1 \neq h' \in H_{i+1}$, by considering the length of $c^{-1}h'c = h'$ as before, we have $c = e_1 \in E' = H_iH_{i+1} *_{H_{i+1}} H_{i+1}H_{i+2}$. Hence $c \in E \cap E' = H_iH_{i+1}$, as required. This proves the condition (C1) in Theorem 2.7.

To prove (C2) in Theorem 2.7, let $1 \neq a \in H_{i-2}$. If $a \sim_{P_i} x$ for $x \in H_iH_{i+1}$, then $a = x$ by Lemma 4.7 [9]. Since $H_{i-2} \cap H_iH_{i+1} = 1$, it is impossible. Thus $\{a\}^{P_i} \cap H_iH_{i+1} = \emptyset$.

Suppose $u^{-1}au = h'ah$, where $1 \neq a \in H_{i-2}$, $u \in P_i$, and $h', h \in H_iH_{i+1}$. Then $a \sim_{P_i} ahh'$. Since $a \in H_{i-2}$, $\{a\}^{P_i} \cap (H_{i-1} * H_{i+1}) = \emptyset$. Hence $a \in F$ has the minimal length 1 in its conjugacy class in $P_i = E *_H F$, where E, F, H are as above. It follows from Theorem 2.6 that $ahh' \in F$ and $a \sim_F ahh'$. Since $a \in F$, $hh' \in F \cap H_iH_{i+1} = H_{i+1}$. Let $\overline{F} = F/M$, where

$$M = \langle S_{i-2}, \ldots, S_1, H_0H_{n-1}, H_{n-1}H_{n-2}, \ldots, H_{i+3}H_{i+2}\rangle^F,$$

where $S_0 = H_0H_{n-1}$, $S_{-1} = H_{n-1}H_{n-2}$, $S_{-2} = H_{n-2}H_{n-3}$ and the subscripts of H_i are taken modulo n. Then $\overline{F} = F/M \cong (S_{i-1}/H_{i-2}) * \overline{H}_{i+1}$. In $\overline{F}$, $1 = \overline{a} \sim_{\overline{F}} \overline{a}\overline{hh'}$. It follows that $\overline{hh'} = 1$. Since $hh' \in H_{i+1}$, $hh' = 1$. The condition (C2) in Theorem 2.7 holds.

Therefore $P_{i+1} = P_i *_{H_iH_{i+1}} S_{i+1}$ has Property E. Inductively, $G = P_n$ has Property E. □

Since polygonal products of finitely generated central subgroup separable and conjugacy separable groups, amalgamating central subgroups with trivial intersections, are conjugacy separable [9], we have the following by Theorem 2.4:

Theorem 5.6. *Let G be a polygonal product of finitely generated central subgroup separable and conjugacy separable groups $S_1, S_2, \ldots, S_n$ $(n > 3)$, amalgamating central subgroups $H_1, H_2, \ldots, H_0$, with trivial intersections. Then* Out G *is residually finite.*

In particular we have the following:

Theorem 5.7. *Let G be a polygonal product of polycyclic-by-finite groups $S_1, S_2, \ldots, S_n$ $(n > 3)$, amalgamating central subgroups $H_1, H_2, \ldots, H_0$, with trivial intersections. Then* Out G *is residually finite.*

Corollary 5.8. *Let G be a polygonal product of finitely generated abelian groups $S_1, S_2, \ldots, S_n$ $(n > 3)$, amalgamating any subgroups $H_1, H_2, \ldots, H_0$, with trivial intersections. Then* Out G *is residually finite.*

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