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THE ZERO-DIVISOR GRAPH UNDER GROUP ACTIONS IN A NONCOMMUTATIVE RING

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Abstract. Let R be a ring with identity, X the set of all nonzero, nonunits of R and G the group of all units of R. First, we investigate some connected conditions of the zero-divisor graph $\Gamma(R)$ of a noncommutative ring R as follows: (1) if $\Gamma(R)$ has no sources and no sinks, then $\Gamma(R)$ is connected and diameter of $\Gamma(R)$, denoted by $\mathrm{diam}(\Gamma(R))$ (resp. girth of $\Gamma(R)$, denoted by $g(\Gamma(R))$) is equal to or less than 3; (2) if X is a union of finite number of orbits under the left (resp. right) regular action on X by G, then $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3, in addition, if R is local, then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertices in $\Gamma(R)$; (3) if R is unit-regular, then $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3. Next, we investigate the graph automorphisms group of $\Gamma(\mathrm{Mat}_2(\mathbb{Z}_p))$ where $\mathrm{Mat}_2(\mathbb{Z}_p)$ is the ring of 2 by 2 matrices over the galois field $\mathbb{Z}_p$ (p is any prime).

1. Introduction and basic definitions

The zero-divisor graph of a commutative ring has been studied extensively by Akbari, Anderson, Frazier, Lauve, Livinston, and Maimani in [1, 2, 3] since its concept had been introduced by Beck in [4]. Recently, the zero-divisor graph of a noncommutative ring (resp. a semigroup) has also been studied by Redmond and Wu (resp. F. DeMeyer and L. DeMeyer) in [12, 13, 14] (resp. [6]). The zero-divisor graph is very useful to find the algebraic structures and properties of rings. In this paper, the zero-divisor graph of a noncommutative ring is also studied by considering some group actions.

Throughout this paper all rings are assumed to be rings with identity. For a ring R, let $Z_\ell(R)$ (resp. $Z_r(R)$) be the set of all left (resp. right) zero-divisors of R, $Z(R) = Z_\ell(R) \cup Z_r(R)$ and $\Gamma(R)$ be the zero-divisor graph of R consisting of all vertices in $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero left or right zero-divisors of R, and edges $x \longrightarrow y$, which means that $xy = 0$ for $x, y \in Z(R)^*$. If there exist vertices $x_0, \ldots, x_n \in Z(R)^*$ such that P:

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$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n$ where $x_i \neq x_j$ for all $i, j = 0, 1, \ldots, n$ $(i \neq j)$ for some positive integer n, then P is called a *path* from x_0 to x_n of length n. We will denote $d(x, y)$ by the length of the shortest path from x to y, otherwise, $d(x, y) = \infty$. Recall that $\Gamma(R)$ is *connected* if for all distinct vertices $x, y \in Z(R)^*$ there exists a path from x to y. The *diameter* of $\Gamma(R)$ (denoted by $\text{diam}(\Gamma(R))$) is defined by the supremum of $d(x, y)$ for all distinct vertices x and y in $\Gamma(R)$. In particular, if $x = y$ and $d(x, x) = k$, then the path is called the *cycle* of length k. Usually vertices of a path may be considered to be distinct, however in a cycle, the initial and the final vertices are the same. If $\Gamma(R)$ contains a cycle, then the *girth* of $\Gamma(R)$ (denoted by $g(\Gamma(R))$) is defined by the length of the shortest cycle in $\Gamma(R)$, otherwise, $g(\Gamma(R)) = \infty$. In [7, Proposition 1.3.2], if $\Gamma(R)$ contains a cycle, then $1 + 2\text{diam}(\Gamma(R)) \geq g(\Gamma(R))$. We say that $\Gamma(R)$ is *complete* if $xy = 0$ for any distinct vertices x, y in $\Gamma(R)$.

For a ring R, let $X(R)$ (simply, denoted by X) be the set of all nonzero, nonunits of R, $G(R)$ (simply, denoted by G) be the group of all units of R and $J(R)$ (simply, denoted by J) be the Jacobson radical of R. In this paper, we will consider some group actions on X by G given by $(g, x) \longrightarrow gx$ (resp. $(g, x) \longrightarrow xg^{-1}$) from $G \times X$ to X, called the left (resp. right) regular action. If $\phi : G \times X \longrightarrow X$ is the left (resp. right) regular action, then for each $x \in X$, we define the *orbit* of x by $o_\ell(x) = \{\phi(g, x) = gx : \forall g \in G\}$ (resp. $o_r(x) = \{\phi(g, x) = xg^{-1} : \forall g \in G\}$). Recall that G is *transitive* on X (or G acts transitively on X) under the regular action on X by G if there is an $x \in X$ with $o_\ell(x) = X$ (resp. $o_r(x) = X$) and the left (resp. right) regular action on X by G is *trivial* if $o_\ell(x) = \{x\}$ (resp. $o_r(x) = \{x\}$) for all $x \in X$. In [8], it has been shown that if X is a union of a finite n number of orbits under the left regular action on X by G, then $x^{n+1} = 0$ for all $x \in J$ and X is the set of all nonzero right zero-divisors of R. Similarly, it is also shown that if X is a union of a finite n number of orbits under the right regular action on X by G, then $x^{n+1} = 0$ for all $x \in J$ and X is the set of all nonzero left zero-divisors of R.

Recall that for all $x \in X$ the set $\text{ann}_\ell(x) = \{y \in X : yx = 0\}$ (resp. $\text{ann}_r(x) = \{z \in X : xz = 0\}$) is called a left (resp. right) annihilator of x. Let $\text{ann}_\ell^*(x) = \text{ann}_\ell(x) \setminus \{0\}$ (resp. $\text{ann}_r^*(x) = \text{ann}_r(x) \setminus \{0\}$). Given a zero-divisor graph $\Gamma(R)$ and a vertex $x \in Z(R)^*$, the *indegree* (resp. *outdegree*) of x (denoted by $\text{indegree}(x)$ (resp. $\text{outdegree}(x)$)) is the number of edges arriving (resp. leaving) at x. That is, $\text{indegree}(x) = |\text{ann}_\ell^*(x)|$ (resp. $\text{outdegree}(x) = |\text{ann}_r^*(x)|$). A vertex of indegree 0 (resp. outdegree 0) is called a *source* (resp. *sink*).

In Section 2, some connected conditions of the zero-divisor graph of a noncommutative ring R are investigated as follows: (1) if $\Gamma(R)$ has no sources and no sinks, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3; (2) if X is a union of finite number of orbits under the left (resp. right) regular action on X by G, then $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3, in addition, if R is a local ring, then there exists a vertex of $\Gamma(R)$ which is adjacent to every other vertices in $\Gamma(R)$;

(4) if R is a unit-regular ring, then $\Gamma(R)$ is connected and $\operatorname{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.

In [3], Anderson and Livingston have shown that distinct ring automorphisms of a finite ring R which is not a field induce distinct graph automorphisms of $\Gamma(R)$ and determined $\operatorname{Aut}(\Gamma(R))$, the graph automorphisms group of $\Gamma(R)$. In particular, they have computed $\operatorname{Aut}(\Gamma(\mathbb{Z}_n))$.

In Section 3, when $R = \operatorname{Mat}_2(\mathbb{Z}_p)$, the ring of 2 by 2 matrices over the Galois field $\mathbb{Z}_p$ (p is any prime), we will show that $\operatorname{Aut}(\Gamma(R))$ is isomorphic to the group S_{p+1}, the symmetric group of degree $p+1$ by investigating that (1) the number of orbits under the left (resp. right) regular action on X by G is $p+1$; (2) the number of nonzero nilpotents in R is p^2-1; (3) $\operatorname{Aut}(\Gamma(R)) \neq \{1\}$; (4) under the left (resp. right) regular action on X by G, $o_\ell(a) \cap N(p) = o_r(a) \cap N(p) = o_\ell(a) \cap o_r(a)$ for all $a \in N(p)$ where $N(p)$ is the set of all nonzero nilpotents in R.

2. Connected zero-divisor graph under the left (resp. right) regular action

For a subset S of $Z(R)^*$, we will denote the subgraph of $\Gamma(R)$ with vertices in S by $\Gamma_S(R)$.

Proposition 2.1. *Let R be a ring. If the left (or right) regular action of G on X is transitive, then $\Gamma_X(R)$ is complete.*

Proof. Since the left regular action of G on X is transitive, R is a local ring and $J^2 = 0$ by [8, Corollary 2.4], and so $Z(R)^* = X$ and $\Gamma_X(R)$ is complete. If the right regular action of G on X is transitive, then $Z(R)^* = X$ and $\Gamma_X(R)$ is also complete by the similar argument. □

Remark 1. In Proposition 2.1, we see that if the left (resp. right) regular action on X by G is transitive, then $x^2 = 0$, i.e., x is a nilpotent element of nilpotency 2 for all $x \in X$.

Theorem 2.2. *Let R be a ring. If $\Gamma(R)$ has no sources and no sinks, then $\Gamma(R)$ is connected and $\operatorname{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3.*

Proof. Let $x, y \in Z(R)^* (x \neq y)$ be arbitrary. Since $\Gamma(R)$ has no sources and no sinks, i.e., $\operatorname{ann}_\ell^*(x) \neq \emptyset$ (resp. $\operatorname{ann}_r^*(x) \neq \emptyset$), there exists an element $a \in X$ (resp. $b \in X$) such that $xa = 0$ (resp. $by = 0$). If $ab = 0$, then $x \longrightarrow a \longrightarrow b \longrightarrow y$ is a path of length 3. If $ab \neq 0$, then $x \longrightarrow ab \longrightarrow y$ is a path of length 2. In particular, if we let $x = y$, then $g(\Gamma(R))$ is equal to or less than 3. □

Example 1 (See Example 1.5, p. 5 in [5])**.** Let

$$R = \left\{ \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix} \right\} \text{ and take } a = \begin{pmatrix} 2 & 0 \\ 0 & \bar{1} \end{pmatrix} \in R.$$

Since the left annihilator of a is equal to $\{0\}$ but the right annihilator of a is not equal to $\{0\}$, a is not a left zero-divisor, and so a is an origin but a is a right zero-divisor. Since there is no path from a to a^2, $\Gamma(R)$ is not connected.

Let
$$S = \left\{ \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix} \right\} \text{ and take } c = \begin{pmatrix} 2 & 0 \\ 0 & \bar{1} \end{pmatrix} \in S.$$
Similarly, we note that c is not a right zero-divisor, and so c is a sink but c is a left zero-divisor. Since there is also no path from c^2 to c, $\Gamma(S)$ is not connected.

Remark 2. In [3, Theorem 2.3], Anderson and Livingston have shown that for every commutative ring R, $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R))$ is equal to or less than 3. But by Example 1 we can note that there is a noncommutative ring in which its zero-divisor graph is not connected and also note that the condition [there are no sources and no sinks in the zero-divisor graph of a noncommutative ring] is not superfluous to be connected.

Theorem 2.3. *Let R be a ring such that X is a union of finite number of orbits under the left and right regular action on X by G. Then $X = Z^*(R)$, and so $\Gamma_X(R)$ is connected and* $\text{diam}(\Gamma_X(R))$ (*resp.* $g(\Gamma(R))$) *is equal to or less than 3.*

Proof. Since X is a union of finite number of orbits under the left regular action on X by G, then $Z_\ell^*(R) \subseteq Z_r^*(R) = X$ by [8, Theorem 2.2]. Similarly, we can show that if X is a union of finite number of orbits under the right regular action on X by G, then $Z_r^*(R) \subseteq Z_\ell^*(R) = X$. Thus $Z^*(R) = Z_\ell^*(R) = Z_r^*(R) = X$, which implies that $\Gamma(R)$ has no sources and no sinks, and so $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$(resp. $g(\Gamma(R))$) is equal to or less than 3 by Theorem 2.2. □

Corollary 2.4. *Let R be a ring such that $X \neq \emptyset$. If X is finite, then $X = Z^*(R)$, and so R is finite and $(|X|+1)^2 \geq |R|$.*

Proof. Since $X \neq \emptyset$ and is finite, X is a union of finite number of orbits under the left and right regular action on X by G, and so we have $X = Z^*(R)$ by the argument given in the proof of Theorem 2.3. Hence R is finite and then $(|X|+1)^2 \geq |R|$ by [11, Theorem I].

□

Corollary 2.5. *Let R be a finite ring. Then $\Gamma_X(R)$ is connected and*
$$\text{diam}(\Gamma_X(R))$$
(*resp.* $g(\Gamma(R))$) *is equal to or less than 3.*

Proof. Since R is a finite ring, X is a union of finite number of orbits under the left and right regular action on X by G. Hence it follows from Theorem 2.3. □

Proposition 2.6. *Let n be any positive integer and R be the matrix ring of all $n \times n$ matrices over a division ring D. Then $X = Z^*(R)$, and so $\Gamma_X(R)$ is connected and* $\text{diam}(\Gamma_X(R))$ (*resp.* $g(\Gamma(R))$) *is equal to or less than 3.*

Proof. Let $x \in X$ be arbitrary. Then there exists $y \in X$ (resp. $z \in X$) such that $xy = 0$ (resp. $zx = 0$), which implies that $\mathrm{ann}_r^*(x) \neq \emptyset$ (resp. $\mathrm{ann}_\ell^*(x) \neq \emptyset$) for all $x \in X$, i.e., $X = Z^*(R)$. Hence $\Gamma_X(R)$ is connected and $\mathrm{diam}(\Gamma(R))$ (resp. $g(\Gamma(R))$) is equal to or less than 3 by Theorem 2.2. □

Lemma 2.7. *Let R and S be two rings. If $\Gamma(R)$ and $\Gamma(S)$ have no sources (resp. no sinks), then $\Gamma(R \times S)$ has no sources (resp. no sinks).*

Proof. Let $(x_R, x_S) \in Z^*(R \times S)$ be arbitrary. Then $x_R \in Z^*(R)$ or $x_S \in Z^*(S)$. If $x_R \in Z^*(R)$, then there is $y_R \in X(R)$ such that $y_R x_R = 0_R$ where 0_R is the additivite identity of R since $\Gamma(R)$ has no origins. Thus $(y_R, 0_S)(x_R, x_S) = (0_R, 0_S)$ where 0_S is the additivite identity of S, and so $\Gamma(R \times S)$ has no sources. Similarly, if $x_S \in Z^*(S)$, then $\Gamma(R \times S)$ has no sources. By the similar argument, if $\Gamma(R)$ and $\Gamma(S)$ have no sinks, then $\Gamma(R \times S)$ has no sinks. □

Corollary 2.8. *Let $R_1, R_2, \ldots, R_n$ be rings for some positive integer n. If all $\Gamma(R_i)$ for $i = 1, 2, \ldots, n$ have no sources (resp. sinks), then $\Gamma(R_1 \times R_2 \times \cdots \times R_n)$ has no sources (resp. no sinks).*

Proof. It follows from the Lemma 2.7 and the mathematical induction on n. □

Proposition 2.9. *Let R be a ring with $X = o_r(x) \cup o_r(x^2) \cup \cdots \cup o_r(x^n)$ (resp. $X = o_\ell(x) \cup o_\ell(x^2) \cup \cdots \cup o_\ell(x^n)$) under the right (resp. left) regular action on X by G for some positive integer n. If $n = 1$ and $|X| \geq 3$, or $n = 2$ and $o_r(x^2) \neq \{x^2\}$, or $n = 3$ and $o_r(x^i) \neq \{x^i\}$ for some $i = 2$ or 3, or $n \geq 4$, then there exists a cycle of length 3 in $\Gamma(R)$.*

Proof. Consider the right regular action of G on X. If $n = 1$, right regular action is transitive, then $\Gamma(R)$ is complete by Proposition 2.1. Since $|X| \geq 3$, there exists a cycle of length 3 in $\Gamma(R)$. If $n = 2$ and $o_r(x^2) \neq \{x^2\}$, then there exists $g \in G$ such that $x^2 g \neq x^2$. Since $X = o(x) \cup o(x^2)$ and $x^2 g \in X$, $x^2 g = hx$ or hx^2 for some $h \in G$. Thus $x^2 \longrightarrow x \longrightarrow x^2 g \longrightarrow x^2$ is a cycle of length 3. If $n = 3$ and $o_r(x^i) \neq \{x^i\}$ for some $i = 2$ or 3, then there exists $g \in G$ such that $x^i g \neq x^i$. Since $X = o(x) \cup o(x^2) \cup o(x^3)$ and $x^i g \in X$, $x^i g = hx$ or hx^2 or $hx^3$for some $h \in G$. Thus $x^3 \longrightarrow x^2 \longrightarrow x^i g \longrightarrow x^3$ is a cycle of length 3. Finally, if $n \geq 4$, then clearly $x^{n-2} \longrightarrow x^{n-1} \longrightarrow x^n \longrightarrow x^{n-2}$ is a cycle of length 3. Similarly, the result holds under the left regular action of G on X. □

Remark 3. Let R be a ring. Then for each $x \in X$, $\mathrm{ann}_\ell^*(x)$ (resp. $\mathrm{ann}_r^*(x)$) is a union of orbits under the left (resp. right) regular action on X by G. Indeed, let $y \in \mathrm{ann}_\ell^*(x)$ be arbitrary. Then we have $o_\ell(y) \subseteq \mathrm{ann}_\ell^*(x)$, and so $\bigcup_{y \in \mathrm{ann}_\ell^*(x)} o_\ell(y) \subseteq \mathrm{ann}_\ell^*(x)$. Clearly, $\mathrm{ann}_\ell^*(x) \subseteq \bigcup_{y \in \mathrm{ann}_\ell^*(x)} o_\ell(y)$. Hence $\mathrm{ann}_\ell^*(x) = \bigcup_{y \in \mathrm{ann}_\ell^*} o_\ell^*(y)$, i.e., $\mathrm{ann}_\ell^*(x)$ is a union of orbits under the left regular action on X by G. By the similar argument, $\mathrm{ann}_r^*(x)$ is a union of orbits under the right regular action on X by G.

Theorem 2.10. *Let R be a ring such that X is a union of finite number of orbits under the left (resp. right) regular action on X by G. If R is a local ring, then there is a vertex of $\Gamma_X(R)$ which is adjacent to every other vertex in $\Gamma_X(R)$.*

Proof. Let X be a union of n orbits under the left (resp. right) regular action on X by G. Since R is a local ring, by [8, Lemma 2.3] there exists $x \in X$ such that $x^n \neq 0 = x^{n+1}$ and $X = o_\ell(x) \cup o_\ell(x^2) \cup \cdots \cup o_\ell(x^n)$. Hence we have $\text{ann}_\ell(x^n) = X$, i.e., $a \longrightarrow x^n$ for all $a \in X$, which means that x^n is adjacent to every other vertex in $\Gamma_X(R)$. By the similar argument, we can show that if X is a union of n orbits under the right regular action on X by G, then there exists $y \in X$ such that $y^n \neq 0 = y^{n+1}$ and $X = o_r(y) \cup o_r(y^2) \cup \cdots \cup o_r(y^n)$. Thus $\text{ann}_r(y^n) = X$, i.e., $y^n \longrightarrow b$ for all $b \in X$, which means that y^n is adjacent to every other vertex in $\Gamma_X(R)$. □

Remark 4. We note that in the proof of Theorem 2.11 if R is a local ring such that $X = o_\ell(x) \cup o_\ell(x^2) \cup \cdots \cup o_\ell(x^n)$ (resp. $X = o_r(x) \cup o_r(x^2) \cup \cdots \cup o_r(x^n)$) with $x^n \neq 0 = x^{n+1}$ under the left (resp. right) regular action on X by G, then the subgraph $\Gamma_{o_\ell(x^n)}$ (resp. $\Gamma_{o_r(x^n)}$) of $\Gamma_X(R)$ is complete.

Corollary 2.11. *If R is a finite local ring, then there is a vertex of $\Gamma_X(R)$ which is adjacent to every other vertex in $\Gamma_X(R)$.*

Proof. Since R is a finite ring, X is a union of finite number of orbits under the left and right regular action on X by G. Hence it follows from Theorem 2.10. □

Recall that a ring R is called *unit-regular* if for every $x \in R$ there exists a unit $g \in R$ such that $xgx = x$. In [10], it has been shown that R is a unit-regular ring if and only if for every orbit $o_\ell(x)$ ($x \in X$) under the left regular action on X by G, there exists some idempotent $e \in X$ such that $o_\ell(x) = o_\ell(e)$. Similarly, we can show that R is a unit-regular ring if and only if for every orbit $o_r(x)$ ($x \in X$) under the right regular action of G on X, there exists some idempotent $e \in X$ such that $o_r(x) = o_r(e)$.

Proposition 2.12. *Let R be a unit-regular ring such that $X \neq \emptyset$. Then $\Gamma_X(R)$ is connected and* $\text{diam}(\Gamma_X(R))$ *(resp.* $g(\Gamma(R))$*) is equal to or less than 3.*

Proof. Let $x \in X$ be arbitrary. Then there exists an idempotent $e_1 \in X$ such that $o_\ell(x) = o_\ell(e_1)$ under the left regular action on X by G by [10, Lemma 2.3]. By the similar argument, there exists an idempotent $e_2 \in X$ such that $o_r(x) = o_r(e_2)$ under the right regular action on X by G. Hence there exists $g_1 \in G$ (resp. $g_2 \in G$) such that $x = g_1 e_1$ (resp. $x = e_2 g_2$). Since $x(1 - e_1) = g_1 e_1 (1 - e_1) = 0$ (resp. $(1 - e_2)x = (1 - e_2)e_2 g_2 = 0$, x is neither source nor sink. Thus $\Gamma_X(R)$ is connected and $\text{diam}(\Gamma_X(R))$ is equal to or less than 3 by Theorem 2.2. □

Proposition 2.13. *Let R be a unit-regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in R is orthogonal and the left regular action on X by G is trivial, i.e., $o_\ell(x) = \{x\}$ for all $x \in X$.*

Proof. ($\Rightarrow$) Suppose that $\Gamma_X(R)$ is complete. Clearly, the set of all idempotents in R is orthogonal. Assume that the left regular action of G on X is not trivial. Then there exists an idempotent $e \in X$ such that $o_\ell(e) \neq \{e\}$ by [10, Lemma 2.3] and so there exists $y(\neq e) \in o_\ell(e)$ such that $y = ge$ for some $g \in G$. Since $\Gamma_X(R)$ is complete and $y, e(y \neq e) \in X$, $0 = ye = (ge)e = ge = y$, a contradiction. Hence the left regular action on X by G is trivial.

($\Leftarrow$) It follows from [10, Lemma 2.3]. □

Corollary 2.14. *Let R be a unit-regular ring. Then $\Gamma_X(R)$ is complete if and only if the set of all idempotents in R is orthogonal and the right regular action on X by G is trivial, i.e., $o_r(x) = \{x\}$ for all $x \in X$.*

Proof. It follows from the similar argument given in the proof of Proposition 2.13. □

Lemma 2.15. *Let R be a ring. If under the left (resp. right) regular action on X by G, $y \in o_\ell(x)$ (resp. $y \in o_r(x)$) for some $x \in X$, then $\mathrm{ann}_r(x) = \mathrm{ann}_r(y)$ (resp. $\mathrm{ann}_\ell(x) = \mathrm{ann}_\ell(y)$).*

Proof. If $y \in o_\ell(x)$ (resp. $y \in o_r(x)$) for some $x \in X$, then there exists $g \in G$ (resp. $h \in G$) such that $y = gx$ (resp. $y = xh$). It is obvious to show that $\mathrm{ann}_r(x) = \mathrm{ann}_r(y)$ (resp. $\mathrm{ann}_\ell(x) = \mathrm{ann}_\ell(y)$). □

Corollary 2.16. *Let R be a unit-regular ring with $X \neq \emptyset$. Then for any $x \in X$ there exists an idempotent $e \in X$ such that $\mathrm{ann}_r(x) = \mathrm{ann}_r(e)$ (resp. $\mathrm{ann}_\ell(x) = \mathrm{ann}_\ell(e)$).*

Proof. It follows from the Lemma 2.15 and [10, Lemma 2.3]. □

Proposition 2.17. *Let R be a unit-regular ring such that $X \neq \emptyset$ and $2 = 2\cdot 1$ is a unit in R. Then there exists a cycle of length 4 in $\Gamma(R)$.*

Proof. Let $e \in X$ be an idempotent. Since $2 = 2 \cdot 1 \in G$, $e \neq 1 - e, -e$. Thus $e \longrightarrow 1 - e \longrightarrow -e \longrightarrow e - 1 \longrightarrow e$ is a cycle of length 4 in $\Gamma(R)$. □

3. Automorphism of graph over $\mathrm{Mat}_2(\mathbb{Z}_p)$

Recall that a *graph automorphism* f of a graph $\Gamma(R)$ is a bijection $f : \Gamma(R) \longrightarrow \Gamma_X(R)$ which preserves adjacency. Of course, the set $\mathrm{Aut}(\Gamma(R))$ of all graph automorphisms of $\Gamma(R)$ forms a group under the usual composition of functions. In [3], Anderson and Livingston computed $\mathrm{Aut}(\Gamma(\mathbb{Z}_n))$. In this section, we compute $\mathrm{Aut}(\Gamma(\mathrm{Mat}_2(\mathbb{Z}_p))$ where $\mathrm{Mat}_2(\mathbb{Z}_p)$ is the matrix ring of all 2×2 matrices over $\mathbb{Z}_p$ for any prime p.

Lemma 3.1. *Let R be a ring and $f : \Gamma_X(R) \longrightarrow \Gamma_X(R)$ be a graph automorphism of $\Gamma_X(R)$. Then for all $x \in X$, $f(\mathrm{ann}_\ell(x)) = \mathrm{ann}_\ell(f(x))$ (resp. $f(\mathrm{ann}_r(x)) = \mathrm{ann}_r(f(x))$).*

Proof. Let $y \in f(\mathrm{ann}_\ell(x))$ be arbitrary. Then $y = f(z)$ for some $z \in \mathrm{ann}_\ell(x)$. Since $zx = 0$, $0 = f(zx) = f(z)f(x) = yf(x)$ and so $y \in \mathrm{ann}_\ell(f(x))$. Hence $f(\mathrm{ann}_\ell(x)) \subseteq \mathrm{ann}_\ell(f(x))$. Let $z \in \mathrm{ann}_\ell(f(x))$ be arbitrary. Then $zf(x) = 0$. Since f is one to one, there exists $z_1 \in X$ such that $f(z_1) = z$. Then $0 = zf(x) = f(z_1)f(x) = f(z_1x)$, and so $z_1x = 0$. Since $z_1 \in \mathrm{ann}_\ell(x)$ and $z = f(z_1) \in f(\mathrm{ann}_\ell(x))$, $\mathrm{ann}_\ell(f(x)) \subseteq f(\mathrm{ann}_\ell(x))$. By the similar argument, we have $f(\mathrm{ann}_r(x)) = \mathrm{ann}_r(f(x))$. □

In a ring R with identity the left (resp. right) regular action of G on X is said to be *half-transitive* if G is transitive on X or if $o_\ell(x)$(resp. $o_r(x)$) is a finite set with $|o_\ell(x)| > 1$ (resp. $|o_r(x)| > 1$) and $|o_\ell(x)| = |o_\ell(y)|$ (resp. $|o_r(x)| = |o_r(y)|$) for all x and $y \in X$. In [9, Theorem 2.4 and Lemma 2.7], it was shown that if R is a matrix ring of all 2×2 matrices over a finite field F, then G is half-transitive on X by the left (resp. right) regular action and $|o_\ell(x)| = |F|^2 - 1$ (resp. $|o_r(x)| = |F|^2 - 1$) for all $x \in X$.

Lemma 3.2. *Let p be a prime and $R = \mathrm{Mat}_2(\mathbb{Z}_p)$. Then for any $x \in X$, $\mathrm{ann}_\ell^*(x) = o_r(y)$ (resp. $\mathrm{ann}_r^*(x)) = o_\ell(z)$) for some $y \in X$ (resp. $z \in X$).*

Proof. By [9, Lemma 2.7], we have $|o_\ell(x)| = p^2 - 1$ (resp. $|o_r(x)| = p^2 - 1$) for all $x \in X$. Since $\mathrm{ann}_\ell^*(x)$ (resp. $\mathrm{ann}_r^*(x)$) is a union of a finite number of orbits under the left (resp. right) regular action of G on X by Remark 3 and since the left (resp. right) regular action of G on X is half-transitive by [9, Theorem 2.4], $|o_\ell(y)|$ (resp.$|o_r(z)|$) for all $y \in \mathrm{ann}_\ell^*(x)$ (resp. all $z \in \mathrm{ann}_r^*(x)$) is a divisor of $|\mathrm{ann}_\ell^*(x)|$ (resp. $|\mathrm{ann}_r^*(x)|$) and then $|\mathrm{ann}_\ell^*(x)| = p^2 - 1$ or $p^3 - 1$ (resp. $|\mathrm{ann}_r^*(x)| = p^2 - 1$ or $p^3 - 1$) since $|\mathrm{ann}_l(x)| = p^2$ or p^3 (resp. $|\mathrm{ann}_r(x)| = p^2$ or p^3) and so $|\mathrm{ann}_\ell^*(x)| = p^2 - 1$ (resp. $|\mathrm{ann}_r^*(x) = p^2 - 1$). Hence we have the result. □

Lemma 3.3. *Let p be a prime and $R = \mathrm{Mat}_2(\mathbb{Z}_p)$. Then the number of orbits under the left (resp. right) regular action on X by G is $p + 1$.*

Proof. Let μ be the number of orbits under the left (resp. right) regular action on X by G. Note that $|G| = (p^2 - 1)(p^2 - p)$. Thus $|X| = |R| - |G| - 1 = p^4 - (p^2 - 1)(p^2 - p) - 1 = (p+1)(p^2 - 1)$. Since the cardinality of any orbit under the left (resp. right) regular action on X by G is $p^2 - 1$ by [9, Lemma 2.7], $\mu = |X|/(p^2 - 1) = p + 1$. □

Lemma 3.4. *Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and let $N(p)$ be the set of nonzero nilpotents in R. Then $|N(p)| = p^2 - 1$.*

Proof. Let

$$N_1(p) = \left\{ \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \in N(p) \mid a, b, \alpha \neq 0 \right\}$$

and

$$N_2(p) = \left\{ \begin{pmatrix} a & -\alpha a \\ b & -\alpha b \end{pmatrix} \in N(p) \,|\, a, b, \alpha \neq 0 \right\}.$$

We will show that $N_1(p) = N_2(p)$. Let

$$\begin{pmatrix} a & -\alpha a \\ b & -\alpha b \end{pmatrix} \in N_2(p)$$

be arbitrary. Since $A^2 = 0$ and $a, b \neq 0$, we have

$$A = \begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} \in N_2(p),$$

and also $(1/\alpha^2)\begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} \in N_2(p)$.

Since

$$(1/\alpha^2)\begin{pmatrix} \alpha b & -\alpha^2 b \\ b & -\alpha b \end{pmatrix} = \begin{pmatrix} (-1/\alpha)(-b) & -b \\ (-1/\alpha^2)(-b) & (-1/\alpha)(-b) \end{pmatrix} \in N_1(p),$$

we have $N_2(p) \subseteq N_1(p)$. By the similar argument, we can have $N_1(p) \subseteq N_2(p)$.

Let A be any nonzero nilpotent in R. Then

$$A = \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} \text{ or } \begin{pmatrix} a & \alpha a \\ a & \alpha b \end{pmatrix}$$

for some $\alpha \in \mathbb{Z}_p$.

Note that since A is a nonzero nilpotent in R, $b \neq 0$. Consider the following cases:

Case 1. $\alpha = 0$;
Since

$$A^2 = 0,\ A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

for all nonzero $b \in \mathbb{Z}_p$.

Case 2. $\alpha \neq 0$;
In this case, $a \neq 0$. Hence we have $N_1(p) = N_2(p)$ by the above argument. Since $A^2 = 0$, we have $A = \begin{pmatrix} -\alpha b & b \\ -\alpha^2 b & \alpha b \end{pmatrix}$.

Consequently, we have

$$\begin{aligned} |N(p)| &= |N_1(p)| + \left| \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in N(p) : b(\neq 0) \right\} \right| \\ &\quad + \left| \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in N(p) : b(\neq 0) \right\} \right| \\ &= (p-1)(p-1) + 2(p-1) = p^2 - 1. \end{aligned}$$

□

Example 2. Let $R = \mathrm{Mat}_2(\mathbb{Z}_2)$. Then $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, where

$$x_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$x_5 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, x_6 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, x_7 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, x_8 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, x_9 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that $\{x_2, x_4, x_9\}$ is the set of nonzero nilpotents in R. Under the left (resp. right) regular action on X by G, there are three orbits $o_\ell(x_2) = \{x_2, x_6, x_7\}$, $o_\ell(x_4) = \{x_1, x_4, x_5\}$, $o_\ell(x_9) = \{x_3, x_8, x_9\}$ (resp. $o_r(x_2) = \{x_1, x_2, x_3\}$, $o_r(x_4) = \{x_4, x_6, x_8\}$, $o_r(x_9) = \{x_5, x_7, x_9\}$).

We can compute Aut$(\Gamma(R))$= $\{1, f, g, g \circ f, f \circ g, g \circ f \circ g\}$, where

$f = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_3 & x_2 & x_1 & x_9 & x_7 & x_5 & x_8 & x_6 & x_4 \end{pmatrix}$,

$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_6 & x_4 & x_8 & x_2 & x_1 & x_3 & x_7 & x_5 & x_9 \end{pmatrix}$ are permutations.

Observe that Aut$(\Gamma(R))$ is isomorphic to S_3, the symmetric group of degree 3.

Theorem 3.5. *Let p be a prime and $R = \mathrm{Mat}_2(\mathbb{Z}_p)$. Then* Aut$(\Gamma(R))$ $\neq \{1\}$.

Proof. If $p = 2$, then Aut$(\Gamma(R)) \neq \{1\}$ by Example 2. Suppose that $p \geq 3$. Let $N(p)$ be the set of nonzero nilpotents in R. Since the number orbits is $p + 1$ by Lemma 3.3 under the left (resp. right) regular action on X by G and $|N(p)| = p^2 - 1$ by Lemma 3.4, there exists $x \in X$ such that $|o_\ell(x) \cap N(p)| \geq 2$. Let $x_1, x_2 \in o_l(x) \cap N(p)$ $(x_1 \neq x_2)$. Since x_1 and x_2 are nilpotents, we have $\mathrm{ann}^*_\ell(x_1) = o_\ell(x_1) = o_\ell(x_2) = \mathrm{ann}^*_\ell(x_2)$ by Lemma 3.2. We have also $\mathrm{ann}^*_r(x_1) = \mathrm{ann}^*_r(x_2)$. Indeed, if $a \in \mathrm{ann}^*_r(x_1)$, then $0 = x_1 a = g x_2 a = 0$ for some $g \in G$ since $x_2 \in o_\ell(x_1)$, which implies that $a \in \mathrm{ann}^*_r(x_2)$, and so $\mathrm{ann}^*_r(x_1) \subseteq \mathrm{ann}^*_r(x_2)$. By the similar argument, we have $\mathrm{ann}^*_r(x_2) \subseteq \mathrm{ann}^*_r(x_1)$. Also we have $\mathrm{ann}^*_r(x_1) = o_r(x_1) = o_r(x_2) = \mathrm{ann}^*_r(x_2)$ by Lemma 3.2. Let $f = (x_1, x_2)$ be a transposition in $S_{|X|}$, the symmetric group of degree $|X|$. Since $x_1 \neq x_2$, $f \neq 1$. We will show that $f \in$ Aut$(\Gamma(R))$. Consider $x_1 y = 0$ for some $y \in X$. If $y = x_1$, then $f(x_1)f(y) = x_2 x_2 = 0$. If $y = x_2$, then $f(x_1)f(y) = x_2 x_1 = g_1 x_1 x_1 = 0$ for some $g_1 \in G$ since $x_2 \in o_l(x_1)$. If $y \neq x_1, x_2$, then $f(x_1)f(y) = x_2 y = g_1 x_1 y = 0$ for some $g_1 \in G$ since $x_2 \in o_l(x_1)$. Also consider $z x_1 = 0$ for some $z \in X$. If $z = x_1$, then $f(z)f(x_1) = x_2 x_2 = 0$. If $z = x_2$, then $f(z)f(x_1) = x_1 x_2 = h_1 x_2 x_2 = 0$ for some $h_1 \in G$ since $x_1 \in o(x_2)$. If $z \neq x_1, x_2$, then $f(z)f(x_1) = z x_2 = z x_1 h_2 = 0$ for some $h_2 \in G$ since $x_2 \in o_r(x_1)$. Consequently, $f \in$ Aut$(\Gamma(R))$, and so Aut$(\Gamma(R)) \neq \{1\}$. □

Remark 5. Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and $N(p)$ be the set of nonzero nilpotents in R. We can choose that $f(\neq 1) \in$ Aut$(\Gamma(R))$ by Theorem 3.5. Then we note that (1) $f(a) \in N(p)$ for all $a \in N(p)$; (2) since f is bijective and the left (resp. right) regular action on X by G is half-transitive with $|o_\ell(x)| = p+1$ (resp. $|o_r(x)| = p+1$) for all $x \in X$, $|o_l(x) \cap N(p)| = p-1$ (resp. $|o_r(x) \cap N(p)| = p-1$) and $f(o_\ell(x)) = o_\ell(f(x))$ (resp. $f(o_r(x)) = o_r(f(x))$) by Lemma 3.1 and Lemma 3.2; (3) every orbit under the left (resp. right) regular action on X by G is $o_\ell(x)$ (resp. $o_r(x)$) for some nilpotent $x \in X$.

Lemma 3.6. *Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and $N(p)$ be the set of all nonzero nilpotents in R. Then under the left (resp. right) regular action on X by G, $o_\ell(a) \cap N(p) = o_r(a) \cap N(p) = o_\ell(a) \cap o_r(a)$ for all $a \in N(p)$.*

Proof. Let $b \in o_\ell(a) \cap N(p)$ be arbitrary. Since $o_\ell(a) = o_\ell(b)$, $ba = ab = 0$, and thus $b \in \mathrm{ann}_r^*(a) = o_r(a)$. Hence $o_\ell(a) \cap N(p) \subseteq o_r(a) \cap N(p)$ and $o_\ell(a) \cap N(p) \subseteq o_\ell(a) \cap o_r(a)$. By the similar argument, we have $o_r(a) \cap N(p) \subseteq o_\ell(a) \cap N(p)$ and $o_r(a) \cap N(p) \subseteq o_\ell(a) \cap o_r(a)$. Therefore, $o_\ell(a) \cap N(p) = o_r(a) \cap N(p) \subseteq o_\ell(a) \cap o_r(a)$. By Remark 4, we already knew that $|o_r(a) \cap N(p)| = |o_\ell(a) \cap N(p)| = p-1$. Next, we will show that $o_\ell(a) \cap N(p) = o_\ell(a) \cap o_r(a)$. Let $S = \mathrm{ann}_\ell(a) \cap \mathrm{ann}_r(a)$. Then $S = (o_\ell(a) \cap o_r(a)) \cup \{0\}$. Since S is an additive subgroup of $\mathrm{ann}_\ell(a)$ and $|\mathrm{ann}_\ell(a)| = p^2$, $|S| = 1$ or p. Since $|o_\ell(a) \cap o_r(a)| \geq |o_\ell(a) \cap N(p)| = p-1 \geq 1$, $|S| = |o_\ell(a) \cap o_r(a)| + 1 \geq 2$, and thus $|S| = p$. Since $|o_\ell(a) \cap o_r(a)| = |S| - 1 = p-1 = |o_\ell(a) \cap N(p)| = |o_r(a) \cap N(p)|$ and $o_\ell(a) \cap N(p), o_r(a) \cap N(p) \subseteq o_\ell(a) \cap o_r(a)$, we have $o_\ell(a) \cap N(p) = o_r(a) \cap N(p) = o_\ell(a) \cap o_r(a)$. □

Remark 6. Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and $N(p)$ be the set of nonzero nilpotents in R. We can choose $a_1, \ldots, a_{p+1} \in N(p)$ such that $X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1})$ (resp. $X = o_r(a_1) \cup \cdots \cup o_r(a_{p+1})$). Note that for each $i = 1, \ldots, p+1$, $o_\ell(a_i) = o_\ell(a_i) \cap X = o_\ell(a_i) \cap [o_r(a_1) \cup \cdots \cup o_r(a_{p+1})] = [o_\ell(a_i) \cap o_r(a_1)] \cup \cdots \cup [o_\ell(a_i) \cap o_r(a_{p+1})]$.

Lemma 3.7. *Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and $N(p)$ be the set of nonzero nilpotents in R. Consider $X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1})$ (resp. $X = o_r(a_1) \cup \cdots \cup o_r(a_{p+1})$) for some $a_1, \ldots, a_{p+1} \in N(p)$ as mentioned in Remark 6. Then under the left (resp. right) regular action on X by G, $|o_\ell(a_i) \cap o_r(a_j)| = p-1$ for all $a_i, a_j \in N(p)$ $(i, j = 1, \ldots, p+1)$.*

Proof. Let $A_{ij} = \mathrm{ann}_\ell(a_i) \cap \mathrm{ann}_r(a_j)$ for all $i, j = 1, \ldots, p+1$. Note that $A_{ij} = [o_{ell}(a_i) \cap o_r(a_j)] \cup \{0\}$. If $i = j$, then $|o_l(a_i) \cap o_r(a_j)| = p-1$ as given in the proof of Lemma 3.6. Suppose that $i \neq j$. Since A_{ij} ia an additive subgroup of $\mathrm{ann}_\ell(a_i)$ with $|\mathrm{ann}_\ell(a_i)| = p^2$, $|A_{ij}| = 1$ or p. Hence $|o_\ell(a_i) \cap o_r(a_j)| = 0$ or $p-1$. Assume that $|A_{ij}| = 1$ (equivalently, $|o_\ell(a_i) \cap o_r(a_j)| = 0$) for some i, j. Then $|A_{ik}| > |A_{ii}|$ for some k. Since $|A_{ii}| = p$ (equivalently, $|o_\ell(a_i) \cap o_r(a_j)| = p-1$) as given in the proof of Lemma 3.6, $|A_{ik}| > p$, a contradiction. Therefore, $|A_{ij}| = p$, and so $|o_\ell(a_i) \cap o_r(a_j)| = p-1$ for all $i, j = 1, \ldots, p+1$. □

Lemma 3.8. *Let p be a prime, $R = \mathrm{Mat}_2(\mathbb{Z}_p)$ and $N(p)$ be the set of nonzero nilpotents in R. Consider $X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1})$ (resp. $X = o_r(a_1) \cup \cdots \cup o_r(a_{p+1})$) for some $a_1, \ldots, a_{p+1} \in N(p)$ as mentioned in Remark 5. If*

$s_j = (1, j)$ is a transposition in S_{p+1}, the symmetric group of degree $p+1$, and $f_{s_j} : \Gamma(R) \longrightarrow \Gamma(R)$ is a bijective map such that $f_{s_j}(o_\ell(a_i)) = o_\ell(a_{s_j(i)})$, then f_{s_j} is a graph automorphism in $\Gamma(R)$.

Proof. Note that since $f_{s_j} : \Gamma(R) \longrightarrow \Gamma(R)$ is a bijective map such that $f_{s_j}(o_\ell(a_i)) = o_\ell(a_{s_j(i)})$, $f_{s_j}(o_\ell(a_i) \cap o_r(a_k)) = o_\ell(a_{s_j(i)}) \cap o_r(a_{s_j(k)})$ for all $i, k = 1, \ldots, p+1$.

Let $x, y \in X$ be arbitrary. Consider the following cases.

Case 1. $x, y \in o_\ell(a_1) \cap o_r(a_1)$.
Since $a_1^2 = 0$, $xy = yx = 0$. Note that $f_{s_j}(x), f_{s_j}(y) \in o_\ell(a_j) = o_r(a_j)$, and so $f_{s_j}(x)f_{s_j}(y) = f_{s_j}(xy) = f_{s_j}(0) = 0$ and also $f_{s_j}(y)f_{s_j}(x) = 0$.

Case 2. $x, y \in o_\ell(a_j) \cap o_r(a_j)$.
By the similar argument given to the case 1, $xy = yx = 0$ and also $f_{s_j}(x)f_{s_j}(y) = f_{s_j}(y)f_{s_j}(x) = 0$.

Case 3. $x \in o_\ell(a_1) \cap o_r(a_1), y \in o_\ell(a_1) \cap o_r(a_j)$ $(j \neq 1)$.
Then $yx = 0$. Note that $f_{s_j}(x) \in o_\ell(a_j) \cap o_r(a_j), f_{s_j}(y) \in o_\ell(a_j) \cap o_r(a_1)$, and so $f_{s_j}(y)f_{s_j}(x) = 0$. Assume that $xy = 0$. Then $a_1 a_j = 0$, which implies that $o_\ell(a_1) = o_\ell(a_j)$, a contradiction. Hence $xy \neq 0$. Assume that $f_{s_j}(x)f_{s_j}(y) = 0$. Since $f_{s_j}(x) \in o_\ell(a_j) \cap o_r(a_j), f_{s_j}(y) \in o_\ell(a_j) \cap o_r(a_1)$, $a_j a_1 = 0$, which implies that $o_\ell(a_1) = o_\ell(a_j)$, also a contradiction. Hence we have $f_{s_j}(x)f_{s_j}(y) \neq 0$.

Case 4. $x \in o_\ell(a_j) \cap o_r(a_j), y \in o_\ell(a_1) \cap o_r(a_1)$.
By the similar argument given to the case 3, $xy = 0$ and also $f_{s_j}(x)f_{s_j}(y) = 0$; $yx \neq 0$ and $f_{s_j}(y)f_{s_j}(x) \neq 0$.

Case 5. $x \in o_\ell(a_1) \cap o_r(a_i), y \in o_\ell(a_1) \cap o_r(a_k), (i, k \neq 1, j)$.
Then $x = g_1 a_1 = a_i h_1, y = g_2 a_1 = a_k h_2$ for some $g_1, g_2, h_1, h_2 \in G$. If $xy = 0$, then $a_1 a_k = 0$, which implies that $o_\ell(a_1) = o_\ell(a_k)$, a contradiction. Hence we have $xy \neq 0$. Since $f(x) \in o_\ell(a_j) \cap o_r(a_i), f(y) \in o_\ell(a_j) \cap o_r(a_k)$, we also have $f(x)f(y) \neq 0$. Similarly, we have $yx \neq 0$ and $f(y)f(x) \neq 0$.

Case 6. $x \in o_\ell(a_i) \cap o_r(a_r), y \in o_\ell(a_k) \cap o_r(a_t), (i, k, r, s \neq 1, j)$.
If $xy = 0$, then $a_i a_t = 0$. Since $f(x) \in o_\ell(a_i) \cap o_r(a_r), f(y) \in o_\ell(a_k) \cap o_r(a_s)$, $f(x)f(y) = 0$. Similarly we have that if $yx = 0$, $f(y)f(x) = 0$.

Consequently, f_{s_j} is a graph automorphism in $\Gamma(R)$. □

Theorem 3.9. *Let p be a prime and let $R = \mathrm{Mat}_2(\mathbb{Z}_p)$. Then $\mathrm{Aut}(\Gamma(R)) \simeq S_{p+1}$ where S_{p+1} is the symmetric group of degree $p+1$.*

Proof. Let $N(p)$ be the set of nonzero nilpotents in R. We can choose $a_1, \ldots, a_{p+1} \in N(p)$ such that $X = o_\ell(a_1) \cup \cdots \cup o_\ell(a_{p+1})$. Define $\sigma : S_{p+1} \longrightarrow \mathrm{Aut}(\Gamma(R))$ by $\sigma(s) = f_s$ for all $s \in S_{p+1}$ where $f_s(o_\ell(a_i)) = o_\ell(a_{s(i)})$ for all $i = 1, \ldots, p+1$. Then σ is well-defined and onto. Indeed, by Lemma 3.1 and Lemma 3.2, we have that if $f \in \mathrm{Aut}(\Gamma(R))$ is arbitrary, then for all $i = 1, \ldots, p+1$, $f(o_\ell(a_i)) = o_\ell(a_{s(i)})$ for some $s \in S_{p+1}$. Since S_{p+1} is generated by the p transpositions $s_1 = (1, 2), \ldots, s_p = (1, p+1)$, and $f_{s_1}, \ldots, f_{s_p} \in$

$\mathrm{Aut}(\Gamma(R))$ by Lemma 3.8, $\mathrm{Aut}(\Gamma(R))$ is generated by the p graph automorphisms $f_{s_1}, \ldots, f_{s_p} \in \mathrm{Aut}(\Gamma(R))$ where $f_{s_j}(o_\ell(a_i)) = o_\ell(a_{s_j(i)})$ for all $i = 1, \ldots, p+1$ and $j = 1, \ldots, p$. Thus $|S_{p+1}| = |\mathrm{Aut}(\Gamma(R))|$, which implies that σ is a bijective map. Also σ is a group homomorphism by observing that for all $s_i, s_j \in S_{p+1}$ $(i, j = 1, \ldots, p)$ and all $o_\ell(a_k)$ $(k = 1, \ldots, p+1)$, $(f_{s_i} \circ f_{s_j})(o_\ell(a_k)) = f_{s_i s_j}(o_\ell(a_k))$. Therefore, $\mathrm{Aut}(\Gamma(R)) \simeq S_{p+1}$. □

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