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# ALGEBRAIC POINTS ON THE PROJECTIVE LINE

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Abstract. Schanuel's formula describes the distribution of *rational* points on projective space. In this paper we will extend it to *algebraic* points of bounded degree in the case of $\mathbb{P}^1$. The estimate formula will also give an explicit error term which is quite small relative to the leading term. It will also lead to a *quasi-asymptotic* formula for the number of points of bounded degree on $\mathbb{P}^1$ according as the height bound goes to $\infty$.

## 1. Introduction

Given a variety defined over a number field, we are usually interested in the problem of counting its rational points. According to Northcott's theorem ([5]), there are only finitely many points of bounded degree and bounded height. So, in particular, according as we increase a height bound, we can count the number of rational points. Then the asymptotic behavior of the counting function reflects geometric properties of the variety.

Above all, along this line, there is Schanuel's formula describing the asymptotic behavior of the counting function for rational points on projective space. It states roughly

$$N\big(\mathbb{P}^n(k):T\big) \sim cT^{n+1} \quad \text{as } T \to \infty,$$

where $k$ is a number field, $c > 0$ is a constant, and $N\big(\mathbb{P}^n(k):T\big)$ is the number of $k$-rational points (on $\mathbb{P}^n$) of height $\le T$ with respect to the *usual* exponential height *relative to* $k$ (We will introduce its more precise form in a while).

Motivated by this formula, Batyrev and Manin later formulated their conjecture on the distribution of rational points on Fano varieties. It says that on a Fano variety $X$ defined over a sufficiently large number field $k$ there exists a Zariski open subset $U \subset X$ such that

$$N\big(U(k), H_{-K_X}:T\big) \sim cT(\log T)^{\text{rank Pic } X-1} \quad \text{as } T \to \infty,$$

where $c > 0$ is a constant, $H_{-K_X}$ is the exponential height relative to $k$ and the anticanonical divisor on $X$, and $N\big(U(k), H_{-K_X}:T\big)$ is the number of $k$-rational points (on $U$) of $H_{-K_X}$-height $\le T$ (Note that $\mathbb{P}^n$ has Picard group

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of rank 1. And for a more detailed exposition, see [5]). The constant $c$ is more related to the arithmetic of $X$ while the exponent of $\log T$ is to its geometry (i.e., numerical invariants of $X(\bar{k})$). By the way, the fact is that this conjecture is not true in its full generality though it has been known to be true for a wide class of Fano varieties. See, e.g., [2], [3], and [4].

In the above we saw the "precise" asymptotic distribution of *rational* points. We may ask whether there is such an asymptotic formula for the number of *algebraic* points of bounded degree. This is the main topic of this paper. We will be only concerned with the projective line $\mathbb{P}^1$. And the result will say something about its *quasi-asymptotic* formula.

It should be noticed that W. M. Schmidt ([10]), D. Masser, and J. D. Vaaler ([8]) considered the same problem. Our main result below will consider the result in [10] in the case of $\mathbb{P}^1$, with an explicit error term, which has a "smaller" exponent of $T$ below in particular when $n$ is big, as well as an identification of the leading coefficient in the estimation of [10] in the case of $\mathbb{P}^1$.

This article gives an elementary new proof to both Schmidt's result above in the case of $\mathbb{P}^1$ and a weaker version of Masser and Vaaler's result above. An advantage of this article over the two is that its proof is much simpler than those of the two, based on a simple geometric analysis of the symmetric product of the projective line. In particular, it provides us with a geometric nature behind the numerical result as a by-product, which in turn motivates us to an expectation or a question of a generalization of the Batyrev-Manin conjecture, cf. Remark (a) at the end of the article. The main theorem is as follows.

**Theorem 1.0.1.** *Let $k$ be a number field of degree $d$ (over $\mathbb{Q}$), let $n \geq 2$ be an integer, and let $N\big(\mathbb{P}^1(k,n) : T\big)$ be the counting function on $\mathbb{P}^1$ relative to the usual (absolute) exponential height, i.e., the number of (algebraic) points (of $\mathbb{P}^1$) of degree $\leq n$ over $k$ and height $\leq T$. And let $\epsilon > 0$. Then we have*

$$\begin{aligned}
&n \cdot a(k,n) 2^{-nd(n+1)} T^{dn(n+1)} - O_\epsilon(T^{dn(n+1)-n+\epsilon}) \\
&\leq N\big(\mathbb{P}^1(k,n) : T\big) \\
&\leq n \cdot a(k,n) 2^{nd(n+1)} T^{dn(n+1)} + O(T^{dn(n+1)-n}),
\end{aligned}$$

*where*

$$a(k,n) = \frac{hR/w}{\zeta_k(n+1)} \Big( \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{D_k}} \Big)^{n+1} (n+1)^{r_1+r_2-1},$$

*with*

$$\begin{aligned}
h &= \text{class number of } k, \\
R &= \text{regulator of } k, \\
w &= \text{number of roots of unity in } k, \\
\zeta_k &= \text{zeta function of } k, \\
r_1 &= \text{number of real embeddings of } k,
\end{aligned}$$

$$
\begin{aligned}
r_2 &= \textit{number of complex embeddings of } k, \\
D_k &= \textit{absolute value of the discriminant of } k/\mathbb{Q}.
\end{aligned}
$$

More notation will be explained in Section 2.2 below. We are not interested in the case of $n = 1$, since it reduces to Schanuel's formula. Note that among the appearing constants only the $O_\epsilon$-constant depends on $\epsilon$ (and that the $O_\epsilon$-constant depends on others, too) and that there is no indeterminate constant in the leading coefficient of the estimation, i.e., that the leading coefficient is explicit.

Then it is immediate to get the following quasi-asymptotic formula.

**Corollary 1.0.2.**

$$2^{-nd(n+1)} + o(1) \le \frac{N\big(\mathbb{P}^1(k,n) : T\big)}{n \cdot a(k,n) T^{dn(n+1)}} \le 2^{nd(n+1)} + o(1) \qquad \textit{as } T \to \infty.$$

See also Remark (a) at the end of this article.

## 2. Preliminaries

### 2.1. A brief review of heights

Unless otherwise stated, by heights we will always mean *absolute* and *logarithmic* Weil heights. For the general theory of heights we refer to Hindry–Silverman [5], Lang [7], Silverman [13], and Vojta [14]. Here we will give a brief survey of the definition and some basic properties we need later.

**Definition.** For a point $P := [x_0, x_1, \dots, x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we define the (*Weil*) *height* of $P$ to be

$$h_{\mathbb{P}^n}(P) = \frac{1}{d} \cdot \sum_v \log\big(\max\nolimits_{0 \le i \le n}\big\{\big|x_i\big|_v^{d_v}\big\}\big),$$

where $x_0, x_1, \dots, x_n \in k$ ($k$ a number field of degree $d$ over $\mathbb{Q}$), $v$ runs over (a complete set of) all the primes, finite or infinite, of $k$ so that $\big|\cdot\big|_v^{d_v}$ are *normalized* (hence satisfying the product formula), and $d_v$ is the local degree of $k_v$ over $\mathbb{Q}_v$.

*Remark.* 1. It is independent of the choice of both the homogeneous coordinates of $P$ and the number field $k$ by the product formula and the contribution of the factor $\frac{1}{d}$, respectively.

2. There is a way called the Weil height machine satisfying various standard properties such as additivity and the functoriality of heights which is to associate a (*Weil*) *height* $h_{X,D} : X(\overline{k}) \to \mathbb{R}$ to a locally principal divisor $D$ on a complete variety $X$ defined over a number field $k$. We do not go into its detail here except to mention the following.

If $D$ is a very ample divisor on $X$, then we define $h_{X,D} := h_{\mathbb{P}^n} \circ \varphi$, where $\varphi : X \hookrightarrow \mathbb{P}^n$ is an embedding associated to $D$. In more general, given a locally principal divisor $D$ on $X$, write $D$ as the difference $D_1 - D_2$ of two very ample divisors $D_i$ $(i = 1,\, 2)$ on $X$. Then we define $h_{X,D} := h_{X,D_1} - h_{X,D_2}$, where $h_{X,D_i}$ $(i = 1,\, 2)$ is as before. Properties of the Weil height machine will be freely used in what follows.

If $X$ is clearly understood from the context, we usually omit $X$ from the notation of $h_{X,D}$. We also sometimes use the linear equivalence class of $D$ or the line bundle $\mathcal{O}_X(D)$ instead of $D$ in the notation of $h_{X,D}$. See any reference mentioned above for details.

3. A height $h_{X,D}$ appearing above is uniquely determined only up to a bounded function. Thus, for example, the equality of two heights will always actually mean the equality up to a bounded function. Note the ambiguity of a bounded function in determining a height does not cause any problem for our usual purposes.

## 2.2. Notation, definitions, and conventions

In this section we will make explicit part of the tools and notation we will use in our subsequent proofs. Unless otherwise stated, by heights we will always mean *absolute and logarithmic* ones as above.

Let $n \geq 1$ be an integer. Let $h$ (resp. $h_{\mathbb{P}^n}$) be the standard height on $\mathbb{P}^1$ (resp. $\mathbb{P}^n$) and $H$ (resp. $H_{\mathbb{P}^n}$) its corresponding exponential height. And let $k$ be a number field.

**Definition.** Let $I$ be an interval in $\mathbb{R}$. Then we write

$$\begin{aligned}
\mathbb{P}^1(k,n) &:= \{P \in \mathbb{P}^1(\bar{k}) :\ [k(P):k] \leq n\},\\
\mathbb{P}^1(k,=n) &:= \{P \in \mathbb{P}^1(\bar{k}) :\ [k(P):k] = n\},\\
\mathbb{P}^1(k,n:I) &:= \{P \in \mathbb{P}^1(\bar{k}) :\ [k(P):k] \leq n \quad \text{and} \quad H(P)\ \in I\}, \quad \text{and}\\
\mathbb{P}^1(k,=n:I) &:= \{P \in \mathbb{P}^1(\bar{k}) :\ [k(P):k] = n \quad \text{and} \quad H(P)\ \in I\}.
\end{aligned}$$

We use their similar corresponding notation for higher dimensional projective spaces. And, in particular, in case $n = 1$ (resp. $I = [0, T]$, $T$ a positive real), we suppress $n$ (resp. replace $I$ by $T$) for brevity.

**Definition.** The *counting function of* $\mathbb{P}^1(k,n)$ is

$$N\big(\mathbb{P}^1(k,n) : T\big) := \#\mathbb{P}^1(k,n:T).$$

(We can also introduce a similar counting function on an arbitrary variety (using a different notion of height, in which case we make its used notion explicit). And we will also use brief notation for the counting function similarly to the above).

## 2.3. The symmetric product of $\mathbb{P}^1$

The elementary symmetric polynomials in $n \geq 1$ variables give rise to a finite covering (of degree $n!$) $\pi : (\mathbb{P}^1)^n \to \mathbb{P}^n$. This factors through $\mathbb{P}^{(n)} := \operatorname{Sym}_n \mathbb{P}^1$, the $n^{\text{th}}$ symmetric product of $\mathbb{P}^1$, so that we can write $\pi : (\mathbb{P}^1)^n \to \mathbb{P}^{(n)} \xrightarrow{\sim} \mathbb{P}^n$. Note that all the varieties and the morphisms here are defined over $\mathbb{Q}$.

First, we then have the $n$-to-1 map

$$\mathbb{P}^1(k,=n) \to \mathbb{P}^n(k), \qquad P \mapsto \pi(P^{(1)}, \dots, P^{(n)}),$$

where $P^{(1)}, \dots, P^{(n)}$ are the $\mathrm{Gal}(\overline{k}/k)$-conjugates of $P$. Note this is not a surjective map unless $n = 1$.

Second, since $\pi^* \mathcal{O}(1) = \mathcal{O}(1, \dots, 1)$, we then easily see that

$$\Big| \sum_{i=1}^{n} h \circ \pi_i - h_{\mathbb{P}^n} \circ \pi \Big| \leq n \ln 2, \tag{1}$$

where $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ is the $i^{\text{th}}$ projection (The fact is that we can also prove this inequality directly by keeping explicit track of $h$ under $\pi$. See [6] and [12]). For notational simplicity and in order to keep track of the dependence of the leading coefficient in Theorem 1.0.1 on the bound in (1), we will write $c_n := n \ln 2$ from now on. The fact is that "$(-1)\cdot$the inner part of the absolute sign in (1)" has a slightly better *upper* bound $(n-1)\ln 2$, which would improve the exponent of 2 of the leading coefficients in Theorem 1.0.1. And yet, for simplicity we will use $c_n = n \ln 2$.

Now let $\sum_{i=1}^{m} n_i = n$ ($m \geq 1$ and $n_i \geq 1$ for all $i = 1, 2, \dots, m$). And for $i = 1, 2, \dots, m$, let $P_i \in \mathbb{P}^1(k, = n_i)$ and let $P_i^{(1)} = P_i, P_i^{(2)}, \dots, P_i^{(n_i)}$ be its $\mathrm{Gal}(\overline{k}/k)$-conjugates. Note $\sum_{j=1}^{n_i} h(P_i^{(j)}) = n_i h(P_i)$. Then it is immediate from (1) that

$$\Big| \sum_{i=1}^{m} n_i h(P_i) - h_{\mathbb{P}^n} \circ \pi \Big(P_1^{(1)}, \dots, P_1^{(n_1)}, \dots, P_m^{(1)}, \dots, P_m^{(n_m)}\Big) \Big| \leq c_n. \tag{2}$$

Also note similarly to the above that $\pi\Big(P_1^{(1)}, \dots, P_1^{(n_1)}, \dots, P_m^{(1)}, \dots, P_m^{(n_m)}\Big) \in \mathbb{P}^n(k)$. And conversely, all the points of $\mathbb{P}^n(k)$ are of this type.

Though we state (1) and (2) above in terms of the logarithmic height for convenience, we will use their multiplicative version later.

## 2.4. Schanuel's formula

The counting function on a variety has been extensively studied. However, it is a rare case that we can describe it explicitly or asymptotically. The best well-known example is Schanuel's result for the asymptotic formula for the number of rational points on an arbitrary projective space. Since we will need it repeatedly in what follows, we introduce it here.

**Theorem 2.4.1** (Scahnuel's formula [5], [7], or [9])**.** *Let $k$ be a number field of degree $d$ over $\mathbb{Q}$, let $n \geq 1$ be an integer, and let $N\big(\mathbb{P}^n(k) : T\big)$ be the counting function of $\mathbb{P}^n(k)$. Then we have*

$$N\big(\mathbb{P}^n(k) : T\big) = \begin{cases} a(k,n) T^{d(n+1)} + O(T \log T) & \textit{if } d = n = 1, \\ O(T^{d(n+1)-1}) & \textit{otherwise}, \end{cases}$$

*where the constant* $a(k,n) > 0$ *is the same as given in Theorem 1.0.1* (*not only for* $n \geq 2$ *but also for* $n = 1$)*.*

Notice this formula is usually written in terms of the usual "relative" (not absolute) height depending on the number field $k$. And yet, we state it in terms of the absolute height, since it is more useful for our application.

## 3. Proof of Theorem 1.0.1

Note that we will keep all the notation introduced before. In particular, $k$ will always mean a number field of degree $d$ over $\mathbb{Q}$. Further, we have $n \geq 2$.

### 3.1. An upper bound

We first prove the relatively easier part that is the upper bound of the estimate.

**Proposition 3.1.1.** *Under the same notation as before, we have*

$$N\big(\mathbb{P}^1(k,n) : T\big) \leq n \cdot a(k,n) 2^{nd(n+1)} T^{dn(n+1)} + O(T^{dn(n+1)-n}).$$

*Proof.* Write

$$\mathbb{P}^1(k,n) = \bigcup_{m=1}^{n} \mathbb{P}^1(k, = m).$$

Let $P \in \mathbb{P}^1(k, = m)$. From (1) of Section 2.3 we know that

$$H_{\mathbb{P}^m} \circ \pi(P^{(1)}, \ldots, P^{(m)}) \leq e^{c_m} H(P)^m,$$

where $c_m = m \ln 2$ and $P^{(1)}, \ldots, P^{(m)}$ are the $\mathrm{Gal}(\overline{k}/k)$-conjugates of $P$.

Hence it follows that (cf. Section 2.3.)

$$\begin{aligned} \#\mathbb{P}^1(k, = m : T) &\leq m \cdot N\big(\mathbb{P}^m(k) : e^{c_m} T^m\big) \\ &= m \cdot a(k,m) e^{c_m d(m+1)} T^{dm(m+1)} + O(T^{dm(m+1)-m}), \end{aligned}$$

unless $d = m = 1$. In case $d = m = 1$, we replace $O(T^{dm(m+1)-m})$ by $O(T \log T)$.

Notice, in any case, that $dn(n+1) - n > d(n-1)n$. So the contribution from the points of degree $\leq n-1$ can be absorbed into the error term $O(T^{dn(n+1)-n})$. Therefore we get the desired inequality. $\square$

As a matter of fact, it may also be possible to obtain more or less the leading power of $T$ of the estimate in Proposition 3.1.1 directly from the usual proof of Northcott's theorem.

### 3.2. Auxiliary results toward a lower bound

It is a little complicated to prove that the upper bound obtained is "essentially" the best one. The problem is that the $n$-to-1 map introduced at the beginning of Section 2.3 is not surjective because $n \geq 2$.

**Proposition 3.2.1.** *Let $m \geq 1$ be an integer, and let $t > 0$ and $T \geq 1$ be real numbers. Then we have*

$$\sum_{P \in \mathbb{P}^1(k,=m\colon T)} H(P)^{-t} \ll \begin{cases} T & \text{if } t \geq dm(m+1), \\ T^{dm(m+1)-t+1} & \text{otherwise.} \end{cases}$$

*Proof.* By $[T]$ we mean the integer part of $T$. Then we observe that

$$\begin{aligned}
&\sum_{P \in \mathbb{P}^1(k,=m\colon T)} H(P)^{-t} \\
&= O(1) + \sum_{j=1}^{[T]-1} \sum_{P \in \mathbb{P}^1(k,=m\colon (j,j+1])} H(P)^{-t} + \sum_{P \in \mathbb{P}^1(k,=m\colon ([T],T])} H(P)^{-t} \\
&\leq O(1) + \sum_{j=1}^{[T]-1} j^{-t} \cdot \#\mathbb{P}^1\big(k,=m : (j,j+1]\big) + [T]^{-t} \cdot \#\mathbb{P}^1\big(k,=m : ([T],T]\big) \\
&\leq O(1) + \sum_{j=1}^{[T]-1} j^{-t} \cdot \#\mathbb{P}^1\big(k,=m : j+1\big) + [T]^{-t} \cdot \#\mathbb{P}^1\big(k,=m : T\big) \\
&\leq O(1) + \sum_{j=1}^{[T]-1} j^{-t} \cdot \big\{m \cdot a(k,m) e^{c_m d(m+1)} (j+1)^{dm(m+1)} + O(j^{dm(m+1)-m})\big\} \\
&\qquad + \ [T]^{-t} \cdot m \cdot a(k,m) e^{c_m d(m+1)} T^{dm(m+1)} + O(T^{dm(m+1)-m}) \\
&\qquad \text{(from Proposition 3.1.1.)} \\
&\leq \sum_{j=1}^{[T]-1} O(j^{dm(m+1)-t}) + O(T^{dm(m+1)-t}),
\end{aligned}$$

unless $d = m = 1$. If $d = m = 1$, we replace

$$O(j^{dm(m+1)-m}) \text{ (resp. } O(T^{dm(m+1)-m}))$$

by $O(j\log(j+1))$ (resp. $O(T\log T)$) and we still have exactly the same last inequality. Therefore it is easy to see, in any case, that the desired inequalities follow. □

Cf. Obviously, we can specify the $\ll$-constant better. But it is not interesting for our purpose, since the sum in Proposition 3.2.1 will be absorbed into an $O$-error term eventually. And the fact is that we get a slightly better bound in case $m = 1$. We will see it in the proof of Lemma 3.2.2 below (Since it is more convenient to have the above unifying inequalities (though weaker in the case of $m = 1$) in our later application, we do not separate it. Furthermore, though we separate it here, it would not produce a better result in Theorem 1.0.1 (and it would just make the needed computation more complicated), since Claim 3.3.1 (hence also Lemma 3.2.2) below eventually should deal with *all* the possible partitions of an *arbitrary* given integer $n \geq 2$, i.e., since we would have to make $\epsilon > 0$ appear for any $n \geq 2$ in Theorem 1.0.1 after all).

Now we go toward a lemma. We start with a basic set-up.

Let $\sum_{i=1}^{m} n_i = n$ where $1 \leq n_i \leq n-1$ (Recall $n \geq 2$). Let $T \geq 1$ be a real number. And let Condition $(*)$ read

$$\begin{aligned}
P_1 &\in \mathbb{P}^1(k, = n_1 : T), \\
P_2 &\in \mathbb{P}^1(k, = n_2 : \sqrt[n-n_1]{T^n H(P_1)^{-n_1}}), \\
P_3 &\in \mathbb{P}^1(k, = n_3 : \sqrt[n-n_1-n_2]{T^n H(P_1)^{-n_1} H(P_2)^{-n_2}}), \\
&\vdots \\
P_m &\in \mathbb{P}^1(k, = n_m : \sqrt[n_m]{T^n H(P_1)^{-n_1} H(P_2)^{-n_2} \cdots H(P_{m-1})^{-n_{m-1}}}).
\end{aligned}$$

**Lemma 3.2.2.** *Let $\epsilon > 0$. Under the same notation as above, we have*

$$\sum_{P_1, \ldots, P_{m-1}} \prod_{i=1}^{m-1} H(P_i)^{-dn_i(n_m+1)} \ll T^{dn(n-n_m)-n+\epsilon},$$

*where $P_1, \ldots, P_{m-1}$ in the summation run over all the possible choices satisfying Condition $(*)$.*

*Proof.* We use induction on $n \geq 2$. For $n = 2$, we have the only possibility that $m = 2$ and that $n_1 = n_2 = 1$ and Condition $(*)$ then reads

$$P_1 \in \mathbb{P}^1(k : T) \quad (\text{and} \quad P_2 \in \mathbb{P}^1(k : T^2 H(P_1)^{-1})).$$

Thus we see that (as in the proof of Proposition 3.2.1)

$$\begin{aligned}
&\sum_{P_1 \in \mathbb{P}^1(k:\, T)} H(P_1)^{-2d} \\
\leq\ & O(1) + \sum_{j=1}^{[T]-1} j^{-2d} \cdot \#\mathbb{P}^1\big(k : (j, j+1]\big) + [T]^{-2d} \cdot \#\mathbb{P}^1\big(k : ([T], T]\big) \\
=\ & O(1) + \sum_{j=1}^{[T]-1} j^{-2d} \cdot \big\{a(k,1)(j+1)^{2d} - a(k,1)j^{2d} + O(j^{2d-1})\big\} \\
&\qquad + [T]^{-2d} \cdot \big\{a(k,1)T^{2d} - a(k,1)[T]^{2d} + O(T^{2d-1})\big\} \\
\leq\ & O(1) + \sum_{j=1}^{[T]-1} O(j^{-1}) + O(T^{-1}) \\
\leq\ & O(T^{\epsilon}) \ \leq\ O(T^{2d-2+\epsilon}), \qquad \text{as desired,}
\end{aligned}$$

unless $d = 1$. In case $d = 1$, we replace $O(j^{2d-1})$ (resp. $O(T^{2d-1})$, $O(j^{-1})$, $O(T^{-1})$) by $O(j\log(j+1))$ (resp. $O(T\log T)$, $O(j^{-1}\log(j+1))$, $O(T^{-1}\log T)$). Then we still have exactly the same last two inequalities.

We assume the desired inequality is true up to $n-1$ (for all possible partitions of the integers $\leq n-1$). We first observe that

$$\begin{aligned}
&\sqrt[n-n_1-n_2]{T^n H(P_1)^{-n_1} H(P_2)^{-n_2}} \\
&= \sqrt[(n-n_1)-n_2]{\Big(\sqrt[n-n_1]{T^n H(P_1)^{-n_1}}\Big)^{n-n_1} H(P_2)^{-n_2}}, \quad \text{etc.}
\end{aligned}$$

Then we see that

$$\begin{aligned}
&\sum_{P_1,\dots,P_{m-1}} \prod_{i=1}^{m-1} H(P_i)^{-dn_i(n_m+1)} \\
&= \sum_{P_1} \Big\{ H(P_1)^{-dn_1(n_m+1)} \sum_{P_2,\dots,P_{m-1}} \prod_{i=2}^{m-1} H(P_i)^{-dn_i(n_m+1)} \Big\} \\
&\leq \sum_{P_1} \Big\{ H(P_1)^{-dn_1(n_m+1)} \cdot O\big( \sqrt[n-n_1]{T^n H(P_1)^{-n_1}}^{\,d(n-n_1)(n-n_1-n_m)-(n-n_1)+\epsilon} \big) \Big\} \\
&\qquad \text{(by induction hypothesis)} \\
&\leq T^{n\{d(n-n_1-n_m)-1+\frac{\epsilon}{n-n_1}\}} \cdot O\Big( \sum_{P_1} H(P_1)^{-n_1\{d(n-n_1+1)-1+\frac{\epsilon}{n-n_1}\}} \Big).
\end{aligned}$$

Now apply Proposition 3.2.1 to get

$$\begin{aligned}
&\sum_{P_1 \in \mathbb{P}^1(k,=n_1:\,T)} H(P_1)^{-n_1\{d(n-n_1+1)-1+\frac{\epsilon}{n-n_1}\}} \\
&\ll \begin{cases} T & \text{if } \; n_1\{d(n-n_1+1)-1+\frac{\epsilon}{n-n_1}\} \geq dn_1(n_1+1), \\ T^{dn_1(n_1+1)-n_1\{d(n-n_1+1)-1+\frac{\epsilon}{n-n_1}\}+1} & \text{otherwise.} \end{cases}
\end{aligned}$$

It is easy to prove that the addition of $n\{d(n-n_1-n_m)-1+\frac{\epsilon}{n-n_1}\}$ to 1 (resp. $dn_1(n_1+1)-n_1\{d(n-n_1+1)-1+\frac{\epsilon}{n-n_1}\}+1$) is at most $dn(n-n_m)-n+\epsilon$. Therefore the desired inequality follows and the proof is complete by induction. □

*Remark.* The induction step of the above proof works even in case $m=2$, in which case we notice that $\sum_{P_2,\dots,P_{m-1}} \prod_{i=2}^{m-1} H(P_i)^{-dn_i(n_m+1)}$ can be replaced by 1 and clearly that

$$1 \leq O\big( \sqrt[n-n_1]{T^n H(P_1)^{-n_1}}^{\,d(n-n_1)(n-n_1-n_m)-(n-n_1)+\epsilon} \big).$$

However, there is an alternative way to deal with this case. Let us see it briefly. If $m=2$, then we have only two possibilities; (i) $n_1=1$ and $n_2=n-1$, and (ii) $n_1=n-1$ and $n_2=1$. In the former, we follow the preceding proof of the case $n=2$ step by step correspondingly and get $\sum_{P_1\in\mathbb{P}^1(k:\,T)} H(P_1)^{-nd} \ll T^{\epsilon}$. On the other hand, in the latter, we instead use Proposition 3.2.1 to get $\sum_{P_1\in\mathbb{P}^1(k,=n-1:\,T)} H(P_1)^{-2(n-1)d} \ll T^{d(n-1)(n-2)+1}$ by noticing $n \geq 3$. It is then easy to see in both cases that $\sum_{P_1} H(P_1)^{-dn_1(n_2+1)} \ll T^{dn(n-n_2)-n+\epsilon}$ as desired, too.

### 3.3. A lower bound

Now we are ready to prove the lower bound and we will finish the proof of Theorem 1.0.1.

*Proof of the lower bound in Theorem 1.0.1.* First recall Condition $(*)$ and consider all the points $(P_1^{(1)},\dots,P_1^{(n_1)},\, P_2^{(1)},\dots,P_2^{(n_2)},\dots,P_m^{(1)},\dots,P_m^{(n_m)})$ satisfying Condition $(*)$. Next we consider all the possible partitions of $n=\sum_{1=1}^m n_i$ $(1\le n_i\le n-1,\ i=1,\dots,n-1,$ and $m\ge 2)$ and collect all such possible corresponding $n$-tuples. We call the resulting set $S$.

**Claim 3.3.1.** *Let $\epsilon>0$. Then we have*

$$\#\pi(S) \ll T^{dn(n+1)-n+\epsilon}.$$

*Proof.* We have

$$\begin{aligned}
\#\pi(S) &\le \sum_{\text{Partitions in }(*)} \sum_{P_1,\dots,P_{m-1}} \#\{P_m\text{'s}\}\\
&\le \sum_{\text{Partitions in }(*)} \sum_{P_1,\dots,P_{m-1}} O\Big(\Big(T^n\prod_{i=1}^{m-1}H(P_i)^{-n_i}\Big)^{\frac{1}{n_m}\cdot dn_m(n_m+1)}\Big)\\
&\quad \text{(from Proposition 3.1.1)}\\
&= \sum_{\text{Partitions in }(*)} T^{dn(n_m+1)}\cdot O\Big(\sum_{P_1,\dots,P_{m-1}}\prod_{i=1}^{m-1}H(P_i)^{-dn_i(n_m+1)}\Big)\\
&\ll \sum_{\text{Partitions in }(*)} T^{dn(n_m+1)}\cdot T^{dn(n-n_m)-n+\epsilon}\\
&\quad \text{(from Lemma 3.2.2)}\\
&= \sum_{\text{Partitions in }(*)} T^{dn(n+1)-n+\epsilon}\\
&\ll T^{dn(n+1)-n+\epsilon},
\end{aligned}$$

where $\sum_{\text{Partitions in }(*)}$ means "sum over all possible partitions of $n$ in $(*)$".

Therefore we get the desired inequality. $\square$

On the other hand, it follows from (2) of Section 2.3 that

$$e^{-c_n}\prod_{i=1}^m H(P_i)^{n_i} \le H_{\mathbb{P}^n}\circ\pi\Big(P_1^{(1)},\dots,P_1^{(n_1)},P_2^{(1)},\dots,P_2^{(n_2)},\dots,P_m^{(1)},\dots,P_m^{(n_m)}\Big).$$

Thus it is easy to see (cf. Section 2.3) that

$$\mathbb{P}^n(k:e^{-c_n}T^n)\subset \pi(S)\ \cup\ \big\{\pi(Q^{(1)},\dots,Q^{(n)}) : Q\in\mathbb{P}^1(k,=n:T)\big\}$$

(As a matter of fact, this is a disjoint union).

Note that

$$(3)\qquad \#\mathbb{P}^n(k : e^{-c_n}T^n) = a(k,n)e^{-c_n d(n+1)}T^{dn(n+1)} + O(T^{dn(n+1)-n})$$
(from Theorem 2.4.1 with $n \geq 2$).

So we get (cf. Section 2.3)

$$\begin{aligned}
n^{-1}\cdot \#\mathbb{P}^1(k, = n : T) &= \#\big\{\pi(Q^{(1)},\dots,Q^{(n)}) : Q \in \mathbb{P}^1(k, = n : T)\big\}\\
&\geq \#\mathbb{P}^n(k : e^{-c_n}T^n) - \#\pi(S)\\
&\geq a(k,n)e^{-c_n d(n+1)}T^{dn(n+1)} - O(T^{dn(n+1)-n+\epsilon})\\
&\qquad \text{(from Claim 3.3.1 and (3) just above),}\\
\text{i.e., } \#\mathbb{P}^1(k, = n : T) &\geq n\cdot a(k,n)e^{-c_n d(n+1)}T^{dn(n+1)} - O(T^{dn(n+1)-n+\epsilon}).
\end{aligned}$$

Therefore we have, in particular,

$$N\big(\mathbb{P}^1(k,n) : T\big) \geq n\cdot a(k,n)e^{-c_n d(n+1)}T^{dn(n+1)} - O(T^{dn(n+1)-n+\epsilon}).$$

Finally, note that $c_n = n\ln 2$. Then this proves the lower bound of the estimate in Theorem 1.0.1 and finishes the whole proof of the theorem. $\square$

*Remarks.*

(a) It is natural to ask about a possible generalization of the Batyrev-Manin conjecture (cf. the introduction to the paper) to algebraic points of bounded degree on Fano varieties of dimension $\geq 2$: For all $n \leq \dim X - 1$,

$$N\big(U(k,n), H_{-K_X} : T\big) \sim cT^{dn}(\log T)^{\text{rank Pic } X - 1} \quad \text{as } T \to \infty$$

under the same notation as before, however, with the distinction that $H_{-K_X}$ is the *absolute* exponential height associated to the anticanonical divisor on $X$. Of course, this would not be expected to be valid in general because of a counterexample ([2]) to the Batyrev-Manin conjecture. So the question is to what extent it could be true; e.g., whether it is the case at least for the cases where the Batyrev-Manin conjecture has been proven to be true.

(b) As should be noticed in the proof of Theorem 1.0.1, "most" of the points of $\mathbb{P}^1(k, n : T)$ come from $\mathbb{P}^1(k, = n : T)$, i.e., exactly the same inequalities in Theorem 1.0.1 are also true of $\#\mathbb{P}^1(k, = n : T)$.

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## References

[1] V. V. Batyrev and Y. I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Math. Ann. **286** (1990), no. 1-3, 27–43.

[2] V. V. Batyrev and Y. Tschinkel, *Rational points on some Fano cubic bundles*, C. R. Acad. Sci. Paris Ser. I Math. **323** (1996), no. 1, 41–46.

[3] ———, *Manin's conjecture for toric varieties*, J. Algebraic Geom. **7** (1998), no. 1, 15–53.

[4] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), no. 2, 421–435. (Erratum: Invent. Math. **102** (1990), no. 2, 463.)

[5] M. Hindry and J. H. Silverman, *Diophantine Geometry. An Introduction*, Graduate Texts in Mathematics, Vol. 201, Springer-Verlag, New York, 2000.

[6] S. Ih, *Height uniformity for algebraic points on curves*, Compositio Math. **134** (2002), no. 1, 35–57.

[7] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, New York, 1983.

[8] D. Masser and J. D. Vaaler, *Counting algebraic numbers with large height. II*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 427–445.

[9] S. H. Schanuel, *Heights in number fields*, Bull. Soc. Math. France **107** (1979), no. 4, 433–449.

[10] W. M. Schmidt, *Northcott's theorem on heights. I. A general estimate*, Monatsh. Math. **115** (1993), no. 1-2, 169–181.

[11] ———, *Northcott's theorem on heights. II. The quadratic case*, Acta Arith. **70** (1995), no. 4, 343–375.

[12] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, 106. Springer-Verlag, New York, 1986.

[13] ———, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994.

[14] P. Vojta, *Diophantine Approximations and Value Distribution Theory*, Lecture Notes in Mathematics, 1239. Springer-Verlag, Berlin, 1987.

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