J. Korean Math. Soc. **45** (2008), No. 6, pp. 1613–1622

RANDOMLY ORTHOGONAL FACTORIZATIONS OF $(0, mf-(m-1)r)$-GRAPHS

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every vertex x of $V(G)$. We use $d_G(x)$ to denote the degree of a vertex x of G. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for every vertex x of $V(F)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A (g, f)-factorization of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors. Let $F = \{F_1, F_2, \ldots, F_m\}$ be a factorization of G and H be a subgraph of G with mr edges. If $F_i, 1 \leq i \leq m$, has exactly r edges in common with H, we say that F is r-orthogonal to H. If for any partition $\{A_1, A_2, \ldots, A_m\}$ of $E(H)$ with $|A_i| = r$ there is a (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$, $1 \leq i \leq m$, then we say that G has (g, f)-factorizations randomly r-orthogonal to H. In this paper it is proved that every $(0, mf-(m-1)r)$-graph has $(0, f)$-factorizations randomly r-orthogonal to any given subgraph with mr edges if $f(x) \geq 3r-1$ for any $x \in V(G)$.

1. Introduction

In this paper we consider finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$. Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every vertex x of $V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G satisfying that $g(x) \leq d_F(x) \leq f(x)$ for every vertex x of $V(F)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has m edges in total. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \ldots, F_m$. If $g(x) = a$ and $f(x) = b$, where a and b are nonnegative integers, then a (g, f)-factorization of G is called an $[a, b]$-factorization of G. Let H be an mr-subgraph of G. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ is r-orthogonal to H if $|E(H) \cap E(F_i)| = r$ for $1 \leq i \leq m$.

Received February 16, 2007; Revised January 1, 2008.
2000 *Mathematics Subject Classification.* 05C70.
Key words and phrases. graph, subgraph, factor, orthogonal factorization.
This research was supported by Jiangsu Provincial Educational Department(07KJD-110048).

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If for any partition $\{A_1, A_2, \ldots, A_m\}$ of $E(H)$ with $|A_i| = r$ there is a (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$, $1 \le i \le m$, then we say that G has (g, f)-factorizations randomly r-orthogonal to H. Other definitions and terminologies can be found in [1].

Recently Tokuda [9] studied the connected factors in $K_{1,n}$-free graphs containing an $[a, b]$-factor. Kano [3] obtained some sufficient conditions for a graph to have $[a, b]$-factorizations. Liu [5, 6] proved that every $(mg + m - 1, mf - m + 1)$-graph has a (g, f)-factorization orthogonal to a star or a matching. Liu [7] showed that every bipartite $(mg + m - 1, mf - m + 1)$-graph has (g, f)-factorizations randomly k-orthogonal to any km-subgraph. Feng [2] proved that every $(0, mf - m + 1)$-graph has a $(0, f)$-factorization orthogonal to any given m-subgraph. Now we consider the r-orthogonal factorizations of graphs. The purpose of this paper is to prove that for any mr-subgraph H of $(0, mf - (m - 1)r)$-graph G, there exist $(0, f)$-factorizations of G which are randomly r-orthogonal to H, where $f(x) \ge 3r - 1$ for each $x \in V(G)$. In the following we give the main theorem in this paper.

Theorem 1. *Let G be a $(0, mf - (m - 1)r)$-graph, and let f be an integer-valued function defined on $V(G)$ such that $f(x) \ge 3r - 1$ for all $x \in V(G)$, and let H be an mr-subgraph of G. Then G has a $(0, f)$-factorization randomly r-orthogonal to H.*

2. Preliminary results

Let S and T be two disjoint subsets of $V(G)$. We denote by $E_G(S, T)$ the set of edges with one end in S and the other in T, and by $e_G(S, T)$ the cardinality of $E_G(S, T)$. For $S \subset V(G)$ and $A \subset E(G)$, $G - S$ is a subgraph obtained from G by deleting the vertices in S together with the edges to which the vertices in S incident, and $G - A$ is a subgraph obtained from G by deleting the edges in A, and $G[S]$ (rep. $G[A]$) is a subgraph of G induced by S (rep. A). For a subset X of $V(G)$, we write $f(X) = \sum_{x \in X} f(x)$ for any function f defined on $V(G)$, and define $f(\emptyset) = 0$. Especially, $d_G(X) = \sum_{x \in X} d_G(x)$.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$, and C a component (i.e., a maximal connected subgraph) of $G - (S \cup T)$. If there is a vertex $x \in V(C)$ such that $g(x) \ne f(x)$, we call C a neutral component; otherwise, i.e., $g(x) = f(x)$ for all $x \in V(C)$, then we call C an even or odd component according to whether $e_G(T, V(C)) + f(C)$ is even or odd. We denoted by $h_G(S, T)$ the number of the odd components of $G - (S \cup T)$. In 1970 Lovász [8] used the symbol $\delta_G(S, T; g, f)$ for the expression $d_{G-S}(T) - g(T) - h_G(S, T) + f(S)$, and found that $\delta_G(S, T; g, f) \ge 0$ is a necessary and sufficient condition for a graph G to have a (g, f)-factor.

Lemma 2.1 ([8]). *Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Then G has* *a (g, f)-factor if and only if*

$$\delta_G(S, T; g, f) \geq 0$$

for any two disjoint subsets S and T of $V(G)$.

Note that if $g(x) < f(x)$ for all $x \in V(G)$, then all components of $G-(S \cup T)$ are neutral. Hence for any two disjoint subsets S and T of $V(G)$, $h_G(S, T) = 0$ provided $g(x) < f(x)$ for all $x \in V(G)$. Thus in the following $\delta_G(S, T; g, f) = d_{G-S}(T) - g(T) + f(S)$ for any two disjoint subsets S and T of $V(G)$.

Let S and T be two disjoint subsets of $V(G)$, and E_1 and E_2 be two disjoint subsets of $E(G)$. Let $D = V(G) - (S \cup T)$, and

$$E(S) = \{xy \in E(G) : x, y \in S\}, \quad E(T) = \{xy \in E(G) : x, y \in T\},$$
$$E_1' = E_1 \cap E(S), \qquad E_1'' = E_1 \cap E_G(S, D),$$
$$E_2' = E_2 \cap E(T), \qquad E_2'' = E_2 \cap E_G(T, D),$$
$$r_S(E_1) = 2|E_1'| + |E_1''|, \qquad r_T(E_2) = 2|E_2'| + |E_2''|.$$

It is easily seen that $r_S(E_1) \leq d_{G-T}(S)$, $r_T(E_2) \leq d_{G-S}(T)$.

The following lemma has been obtained independently by Yuan and Yu [10] and Li and Liu [4], respectively.

Lemma 2.2 ([4, 10])**.** *Let G be a graph, and g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for all $x \in V(G)$, and E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if*

$$\delta_G(S, T; g, f) = d_{G-S}(T) - g(T) + f(S) \geq r_S(E_1) + r_T(E_2)$$

for any two disjoint subsets S and T of $V(G)$.

Lemma 2.3 ([2])**.** *Let G be a $(0, mf - m + 1)$-graph, and let f be one integer-valued function defined on $V(G)$ such that $f(x) \geq 0$, and let H be an m-subgraph of G. Then G has a $(0, f)$-factorization orthogonal to H.*

In the following, we always assume that G is a $(0, mf - (m-1)r)$-graph, where $m \geq 1$ and $r \geq 1$ are two integers. Define

$$g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-2)r)\},$$
$$\triangle_1 (x) = \frac{1}{m} d_G(x) - g(x),$$
$$\triangle_2 (x) = f(x) - \frac{1}{m} d_G(x).$$

By the definitions of $g(x)$, $\triangle_1 (x)$ and $\triangle_2 (x)$, we have the following lemma.

Lemma 2.4. *For all $x \in V(G)$, the following inequalities hold*:

(1) *If $m \geq 2$, then $0 \leq g(x) < f(x)$,*
(2) *If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then $\triangle_1 (x) \geq \frac{r}{m}$,*
(3) $\triangle_2 (x) \geq \frac{(m-1)r}{m}$.

Proof. (1) Note that G is a $(0, mf-(m-1)r)$-graph, where $m \geq 2$ is an integer. Then $0 \leq mf-(m-1)r$ implies that $f(x) \geq \frac{(m-1)r}{m}$. Note that $f(x)$ is a nonnegative integer-valued function. Then $f(x) \geq 1$.

If $g(x) = 0$, then $0 \leq g(x) < f(x)$.

If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then

$$\begin{aligned} f(x) - g(x) &= f(x) - d_G(x) + (m-1)f(x) - (m-2)r \\ &= mf(x) - (m-2)r - d_G(x) \\ &\geq mf(x) - (m-2)r - (mf(x) - (m-1)r) = r \geq 1. \end{aligned}$$

Hence we obtain that

$$0 \leq g(x) < f(x).$$

(2) If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then

$$\begin{aligned} \triangle_1(x) &= \frac{1}{m}d_G(x) - g(x) \\ &= \frac{1}{m}d_G(x) - [d_G(x) - ((m-1)f(x) - (m-2)r)] \\ &= \frac{1-m}{m}d_G(x) + (m-1)f(x) - (m-2)r \\ &\geq \frac{1-m}{m}(mf(x) - (m-1)r) + (m-1)f(x) - (m-2)r \\ &= (1-m)f(x) + (m-1)r - \frac{(m-1)r}{m} + (m-1)f(x) - (m-2)r \\ &= \frac{r}{m}. \end{aligned}$$

(3) Obviously, we have

$$\begin{aligned} \triangle_2(x) &= f(x) - \frac{1}{m}d_G(x) \geq f(x) - \frac{1}{m}(mf(x) - (m-1)r) \\ &= f(x) - f(x) + \frac{(m-1)r}{m} = \frac{(m-1)r}{m}. \end{aligned}$$

This completes the proof. □

3. Proof of the main result

In this section, we are going to prove our main theorem.

Let E_1 be an arbitrary subset of $E(H)$ with $|E_1| = r$. Put $E_2 = E(H) \setminus E_1$. Then $|E_2| = (m-1)r$. For any two disjoint subsets $S \subseteq V(G)$ and $T \subseteq V(G)$, let $T_0 = \{x | x \in T, g(x) = 0\}$ and $T_1 = T \setminus T_0$, it is easily seen that $T = T_0 \cup T_1$ and $T_0 \cap T_1 = \emptyset$. Let $g(x)$, E_1', E_1'', E_2', E_2'', $r_S(E_1)$ and $r_T(E_2)$ be defined as in Section 2. It follows instantly from the definitions of $r_S(E_1)$ and $r_T(E_2)$ that

$$r_S(E_1) \leq \min\{2r, r|S|\},$$

$$r_T(E_2) \leq \min\{2(m-1)r, (m-1)r|T|\},$$

$$r_{T_1}(E_2) \leq \min\{2(m-1)r, (m-1)r|T_1|\},$$

$$r_T(E_2) = r_{T_0}(E_2) + r_{T_1}(E_2),$$
$$r_{T_0}(E_2) \leq d_{G-S}(T_0).$$

The proof of theorem relies heavily on the following lemma.

Lemma 3.1. *Let G be a $(0, mf-(m-1)r)$-graph with $m \geq 2$ and $f(x) \geq 3r-1$ with $r \geq 2$. Then G admits a (g, f)-factor F_1 such that $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \emptyset$.*

Proof. By Lemma 2.2 and Lemma 2.4(1), it suffices to show that for any two disjoint subsets S and T of $V(G)$, we have

$$\delta_G(S, T; g, f) \geq r_S(E_1) + r_T(E_2).$$

For S and T, we obtain

$$\begin{aligned}
&\delta_G(S, T; g, f) \\
&= d_{G-S}(T) - g(T) + f(S) \\
&= d_{G-S}(T_1) - g(T_1) + f(S) + d_{G-S}(T_0) - g(T_0) \\
&\geq d_{G-S}(T_1) - g(T_1) + f(S) + r_{T_0}(E_2) \\
&= \frac{1}{m}d_G(T_1) - g(T_1) + f(S) - \frac{1}{m}d_G(S) + \frac{m-1}{m}d_{G-S}(T_1) \\
&\quad + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&= \triangle_1(T_1) + \triangle_2(S) + \frac{m-1}{m}d_{G-S}(T_1) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2).
\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}
&\delta_G(S, T; g, f) \\
&\geq \frac{r}{m}|T_1| + \frac{(m-1)r}{m}|S| + \frac{m-1}{m}d_{G-S}(T_1) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2).
\end{aligned} \tag{1}$$

Now let us distinguish among four cases.

Case 1. $S = \emptyset$, $T_1 = \emptyset$.

Thus we have $r_S(E_1) = 0$ and $r_{T_1}(E_2) = 0$. In view of (1), we have

$$\begin{aligned}
&\delta_G(S, T; g, f) \\
\geq\ & \frac{r}{m}|T_1| + \frac{(m-1)r}{m}|S| + \frac{m-1}{m}d_{G-S}(T_1) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
=\ & r_{T_0}(E_2) = r_S(E_1) + r_{T_1}(E_2) + r_{T_0}(E_2) = r_S(E_1) + r_T(E_2).
\end{aligned}$$

Case 2. $S = \emptyset$, $T_1 \neq \emptyset$.

Thus we have $r_S(E_1) = 0$. By the definition of T_1, it is easy to see that

$$g(x) \geq 1$$

for all $x \in T_1$.

Note that $g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-2)r)\}$. For all $x \in T_1$, we have

$$g(x) = d_G(x) - ((m-1)f(x) - (m-2)r) \geq 1.$$

Thus, we obtain

$$\begin{aligned} d_G(x) &\geq (m-1)f(x) - (m-2)r + 1 \\ &\geq (m-1)(3r-1) - (m-2)r + 1 \\ &= 2mr - m - r + 2 \end{aligned} \tag{2}$$

for all $x \in T_1$.

In view of (1), (2), $m \geq 2$ and $r \geq 2$, we have

$$\begin{aligned} \delta_G(S,T;g,f) &\geq \frac{r}{m}|T_1| + \frac{m-1}{m}d_G(T_1) + r_{T_0}(E_2) \\ &\geq \frac{r}{m}|T_1| + \frac{m-1}{m}(2mr-m-r+2)|T_1| + r_{T_0}(E_2) \\ &= \frac{r}{m}|T_1| + (m-1)r|T_1| + \frac{m-1}{m}(mr-m-r+2)|T_1| \\ &\quad + r_{T_0}(E_2) \\ &\geq \frac{r}{m}|T_1| + (m-1)r|T_1| + \frac{m-1}{m}[2(m-1)-m+2]|T_1| \\ &\quad + r_{T_0}(E_2) \\ &= \frac{r}{m}|T_1| + (m-1)r|T_1| + (m-1)|T_1| + r_{T_0}(E_2) \\ &> (m-1)r|T_1| + r_{T_0}(E_2) \\ &\geq r_{T_1}(E_2) + r_{T_0}(E_2) = r_T(E_2) \\ &= r_S(E_1) + r_T(E_2). \end{aligned}$$

Case 3. $S \neq \emptyset$, $T_1 = \emptyset$.

Thus we have $r_{T_1}(E_2) = 0$. According to $r_{T_0}(E_2) \leq d_{G-S}(T_0)$ and $g(T_0) = 0$, we obtain

$$\begin{aligned} \delta_G(S,T;g,f) &= d_{G-S}(T) - g(T) + f(S) \\ &= d_{G-S}(T_1) - g(T_1) + f(S) + d_{G-S}(T_0) - g(T_0) \\ &\geq d_{G-S}(T_1) - g(T_1) + f(S) + r_{T_0}(E_2) \\ &= f(S) + r_{T_0}(E_2) \\ &\geq (3r-1)|S| + r_{T_0}(E_2) \\ &\geq r|S| + r_{T_0}(E_2) \\ &\geq r_S(E_1) + r_{T_0}(E_2) \\ &= r_S(E_1) + r_{T_0}(E_2) + r_{T_1}(E_2) \\ &= r_S(E_1) + r_T(E_2). \end{aligned}$$

Case 4. $S \neq \emptyset$, $T_1 \neq \emptyset$.

Note that $d_{G-T_1}(S) \geq r_S(E_1)$.

Subcase 4.1. $|T_1| = 1$.

Thus we have

$$r_{T_1}(E_2) \le \min\{2(m-1)r, (m-1)r|T_1|\} = (m-1)r.$$

In view of (1), (2), $m \ge 2$ and $r \ge 2$, we have

$$\begin{aligned}
\delta_G(S,T;g,f) &\ge \frac{r}{m}|T_1| + \frac{(m-1)r}{m}|S| + \frac{m-1}{m}d_{G-S}(T_1) \\
&\quad + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&= \frac{r}{m}|T_1| + \frac{(m-1)(r-1)}{m}|S| + \frac{m-1}{m}(d_{G-S}(T_1) + |S|) \\
&\quad + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&\ge \frac{r}{m}|T_1| + \frac{(m-1)(r-1)}{m}|S| + \frac{1}{m}d_{G-T_1}(S) \\
&\quad + \frac{m-1}{m}d_G(x) + r_{T_0}(E_2) \qquad\qquad (x \in T_1) \\
&\ge \frac{r}{m}|T_1| + \frac{(m-1)(r-1)}{m}|S| + \frac{1}{m}d_{G-T_1}(S) \\
&\quad + \frac{m-1}{m}(2mr - m - r + 2) + r_{T_0}(E_2) \\
&= \frac{r}{m}|T_1| + \frac{(m-1)(r-1)}{m}|S| + \frac{1}{m}d_{G-T_1}(S) \\
&\quad + (m-1)r + \frac{(m-1)(mr-m-r+2)}{m} + r_{T_0}(E_2) \\
&\ge \frac{1}{m}d_{G-T_1}(S) + \frac{2r(m-1)}{m} + (m-1)r + \frac{r}{m} + \frac{(m-1)(r-1)}{m} \\
&\quad + \frac{(m-1)(mr-m-r+2)}{m} - \frac{2r(m-1)}{m} + r_{T_0}(E_2) \\
&\ge \frac{1}{m}r_S(E_1) + \frac{m-1}{m}r_S(E_1) + r_{T_1}(E_2) + r_{T_0}(E_2) \\
&\quad + \frac{(m-1)[(mr-m-r+2)+(r-1)-2r]+r}{m} \\
&= r_S(E_1) + r_T(E_2) + \frac{(m-1)[mr-2r-m+1]+r}{m} \\
&\ge r_S(E_1) + r_T(E_2) + \frac{(m-1)[2(m-2)-m+1]+r}{m} \\
&= r_S(E_1) + r_T(E_2) + \frac{(m-1)(m-3)+r}{m} \\
&\ge r_S(E_1) + r_T(E_2).
\end{aligned}$$

Subcase 4.2. $|T_1| \ge 2$.

Thus we have

$$r_{T_1}(E_2) \le \min\{2(m-1)r, (m-1)r|T_1|\} = 2(m-1)r$$

and $\exists x, y \in T_1$. By (1), (2), $m \geq 2$ and $r \geq 2$, we obtain

$$
\begin{aligned}
\delta_G(S,T;g,f) &\geq \frac{r}{m}|T_1| + \frac{(m-1)r}{m}|S| + \frac{m-1}{m}d_{G-S}(T_1) \\
&\quad + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&\geq \frac{2r}{m} + \frac{(m-1)r}{m}|S| + \frac{m-1}{m}d_{G-S}(T_1 \setminus x) \\
&\quad + \frac{m-1}{m}d_{G-S}(x) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&= \frac{2r}{m} + \frac{(m-1)(r-1)}{m}|S| + \frac{m-1}{m}d_{G-S}(x) \\
&\quad + \frac{m-1}{m}(d_{G-S}(T_1 \setminus x) + |S|) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&\geq \frac{2r}{m} + \frac{m-1}{m}|S| + \frac{m-1}{m}d_{G-S}(x) \\
&\quad + \frac{m-1}{m}(d_{G-S}(T_1 \setminus x) + |S|) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&\geq \frac{2r}{m} + \frac{m-1}{m}d_G(x) + \frac{m-1}{m}d_G(y) + \frac{1}{m}d_{G-T_1}(S) \\
&\quad + r_{T_0}(E_2) \qquad (x, y \in T_1) \\
&\geq \frac{2r}{m} + \frac{m-1}{m}(2mr - m - r + 2) \\
&\quad + \frac{m-1}{m}(2mr - m - r + 2) + \frac{1}{m}d_{G-T_1}(S) + r_{T_0}(E_2) \\
&\geq \frac{2r}{m} + \frac{2(m-1)}{m}(2mr - m - r + 2) \\
&\quad + \frac{1}{m}r_S(E_1) + r_{T_0}(E_2) \\
&= \frac{1}{m}r_S(E_1) + r_{T_0}(E_2) + \frac{2r(m-1)}{m} + 2(m-1)r \\
&\quad + \frac{2(m-1)}{m}(mr - m - 2r + 2) + \frac{2r}{m} \\
&\geq \frac{1}{m}r_S(E_1) + r_{T_0}(E_2) + \frac{m-1}{m}r_S(E_1) + r_{T_1}(E_2) \\
&\quad + \frac{2(m-1)}{m}((m-2)r - m + 2) + \frac{2r}{m} \\
&\geq r_S(E_1) + r_T(E_2) + \frac{2(m-1)}{m}(2(m-2) - m + 2) + \frac{2r}{m} \\
&= r_S(E_1) + r_T(E_2) + \frac{2(m-1)}{m}(m-2) + \frac{2r}{m} \\
&\geq r_S(E_1) + r_T(E_2).
\end{aligned}
$$

For any two disjoint subsets S and T of $V(G)$, we have

$$\delta_G(S,T;g,f) \geq r_S(E_1) + r_T(E_2).$$

By Lemma 2.2, G admits a (g,f)-factor F_1 such that $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \emptyset$. This completes the proof. □

Now we are ready to prove the main theorem.

Proof of Theorem 1. According to Lemma 2.3, the theorem is trivial for $r = 1$. In the following, we consider $r \geq 2$. Let $\{A_1, A_2, \ldots, A_m\}$ be any partition of $E(H)$ with $|A_i| = r$, $1 \leq i \leq m$. We prove that there is a $(0,f)$-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$ for all $1 \leq i \leq m$. We apply induction on m. The assertion is trivial for $m = 1$. Supposing the statement holds for $m-1$, let us proceed to the induction step.

Let $E_2 = E(H) \setminus A_1$. By Lemma 3.1, G has a (g,f)-factor F_1 such that $A_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \emptyset$. According to the definition of $g(x)$, obviously, F_1 is also a $(0,f)$-factor of G. Set $G' = G - E(F_1)$. It follows from the definition of $g(x)$ that

$$\begin{aligned} 0 \leq d_{G'}(x) &= d_G(x) - d_{F_1}(x) \leq d_G(x) - g(x) \\ &\leq d_G(x) - [d_G(x) - ((m-1)f(x) - (m-2)r)] \\ &= (m-1)f(x) - (m-2)r. \end{aligned}$$

Hence G' is a $(0,(m-1)f-(m-2)r)$-graph. Let $H' = G[E_2]$. Then the induction hypothesis guarantees the existence of a $(0,f)$-factorization $F' = \{F_2, \ldots, F_m\}$ in G' which satisfies $A_i \subseteq E(F_i)$, $2 \leq i \leq m$. Hence G has a $(0,f)$-factorization which is randomly r-orthogonal to H. This completes the proof. □

Acknowledgments. The authors would like to express their gratitude to the referees for their very helpful and detailed comments in improving this paper.

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