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STRONG CONVERGENCE THEOREMS FOR INFINITE COUNTABLE NONEXPANSIVE MAPPINGS AND IMAGE RECOVERY PROBLEM

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Abstract. In this paper, we introduce an iterative scheme given by infinite nonexpansive mappings in Banach spaces. We prove strong convergence theorems which are connected with the problem of image recovery. Our results enrich and complement the recent many results.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings from C into itself (recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$).

It is well known that the so-called problem of image recovery is essentially to find a common element of nonexpansive retracts $C_1, C_2, \ldots, C_N$ of C with $\cap_{i=1}^N C_i \neq \emptyset$. It is easy to see that every nonexpansive retraction P_i of C onto C_i is a nonexpansive mapping of C into itself. There is no doubt that the problem of image recovery is equivalent to finding a common fixed point of nonexpansive mappings $P_1, P_2, \ldots, P_N$ of C into itself. Now we recall some significant results in the literature concerning the problem of image recovery as follows.

In 1993, Kitahara and Takahashi [4] considered and studied the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space E; see also [12]. Moreover, they proved that an operator given by a convex combination of nonexpansive retractions in E is asymptotically regular and the set of fixed points of the operator is equal to the intersection of the ranges of nonexpansive retractions. Furthermore, using the results they proved some weak convergence theorems which are connected with the problem of image recovery. In 1997, Takahashi and Tamura [15] also considered the feasibility problem in the situation where the constraints are inconsistent. In 2000, Takahashi and Shimoji [14] introduced an iteration scheme,

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given by finitely many nonexpansive mappings, which generalizes Das and Debata's scheme [3], and then they proved weak convergence theorems which are connected with the problem of image recovery in a Banach space. We remark that the problem of finding a common fixed point of nonexpansive mappings of C into itself includes the problem of image recovery as a special case. Therefore, there is no doubt that it is very interesting and quite significant to establish the strong convergence or weak convergence results on the iteration schemes for finding a common fixed point of nonexpansive mappings $T_1, T_2, \ldots, T_N$ of C into itself.

Recently, constructing iterative schemes for finite or infinitely many nonexpansive mappings $T_1, T_2, \ldots$ in the settings of Hilbert spaces or some special Banach spaces have been considered by many authors. Especially, the convergence problem of the following iterative scheme

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad \forall n \geq 0$$

has been studied extensively. See, for example [2, 6, 7, 9, 10] and the references therein. But we note that the authors have imposed some additional assumptions on parameters or mappings $\{T_n\}$.

The purpose of this paper is to propose a new iterative scheme for finding common fixed points of an infinite countable nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$. Under very mild conditions on the parameters, it is proved that the sequence generated by our iterative scheme converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$. Our results enrich and complement the recent many results.

2. Preliminaries

Throughout this paper, we assume that E is a reflexive Banach space, C is a nonempty closed convex subset of E. E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued normalized duality mapping J by j and denote the fixed points set of a mapping T by $F(T)$.

Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. Recall that E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$, and E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Recall that a Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \ x \neq y \text{ implies } \frac{\|x + y\|}{2} < 1.$$

Let C be a nonempty closed convex subset of Banach space E. Let $\{T_i\}_{i=1}^{\infty}$ be infinite mappings of C into itself and let $\alpha_1, \alpha_2, \dots$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of C into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \alpha_n T_n U_{n,n+1} + (1-\alpha_n)I,\\
&U_{n,n-1} = \alpha_{n-1} T_{n-1} U_{n,n} + (1-\alpha_{n-1})I,\\
&\quad\vdots\\
&U_{n,k} = \alpha_k T_k U_{n,k+1} + (1-\alpha_k)I,\\
&U_{n,k-1} = \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})I,\\
&\quad\vdots\\
&U_{n,2} = \alpha_2 T_2 U_{n,3} + (1-\alpha_2)I,\\
&W_n = U_{n,1} = \alpha_1 T_1 U_{n,2} + (1-\alpha_1)I.
\end{aligned} \tag{1}
$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$; see [8, 5] for more details. We can find two important results in [8] concerning mappings W_n as follows.

Lemma 2.1 ([8, Lemma 3.2]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, and let $\alpha_1, \alpha_2, \dots$ be real numbers such that $0 < \alpha_i \le b < 1$ for any $i \in N$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \dots$ and $\alpha_1, \alpha_2, \dots$. Throughout this paper, we will assume that $0 < \alpha_i \le b < 1$ for every $i \in N$.

Lemma 2.2 ([8, Lemma 3.3]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, and let $\alpha_1, \alpha_2, \dots$ be real numbers such that $0 < \alpha_i \le b < 1$ for any $i \in N$. Then, $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.*

We will also make use of the following lemmas.

Lemma 2.3 ([6]). *Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose C is a nonempty closed convex subset of E. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \ne \emptyset$. Let $\{x_t\}$ is defined by $x_t = tu + (1-t)Tx_t$ where $u \in C$ is a fixed point. Then as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point of T.*

Lemma 2.4 ([1]). *Let E be a real Banach space. Then for all $x, y \in E$*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$ *for all* $j(x+y) \in J(x+y)$;
(ii) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle$ *for all* $j(x) \in J(x)$.

Lemma 2.5 ([11]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.6 ([16]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,
(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

Remark 2.1. If $s = (a_0, a_1, \ldots) \in l^\infty$ with $a_n \to a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit μ by (iii). For more details on Banach limits, we refer readers to [13].

Lemma 2.7 ([17]). *Let $a \in R$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+r} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

3. Main results

Now we state and prove the main results of this paper.

Theorem 3.1. *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of E, $\{T_n\}_{n=1}^{\infty}$ be infinite countable nonexpansive mappings from C to C such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For fixed $u \in C$ and any given $x_0 \in C$, let $\{x_n\}$ be the iterative sequence defined by*

$$x_{n+1} = \beta x_n + (1-\beta)W_n(\lambda_{n+1}u + (1-\lambda_{n+1})x_n), \quad \forall n \geq 0, \tag{2}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$, β is a constant in $(0,1)$ and W_n is the W-mapping defined by (1). Suppose the following conditions are satisfied

(C1) $\lim_{n\to\infty} \lambda_n = 0$;

(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequence $\{x_n\}$ *defined by* (2) *converges strongly to some common fixed point* $p \in F$.

Proof. First, we claim that for all $n \geq 0$,

$$\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \|u - x^*\|\}, \quad \forall x^* \in F.$$

Indeed, observe that the mappings $\{W_n\}$ are nonexpansive, then it follows from (2) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta(x_n - x^*) + (1-\beta)[W_n(\lambda_{n+1}u + (1-\lambda_{n+1})x_n) - x^*\| \\
&\leq \beta\|x_n - x^*\| + (1-\beta)\|W_n(\lambda_{n+1}u + (1-\lambda_{n+1})x_n) - x^*\| \\
&\leq \beta\|x_n - x^*\| + (1-\beta)[\lambda_{n+1}\|u - x^*\| + (1-\lambda_{n+1})\|x_n - x^*\|] \\
&= [1 - (1-\beta)\lambda_{n+1}]\|x_n - x^*\| + (1-\beta)\lambda_{n+1}\|u - x^*\| \\
&\leq \max\{\|x_n - x^*\|, \|u - x^*\|\}.
\end{aligned}$$

By induction,

$$\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \|u - x^*\|\}, \quad \forall n \geq 0,$$

and hence $\{x_n\}$ is bounded which leads to the boundedness of $\{W_n x_n\}$.

Setting $y_n = \lambda_{n+1}u + (1-\lambda_{n+1})x_n, \forall n \geq 0$. From (1), we have

$$\begin{aligned}
&\|W_{n+1}y_{n+1} - W_n y_n\| \\
&\leq \|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_n y_n\| \\
&\leq \|y_{n+1} - y_n\| + \|\alpha_1 T_1 U_{n+1,2} y_n - \alpha_1 T_1 U_{n,2} y_n\| \\
&\leq |\lambda_{n+2} - \lambda_{n+1}|(\|u\| + \|x_n\|) + (1-\lambda_{n+2})\|x_{n+1} - x_n\| \\
&\quad + \alpha_1\|U_{n+1,2}y_n - U_{n,2}y_n\| \\
&= |\lambda_{n+2} - \lambda_{n+1}|(\|u\| + \|x_n\|) + (1-\lambda_{n+2})\|x_{n+1} - x_n\| \\
&\quad + \alpha_1\|\alpha_2 T_2 U_{n+1,3} y_n - \alpha_2 T_2 U_{n,3} y_n\| \\
&\leq |\lambda_{n+2} - \lambda_{n+1}|(\|u\| + \|x_n\|) + (1-\lambda_{n+2})\|x_{n+1} - x_n\| \\
&\quad + \alpha_1\alpha_2\|U_{n+1,3}y_n - U_{n,3}y_n\| \\
&\leq \cdots \\
&\leq |\lambda_{n+2} - \lambda_{n+1}|(\|u\| + \|x_n\|) + (1-\lambda_{n+2})\|x_{n+1} - x_n\| \\
&\quad + M\prod_{i=1}^{n} \alpha_i,
\end{aligned} \tag{3}$$

where $M \geq 0$ is a constant such that $\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \leq M$ for all $n \geq 0$.

Therefore, from (3), we obtain

$$\begin{aligned}&\|W_{n+1}y_{n+1} - W_ny_n\| - \|x_{n+1} - x_n\| \\ &\leq |\lambda_{n+2} - \lambda_{n+1}|(\|u\| + \|x_n\|) + M\prod_{i=1}^{n}\alpha_i,\end{aligned}$$

which implies that (noting that (C1) and $0 < \alpha_i \leq b < 1$ for all $i \in N$)

$$\limsup_{n\to\infty}(\|W_{n+1}y_{n+1} - W_ny_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\|W_ny_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta)\|W_ny_n - x_n\| = 0. \tag{4}$$

For each $k \in N$, let u_k be a unique element of C such that

$$u_k = \frac{1}{k}u + (1 - \frac{1}{k})Wu_k. \tag{5}$$

From Lemma 2.2 and Lemma 2.3, we obtain that

$$u_k \to p \in F(W) = \bigcap_{n=1}^{\infty} F(T_n) \text{ as } k \to \infty.$$

For every $n, k \in N$, we have

$$\begin{aligned}
&\|x_{n+1} - Wu_k\| \\
&= \|\beta(x_n - Wu_k) + (1-\beta)(W_ny_n - Wu_k)\| \\
&\leq \beta\|x_n - Wu_k\| + (1-\beta)\|W_ny_n - Wu_k\| \\
&\leq \beta\|x_n - Wu_k\| + (1-\beta)\|W_ny_n - W_nu_k\| + (1-\beta)\|W_nu_k - Wu_k\| \\
&\leq \beta\|x_n - Wu_k\| + (1-\beta)\|y_n - u_k\| + (1-\beta)\|W_nu_k - Wu_k\| \\
&\leq \beta\|x_n - Wu_k\| + (1-\beta)\lambda_{n+1}\|u - u_k\| \\
&\quad + (1-\beta)(1-\lambda_{n+1})\|x_n - u_k\| + (1-\beta)\|W_nu_k - Wu_k\| \\
&\leq \beta\|x_n - Wu_k\| + (1-\beta)\|x_n - u_k\| + (1-\beta)\lambda_{n+1}[\|u - u_k\| \\
&\quad + \|x_n - u_k\|] + (1-\beta)\|W_nu_k - Wu_k\| \\
&= \beta\|x_n - Wu_k\| + (1-\beta)\|x_n - u_k\| + \gamma_n,
\end{aligned} \tag{6}$$

where $\gamma_n = (1-\beta)\lambda_{n+1}[\|u - u_k\| + \|x_n - u_k\|] + (1-\beta)\|W_nu_k - Wu_k\|$. Since $\lim_{n\to\infty}\lambda_{n+1} = 0$ and $\lim_{n\to\infty}W_nu_k = Wu_k$, then $\gamma_n \to 0$ as $n \to \infty$.

From (6), we obtain

$$
\begin{aligned}
\|x_{n+1} - Wu_k\|^2 &\le (\beta\|x_n - Wu_k\| + (1-\beta)\|x_n - u_k\|)^2 \\
&\quad + \gamma_n[2(\beta\|x_n - Wu_k\| + (1-\beta)\|x_n - u_k\|) + \gamma_n] \\
&= \beta^2\|x_n - Wu_k\|^2 + (1-\beta)^2\|x_n - u_k\|^2 \\
&\quad + 2\beta(1-\beta)\|x_n - Wu_k\|\|x_n - u_k\| + \sigma_n \\
&\le \beta^2\|x_n - Wu_k\|^2 + (1-\beta)^2\|x_n - u_k\|^2 \\
&\quad + \beta(1-\beta)(\|x_n - Wu_k\|^2 + \|x_n - u_k\|^2) + \sigma_n \\
&= \beta\|x_n - Wu_k\|^2 + (1-\beta)\|x_n - u_k\|^2 + \sigma_n,
\end{aligned} \tag{7}
$$

where $\sigma_n = \gamma_n[2(\beta\|x_n - Wu_k\| + (1-\beta)\|x_n - u_k\|) + \gamma_n] \to 0$ as $n \to \infty$.

For any Banach limit μ, from (7), we obtain

$$\mu_n\|x_n - Wu_k\|^2 = \mu_n\|x_{n+1} - Wu_k\|^2 \le \mu_n\|x_n - u_k\|^2. \tag{8}$$

Noting that $x_n - u_k = \frac{1}{k}(x_n - u) + (1 - \frac{1}{k})(x_n - Wu_k)$, that is

$$(1 - \frac{1}{k})(x_n - Wu_k) = (x_n - u_k) - \frac{1}{k}(x_n - u). \tag{9}$$

It follows Lemma 2.4(ii) and (9) that

$$
\begin{aligned}
(1 - \frac{1}{k})^2\|x_n - Wu_k\|^2 &\ge \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u, j(x_n - u_k)\rangle \\
&= \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u_k + u_k - u, j(x_n - u_k)\rangle \\
&= (1 - \frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle u - u_k, j(x_n - u_k)\rangle.
\end{aligned} \tag{10}
$$

So by (8) and (10), we have

$$
\begin{aligned}
(1 - \frac{1}{k})^2\mu_n\|x_n - u_k\|^2 &\ge (1 - \frac{1}{k})^2\mu_n\|x_n - Wu_k\|^2 \\
&\ge (1 - \frac{2}{k})\mu_n\|x_n - u_k\|^2 + \frac{2}{k}\mu_n\langle u - u_k, j(x_n - u_k)\rangle,
\end{aligned}
$$

and hence

$$\frac{1}{k^2}\mu_n\|x_n - u_k\|^2 \ge \frac{2}{k}\mu_n\langle u - u_k, j(x_n - u_k)\rangle.$$

This implies

$$\frac{1}{2k}\mu_n\|x_n - u_k\|^2 \ge \mu_n\langle u - u_k, j(x_n - u_k)\rangle.$$

Since $u_k \to p \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$ as $k \to \infty$, from the uniformly Gâteaux differentiability of the norm of E and the above inequality, we get

$$\mu_n\langle u - p, j(x_n - p)\rangle \le 0. \tag{11}$$

On the other hand, from (4), we have

$$\lim_{n\to\infty} |\langle u - p, j(x_{n+1} - p)\rangle - \langle u - p, j(x_n - p)\rangle| = 0. \tag{12}$$

Hence it follows from Lemma 2.7, (11), and (12) that

$$\limsup_{n\to\infty}\langle u-p, j(x_n-p)\rangle \le 0,$$

this together with $\lim_{n\to\infty}\|y_n-x_n\| = \lim_{n\to\infty}\lambda_{n+1}\|u-x_n\| = 0$ imply that

$$\limsup_{n\to\infty}\langle u-p, j(y_n-p)\rangle \le 0. \tag{13}$$

Finally we prove that $x_n \to p$ as $n\to\infty$. From Lemma 2.4(i) and (2), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&= \|\beta(x_n-p)+(1-\beta)(W_ny_n-p)\|^2\\
&\le [\beta\|x_n-p\|+(1-\beta)\|y_n-p\|]^2\\
&\le \beta^2\|(x_n-p)\|^2+(1-\beta)^2\|y_n-p\|^2+2\beta(1-\beta)\|x_n-p\|\|y_n-p\|\\
&\le \beta^2\|(x_n-p)\|^2+(1-\beta)^2\|y_n-p\|^2+\beta(1-\beta)(\|x_n-p\|^2+\|y_n-p\|^2)\\
&= \beta\|x_n-p\|^2+(1-\beta)\|y_n-p\|^2\\
&\le \beta\|x_n-p\|^2+(1-\beta)[(1-\lambda_{n+1})^2\|x_n-p\|^2+2\lambda_{n+1}\langle u-p, j(y_n-p)\rangle]\\
&= [1-2(1-\beta)\lambda_{n+1}]\|x_n-p\|^2+(1-\beta)\lambda_{n+1}^2\|x_n-p\|^2\\
&\quad +2(1-\beta)\lambda_{n+1}\langle u-p, j(y_n-p)\rangle\\
&= [1-2(1-\beta)\lambda_{n+1}]\|x_n-p\|^2+2(1-\beta)\lambda_{n+1}\{\frac{1}{2}\lambda_{n+1}\|x_n-p\|\\
&\quad +\langle u-p, j(y_n-p)\rangle\}\\
&= (1-a_n)\|x_n-p\|+a_nb_n,
\end{aligned} \tag{14}$$

where $a_n = 2(1-\beta)\lambda_{n+1}$ and $b_n = \frac{1}{2}\lambda_{n+1}\|x_n-p\| + \langle u-p, j(y_n-p)\rangle$. We note that $\{\|x_n-p\|\}$ are bounded sequences and $\lim_{n\to\infty}\lambda_{n+1} = 0$. It is easily seen that $\sum_{n=0}^{\infty} a_n = \infty$ and $\limsup_{n\to\infty} b_n \le 0$. Hence the conditions in Lemma 2.6 are satisfied and so we can conclude from (14) that $x_n \to p$ as $n\to\infty$. This completes the proof. □

Next, we give a strong convergence theorem which is connected with the problem of image recovery in a Banach space setting. The proof is obvious from Theorem 3.1 and hence will be omitted.

Theorem 3.2. *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of E. Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive retracts of C such that $\cap_{n=1}^{\infty} C_n$ is nonempty. Let W_n be the W-mapping generated by $P_n, P_{n-1}, \ldots, P_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_n$, where P_k is a nonexpansive retraction of C onto C_k. For fixed $u \in C$ and any given $x_0 \in C$, let $\{x_n\}$ be the iterative sequence defined by*

$$x_{n+1} = \beta x_n + (1-\beta)W_n(\lambda_{n+1}u + (1-\lambda_{n+1})x_n), \quad \forall n \ge 0, \tag{15}$$

where $\{\lambda_n\}$ *is a sequence in* $(0,1)$*,* β *is a constant in* $(0,1)$*. Suppose the following conditions are satisfied*

(C1) $\lim_{n\to\infty}\lambda_n = 0$;

(C2) $\sum_{n=0}^{\infty}\lambda_n = \infty$.

Then the sequence $\{x_n\}$ *defined by* (15) *converges strongly to* $x^* \in \cap_{n=1}^{\infty} C_n$.

Remark 3.1. (i) We note that the authors in [2, 6, 7, 9, 10] have imposed some additional assumptions on parameters $\{\lambda_{n+1}\}$ or mappings $\{T_n\}$ as follows:

(A1) $\sum_{n=1}^{\infty}|\lambda_{n+N}-\lambda_n| < \infty$;

(A2) $\lim_{n\to\infty}(\lambda_{n+N}-\lambda_n)/\lambda_{n+N} = 0$ or equivalently, $\lim_{n\to\infty}\lambda_n/\lambda_{n+N} = 1$;

(A3) $\lim_{n\to\infty}\sup_{x\in C}\|T(T_n x) - T_n x\| = 0$;

(A4) $\lim_{n\to\infty}\sup_{x\in C}\|T_m(T_n x) - T_n x\| = 0$.

(ii) The advantages of these results in this paper are that less restrictions on the parameters $\{\lambda_n\}$ are imposed.

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