Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 757–762

A NOTE ON LOWER RADICALS OF HEMIRINGS

Muhammad Zulfiqar

Abstract. In this paper, we generalize a few results of [7, 10] for lower radical classes of rings, by using the limit ordinal construction for lower radical classes of hemirings.

1. Introduction and preliminaries

D. M. Olson and T. L. Jenksins [8] discussed general Radical Theory of Hemirings. The theory was further enriched by many authors (see [4, 12, 13]). The lower radicals were investigated by (see [5, 6, 9, 10, 11]) for radical classes of rings. Here we are interesting to generalize a several results of (see [3, 7, 10]) in the frame work of hemiring which is quite different from ring theoretical approach discussed in (see [3, 7, 10]).

A semiring $(R, +, \cdot)$ is called a *hemiring* if

(i) '+' is commutative,

(ii) there exists an element $0 \in R$ such that 0 is the identity of $(R, +)$ and the zero element of $(R, \cdot)$, i.e., $0r = r0 = 0$, $\forall r \in R$.

If I is an semi-ideals of R, then we denote $I \leq R$.

Lower radical classes for hemirings can be constructed similar to the construction of lower radicals for rings (see [3, 5, 6, 9, 10,11]).

First we include necessary preliminary, let ω be the universal class of all hemirings and M be a sub-class of ω and let M_0 be the homomorphic closure of M in ω. For each $A \in \omega$, let $D_1(R)$ be the set of all semi-ideals of R. Inductively we define

$$D_{n+1}(R) = \{I : I \text{ is an semi-ideal of some hemiring in } D_n(R)\}.$$

Let $D(R) = \bigcup_{n \in N} D_n(R)$, $n = 1, 2, 3, \ldots$. By using ring theoretical approach discussed in [11], we have

$$\pounds M = \{R \in \omega : D(R/I) \cap M_0 \neq 0 \text{ for each proper semi-ideal } I \text{ of } R\},$$

Received March 20, 2008.

2000 *Mathematics Subject Classification.* 16N80.

Key words and phrases. hemiring, radical classes, universal class, semisimple class, homomorphically closed, p-semi-ideal, k-ideal, hereditary class, lower radical.

is the Lee construction for lower radical determined by M, and $M \subseteq \pounds M$ (see also [8, 12, 13]).

Let ω be a universal class of (not necessarily associative) hemirings and let $A \subseteq \omega$. R. Wiegandt has given in [11] a construction for $\pounds A$, the lower radical class determined by A in ω . Using this construction, Leavitt and Hoffmann have proved in [3] that if A is a hereditary class (if $K \in A$ and $I \leq K$, then $I \in A$), then $\pounds A$ is also hereditary. In this paper lower radical construction is given. As applications, a simple proof is given of the theorem of Leavitt and Hoffmann and a result of Y. L. Lee and R. E. Propes [7] for hemirings is extended to not necessarily associative hemirings.

Let $A \subseteq \omega$ be any class of hemirings. Define $R_1(A)$ to be the homomorphic closure of A. Proceeding inductively, let μ be an ordinal exceeding one and suppose the classes $R_\lambda(A)$ have been defined for all $\lambda < \mu$. If μ is not a limit ordinal, define

$$R_\mu(A) = \{K \in \omega : I, K/I \in R_{\mu-1}(A) \text{ for some } I < K\}.$$

If μ is a limit ordinal, define $R_\mu(A) = \{K \in \omega : K$ contains a chain I_γ of semi-ideals such that each $I_\gamma \in \bigcup_{\lambda<\mu} R_\lambda(A)$, and $K = \cup I_\gamma\}$. Finally define $R(A) = \cup R_\lambda(A)$, where the union is taken over all ordinals λ . For undefined terms of hemirings we may refer (see [1, 2, 8]).

2. Lower radicals

We extend the result of [7] by using the above limit ordinal construction of lower radical for hemiring which is indeed provides an excellent and different approach to handle the many results of [7] in the frame work of hemiring.

The following theorem was proved by F. A. Szasz [10] for rings. Here we generalize it for hemiring, which can be obtained on the line of rings theoretical approach.

Theorem 2.1. *Let ω be a universal class of hemiring and let $A \subseteq \omega$. Then A is a radical class in ω if and only if the following conditions are satisfied*:

(1) *A is homomorphically closed,*

(2) *If I, $K/I \in A$, then $K \in A$,*

(3) *The union of a chain of A-semi-ideals of a ω-hemiring K is again an A-semi-ideal of K.*

The following theorem is obvious.

Theorem 2.2. *If λ and μ are ordinals with $\lambda \leq \mu$, then $R_\lambda(A) \subseteq R_\mu(A)$.*

Theorem 2.3. *For every ordinal $\lambda \geq 1$, $R_\lambda(A)$ is homomorphically closed. Hence $R(A)$ is homomorphically closed.*

Proof. $R_1(A)$ is homomorphically closed. Let $\mu > 1$ be an ordinal, and suppose $R_\lambda(A)$ is homomorphically closed for all $\lambda < \mu$. Let $K \in R_\mu(A)$ and let $I < K$. If μ is a limit ordinal, there is a chain $\{I_\gamma\}$ of semi-ideals of K such that

$I_\gamma \in R_\lambda(A)$ with $\lambda < \mu$ and such that $K = \cup I_\gamma$. But $\{(I + I_\gamma)/I\}$ is a chain of semi-ideals of K/I, and K/I is its union. Since $(I + I_\gamma)/I \cong I_\gamma/(I^* \cap I_\gamma)$, where I^* is a k-ideal generated by I (see [8, 13])) each of these semi-ideals is a homomorphism of some I_γ, and thus by induction hypothesis each $(I+I_\gamma)/I \in R_\lambda(A)$ with $\lambda < \mu$. This implies that $K/I \in R_\mu(A)$.

Now suppose $\mu-1$ exists. Then K contains an semi-ideal J so that $J, K/J \in R_{\mu-1}(A)$. By the induction hypothesis, $(J+I)/I \in R_{\mu-1}(A)$ and $K/(I+J) \in R_{\mu-1}(A)$, since the former is a homomorphic image of J and latter of K/J. Since $[R/I]/[(J+I)^*/I] \cong R/(J+I)$, where $(J+I)^*$ is a k-ideal generated by $(J+I)$ (see [12]), $K/I \in R_\mu(A)$. Thus by transfinite induction $R_\mu(A)$ is homomorphically closed for ordinal μ. It follows immediately that $R(A)$ is homomorphically closed. □

We now show that $R(A)$ satisfies Conditions 2 and 3 of Theorem 2.1.

Theorem 2.4. *Let $K \in \omega$, and let $\{I_\lambda\}$ be a chain of $R(A)$-semi-ideals of K. Then $\cup I_\lambda$ is an $R(A)$-semi-ideal of K.*

Proof. Since K is a set, there is by Theorem 2.2 an ordinal μ with the property that $I_\lambda \in R_\mu(A)$ for each λ. Let δ be a limit ordinal exceeding μ, then $\cup\, I_\lambda \in R_\delta(A)$. □

Theorem 2.5. *Let $K \in \omega$, and suppose K contains an semi-ideal $I \in R(A)$ such that $K/I \in R(A)$. Then $K \in R(A)$.*

Proof. By Theorem 2.2, there is an ordinal μ such that I, $K/I \in R_\mu(A)$. This implies that $K \in R_{\mu+1}(A)$. □

Theorem 2.6. $R(A) = \pounds(A)$.

Proof. By Theorem 2.1 and Theorems 2.3, 2.4, and 2.5, $R(A)$ is a radical class in ω . By the minimality of $\pounds(A)$ among radical classes in ω which contain A, it is enough to show $R(A) \subseteq \pounds(A)$. This is accomplished by proving $R_\lambda(A) \subseteq \pounds(A)$ for every ordinal λ. Clearly $R_1(A) \subseteq \pounds(\mathrm{A})$. Let μ be an ordinal exceeding one, and assume $R_\lambda(A) \subseteq \pounds(A)$ for all ordinals $\lambda < \mu$. Let $K \in R_\mu(A)$. If μ is a limit ordinal, K be the union of a chain of semi-ideals from the classes $R_\lambda(A)$, where $\lambda < \mu$. Thus by induction hypothesis K is the union of $\pounds(A)$-semi-ideals, so $K \in \pounds(A)$ by Theorem 2.1. If μ is not a limit ordinal, there is an semi-ideal I of K such that $I \in R_{\mu-1}(A) \subseteq \pounds(A)$ and $K/I \in R_{\mu-1}(A) \subseteq \pounds(A)$. Again, $K \in \pounds(A)$ by Theorem 2.1. Thus $R_\mu(A) \subseteq \pounds(A)$ for all ordinals $\mu \geq 1$. □

We now give a simple proof of the following theorem which appears in [3]. Other results of the form A has property ρ implies $\pounds A$ has property may, perhaps, be provable in a similar way. The following theorem was proved by A. E. Hoffman and W. G. Leavitt [3] and we generalize it in the frame work of hemirings. Here we give a proof of this theorem which is entirely different from [3].

Theorem 2.7. *Let $A \subseteq \omega$ where ω is some universal class of hemiring. Then if A is hereditary, so is $\pounds(A)$.*

Proof. We prove that $R_\mu(A)$ is hereditary for each $\mu \geq 1$. This is easily seen to be true if $\mu = 1$. Thus, assume $\mu > 1$, and suppose $R_\lambda(A)$ is a hereditary class for each $\lambda < \mu$. Let $K \in R_\mu(A)$, and suppose $I \leq K$. If μ is a limit ordinal, $K = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of semi-ideals each belonging to one of the (hereditary) classes $R_\lambda(A)$, $\lambda < \mu$. But then $I = \cup(I_\gamma \cap I)$ so $I \in R_\mu(A)$. If μ is not a limit ordinal, there is an semi-ideal J of K so that J, $K/J \in R_{\mu-1}(A)$. Since $R_{\mu-1}(A)$ is hereditary, $I \cap J \in R_{\mu-1}(A)$ and $(J + I)/J \cong I/(I \cap J^*) \in R_{\mu-1}(A)$, where J^* is a k-ideal generated by J (see [8, 13])). This implies that $I \in R_\mu(A)$. □

If ρ is a radical class of hemiring then its semisimple class is denoted by Sρ. The proof of Theorem 2.9 requires the following lemma.

Lemma 2.8. *If ρ is a radical class in ω and for some $\acute{K} \in \omega$ a subhemiring $K \subseteq \acute{K}$ is the set theoretic union of ρ-semi-ideals of $\acute{K}$, then $K \in \rho$.*

Proof. If $K = \cup I_\lambda$ not belong to ρ, then $K/I \in S\rho = \{T \in \omega : T$ has no nonzero ρ-semi-ideals$\}$ for some $I \neq K$. By λ, we have $I_\lambda \nsubseteq I$, so

$$(I_\lambda + I)/I \cong I_\lambda/(I^* \cap I_\lambda),$$

where I^* is a k-ideal generated by I (see [8, 13])) is a nonzero ρ-semi-ideal of K/I. This contradiction proves that $K \in \rho$. □

The following theorem was proved by Y. L. Lee and R. E. Propes [7] and we generalize it in the frame work of hemirings. Here we give a proof of this theorem which is entirely different from [7].

Theorem 2.9. *If A_1 and A_2 are homomorphically closed, hereditary classes of ω-hemirings, then $\pounds(A_1 \cap A_2) = \pounds A_1 \cap \pounds A_2$.*

Proof. Trivially $\pounds(A_1 \cap A_2) \subseteq \pounds A_1 \cap \pounds A_2$. Since $K \in \pounds A_1 \cap \pounds A_2$ if and only if $K \in R_\gamma(A_1) \cap R_\gamma(A_2)$ for some ordinal number γ. It suffices to prove $R_\gamma(A_1) \cap R_\gamma(A_2) \subseteq \pounds A_1 \cap \pounds A_2$ for each ordinal $\gamma \geq 1$. This is clear for $\gamma = 1$. Let μ be an ordinal number greater 1 and suppose $R_\lambda(A_1) \cap R_\lambda(A_2) \subseteq \pounds(A_1 \cap A_2)$ for each ordinal $\lambda < \mu$. Let $K \in R_\mu(A_1) \cap R_\mu(A_2)$. If μ is a limit ordinal, K is the union of a chain $\{I_\gamma\}$, $\gamma \in C$ of semi-ideals each belonging to one of the classes $R_\lambda(A_1)$ for $\lambda < \mu$. Also K is the union of a chain $\{J_\delta\}$, $\delta \in D$ of semi-ideals each belonging to one of the classes $R_\lambda(A_2)$ for $\lambda < \mu$. If $x \in K$, $x \in J_\delta$ for some $\delta \in D$ and $x \in I_\gamma$ for some $\gamma \in C$, so $x \in J_\delta \cap I_\gamma$ for some $(\delta, \gamma) \in D \times C$. Since $J_\delta \in R_\lambda(A_2)$ for $\lambda < \mu$, and since $R_\lambda(A_2)$ is hereditary (see proof of Theorem 2.7), $J_\delta \cap I_\alpha \in R_\lambda(A_2)$. Similarly $J_\delta \cap I_\gamma \in R_\eta(A_1)$ for some $\eta < \mu$. Thus $J_\delta \cap I_\gamma \in R_\beta(A_1) \cap R_\beta(A_2)$, where $\beta = \max[\eta, \lambda]$. Since $\beta < \mu$, the induction hypothesis so that K is the set-theoretic union of $\pounds(A_1 \cap A_2)$-semi-ideals. So, by Lemma 2.8, $K \in \pounds(A_1 \cap A_2)$. Now suppose $\mu - 1$ exists, and let $K \in R_\mu(A_1) \cap R_\mu(A_2)$. Then there exist semi-ideals I and

J such that I, $K/I \in R_{\mu-1}(A_1)$ and J, K/J $R_{\mu-1}(A_2)$. Since $R_{\mu-1}(A_1)$ and $R_{\mu-1}(A_2)$ are hereditary, $I \cap J \in R_{\mu-1}(A_1) \cap R_{\mu-1}(A_2)$ so $I \cap J \in \pounds(A_1 \cap A_2)$. Since $R_{\mu-1}(A_1)$ is homomorphically closed (Theorem 2.3),

$$I/(I \cap J^*) \cong (I + J)/J \in R_{\mu-1}(A_1),$$

where J^* is a k-ideal generated by J (see [8, 13]). Since $R_{\mu-1}(A_2)$ is hereditary, $(I + J)/J$, as an semi-ideal of K/J is a member of $R_{\mu-1}(A_2)$. Thus

$$I/(I \cap J^*) \cong (I + J)/J \in R_{\mu-1}(A_1) \cap R_{\mu-1}(A_2) \subseteq \pounds(A_1 \cap A_2),$$

where J^* is a k-ideal generated by J (see [8, 13]). Thus $I \cap J \in R_{\mu-1}(A_1) \cap R_{\mu-1}(A_2) \subseteq \pounds(A_1 \cap A_2)$ and $I/(I + J) \in R_{\mu-1}(A_1) \cap R_{\mu-1}(A_2) \subseteq \pounds(A_1 \cap A_2)$.

Thus since $I \cap J \in \pounds(A_1 \cap A_2)$ and $I/(I + J) \in \pounds(A_1 \cap A_2)$, $I \in \pounds(A_1 \cap A_2)$. Similarly, $J \in \pounds(A_1 \cap A_2)$ so that $I + J$ is an $\pounds(A_1 \cap A_2)$-semi-ideal of K. Also $K/(I + J) \in R_{\mu-1}(A_1) \cap R_{\mu-1}(A_2) \subseteq \pounds(A_1 \cap A_2)$ since it is homomorphically image of both K/J and K/I. Thus, since $I + J \in \pounds(A_1 \cap A_2)$ and $K/(I + J) \in \pounds(A_1 \cap A_2)$, we have that $K \in \pounds(A_1 \cap A_2)$. We have shown that $R_\mu(A_1) \cap R_\mu(A_2) \subseteq \pounds(A_1 \cap A_2)$ which proves the theorem. □

References

[1] J. S. Golan, *The theory of semirings with applications in mathematics and theoretical computer science*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 54. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.

[2] U. Hebisch and H. J. Weinert, *Semirings: algebraic theory and applications in computer science*, Translated from the 1993 German original. Series in Algebra, 5. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[3] A. E. Hoffman and W. G. Leavitt, *Properties inherited by the lower radical*, Portugal. Math. **27** (1968), 63–66.

[4] D. R. LaTorre, *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen **12** (1965), 219–226.

[5] Y. L. Lee, *On the construction of upper radical properties*, Proc. Amer. Math. Soc. **19** (1968), 1165–1166.

[6] ———, *On the construction of lower radical properties*, Pacific J. Math. **28** (1969), 393–395.

[7] Y. L. Lee and R. E. Propes, *On intersections and unions of radical classes*, J. Austral. Math. Soc. **13** (1972), 354–356.

[8] D. M. Olson and T. L. Jenksins, *Radical theory for hemirings*, J. Natur. Sci. Math. **23** (1983), no. 1, 23–32.

[9] H. J. le Roux and G. A. P. Heyman, *A note on the lower radical*, Publ. Math. Debrecen **28** (1981), no. 1-2, 11–13.

[10] F. A. Szász, *Radicals of Rings*, Mathematical Institute Hungarian Academy of Sciences, 1981.

[11] R. Wiegandt, *Radical and Semisimple Classes of Rings*, Queen's Papers in Pure and Applied Mathematics, No. 37. Queen's University, Kingston, Ont., 1974.

[12] S. M. Yusuf and M. Shabir, *Radical classes and semisimple classes for hemirings*, Studia Sci. Math. Hungar. **23** (1988), no. 1-2, 231–235.

[13] M. Zulfiqar, *The sum of two radical classes of hemirings*, Kyungpook Math. J. **43** (2003), no. 3, 371–374.

Department of Mathematics
Govt. College University Lahore
Lahore 54000, Pakistan
E-mail address: mzulfiqarshafi@hotmail.com