

ON THE STABILITY OF A GENERALIZED CUBIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we obtain the general solution of a generalized cubic functional equation, the Hyers-Ulam-Rassias stability, and the stability by using the alternative fixed point for a generalized cubic functional equation

$$\begin{aligned} & 4f\left(\sum_{j=1}^{n-1} x_j + mx_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - mx_n\right) + m^2 \sum_{j=1}^{n-1} f(2x_j) \\ &= 8f\left(\sum_{j=1}^{n-1} x_j\right) + 4m^2 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned}$$

for a positive integer $m \geq 1$.

1. Introduction

The stability theory of functional equations started with the talk of S. M. Ulam held at the Wisconsin University in 1940 as follows: Under what condition does there exist an additive mapping near an approximately additive mapping? see [19].

The first partial solution to Ulam's question was provided by D. H. Hyers [7]. Let X and Y are Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \rightarrow Y$ satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \rightarrow Y$ is the unique additive function such that

$$\|f(x) - a(x)\| \leq \epsilon$$

for any $x \in X$.

Thirty seven years after Hyers's Theorem, Th. M. Rassias in his paper [12], provided a remarkable generalization of Hyers's result by allowing for the first

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time in the subject of functional equations and inequalities the Cauchy difference to be unbounded. This fact rekindled interest of several mathematicians worldwide in the study of several important functional equations of several variables. In 1990, Th. M. Rassias [16] during the 27th International Symposium on Functional Equations asked the question whether his Theorem can also be proved for all real values of p that are greater or equal to one. In 1991, Z. Gajda following Th. M. Rassias' approach to stability of functional equations obtained an affirmative solution to this question for all real values of p that are strictly greater than one; see [5].

The cubic function $f(x) = cx^3$ ($c \in \mathbb{R}$) satisfies the functional equation

$$(1.1) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

The equation (1.1) was solved by Jun and Kim [11]. They proved that a function $f : X \rightarrow Y$ is a solution of the equation (1.1) if and only if there exists a function $F : X \times X \times X \rightarrow Y$ such that $f(x) = F(x, x, x)$ for all $x \in X$, and F is symmetric for each fixed one variable and is additive for fixed two variables; see [11]. We promise that by a cubic function we mean every solution of the equation (1.1) is called a cubic function. Also, the equation (1.1) is equivalent to the following equation (see [2, Lemma 2.1]);

$$(1.2) \quad f(x + 2y) + f(x - 2y) + f(2x) = 2f(x) + 4f(x + y) + 4f(x - y).$$

Recently, the n -dimensional cubic functional equation was investigated by Chu and Kang [2]. Also, G. Isac and Th. M. Rassias [10] were the first mathematicians to apply the Hyers-Ulam-Rassias stability approach for the proof of new fixed point theorems. For an extensive account on the development of fixed point theory and related topics the reader is referred to the book; see [8].

In this paper, we will investigate the Hyers-Ulam-Rassias stability and the stability by using the alternative fixed point for a generalized cubic functional equation as follows:

$$(1.3) \quad \begin{aligned} & 4f\left(\sum_{j=1}^{n-1} x_j + mx_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - mx_n\right) + m^2 \sum_{j=1}^{n-1} f(2x_j) \\ & = 8f\left(\sum_{j=1}^{n-1} x_j\right) + 4m^2 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned}$$

for all $x_1, \dots, x_n \in X$, where $m \geq 1$ is an integer number.

2. Generalized cubic functional equation

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \rightarrow Y$ satisfies the functional equation (1.3) if and only if f is cubic. Therefore, every solution of functional equations (1.3) is also a cubic function.*

Proof. Suppose that f satisfies the equation (1.3) for each integer $m \geq 1$. In particular, when $m = 2$, we know that f satisfying (1.3) is cubic; see [2, Lemma 2.2]. Hence we note that the case $m = 2$ can implies the equation (1.2). It is easy to check that $f(0) = 0$ and $f(2x) = 8f(x)$ for all $x \in X$. First, we start with the case where $m = 1$; by letting $x_1 = x_1 + x_n$ and $x_1 = x_1 - x_n$, we have

$$\begin{aligned} & 4f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j\right) + f(2(x_1 + x_n)) + \sum_{j=2}^{n-1} f(2x_j) \\ &= 8f\left(\sum_{j=1}^{n-1} x_j + x_n\right) + 4(f(x_1 + 2x_n) + f(x_1)) \\ & \quad + 4\sum_{j=2}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right), \end{aligned}$$

and,

$$\begin{aligned} & 4f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j\right) + f(2(x_1 - x_n)) + \sum_{j=2}^{n-1} f(2x_j) \\ &= 8f\left(\sum_{j=1}^{n-1} x_j - x_n\right) + 4(f(x_1 - 2x_n) + f(x_1)) \\ & \quad + 4\sum_{j=2}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned}$$

for all $x_1, \dots, x_n \in X$, respectively.

Adding two above equations and using cases where $m = 1, 2$ and $x_1 = x, x_n = y$, we have

$$f(x + 2y) + f(x - 2y) + f(2x) = 2f(x) + 4\left(f(x + y) + f(x - y)\right)$$

for all $x, y \in X$. Hence f is cubic when $m = 1$. Now for any integer $m \geq 3$, letting $x_1 = x_1 + x_n$ and $x_1 = x_1 - x_n$ in the equation (1.3) we have

$$\begin{aligned} & 4f\left(\sum_{j=1}^{n-1} x_j + (m+1)x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - (m-1)x_n\right) \\ & \quad + m^2 f(2(x_1 + x_n)) + m^2 \sum_{j=2}^{n-1} f(2x_j) \\ &= 8f\left(\sum_{j=1}^{n-1} x_j + x_n\right) + 4m^2(f(x_1 + 2x_n) + f(x_1)) \end{aligned}$$

$$\begin{aligned}
& + 4m^2 \sum_{j=2}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n) \right), \\
& 4f\left(\sum_{j=1}^{n-1} x_j + (m-1)x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - (m+1)x_n\right) \\
& + m^2 f(2(x_1 - x_n)) + m^2 \sum_{j=2}^{n-1} f(2x_j) \\
& = 8f\left(\sum_{j=1}^{n-1} x_j - x_n\right) + 4m^2(f(x_1 - 2x_n) + f(x_1)) \\
& + 4m^2 \sum_{j=2}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n) \right)
\end{aligned}$$

for all $x_1, \dots, x_n \in X$, respectively. Adding two above equations and using cases where $m = 1, 2, m + 1$ we have

$$\begin{aligned}
& 4f\left(\sum_{j=1}^{n-1} x_j + (m-1)x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - (m-1)x_n\right) + (m-1)^2 \sum_{j=1}^{n-1} f(2x_j) \\
& = 8f\left(\sum_{j=1}^{n-1} x_j\right) + 4(m-1)^2 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n) \right)
\end{aligned}$$

for all $x_1, \dots, x_n \in X$. By using above method and induction, we infer the cases where $m = 1$ or $m = 2$, that is, it is cubic. Conversely, suppose that f satisfies the equation (1.2). Because of Lemma 2.2 of [2], we will show that the cases $m = 1$ and then $m \geq 3$ integers. Since f satisfies the equation (1.2), we may use the equation (1.3) with $m = 2$; see [2, Lemma 2.2]. By putting $x = \sum_{j=1}^{n-1} x_j$ and $y = x_n$, we have

$$\begin{aligned}
& f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + f\left(2 \sum_{j=1}^{n-1} x_j\right) \\
& = 2f\left(\sum_{j=1}^{n-1} x_j\right) + 4f\left(\sum_{j=1}^{n-1} x_j + x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - x_n\right).
\end{aligned}$$

By using case $m = 2$ and $f(2x) = 8f(x)$, we have the desired result when $m = 1$ in the equation (1.3). Next, similar to the previous step and induction with the cases $m = 1, 2$ and $m = t$, it is easy to show that

$$4f\left(\sum_{j=1}^{n-1} x_j + (t+1)x_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - (t+1)x_n\right) + (t+1)^2 \sum_{j=1}^{n-1} f(2x_j)$$

$$= 8f\left(\sum_{j=1}^{n-1} x_j\right) + 4(t+1)^2 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right)$$

for all $x_1, \dots, x_n \in X$. Thus the f satisfying the equation (1.2) implies the equation (1.3) for each integer $m \geq 1$. \square

3. Stability

Throughout in this section, let X be a normed vector space with norm $\|\cdot\|$ and Y a Banach space with norm $\|\cdot\|$. For the given mapping $f : X \rightarrow Y$, we define

$$(3.1) \quad \begin{aligned} Df(x_1, \dots, x_n) := & 4f\left(\sum_{j=1}^{n-1} x_j + mx_n\right) + 4f\left(\sum_{j=1}^{n-1} x_j - mx_n\right) + m^2 \sum_{j=1}^{n-1} f(2x_j) \\ & - 8f\left(\sum_{j=1}^{n-1} x_j\right) - 4m^2 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned}$$

for all $x_1, \dots, x_n \in X$ and each integer $m \geq 1$.

Theorem 3.1. *Let $f : X \rightarrow Y$ be a mapping for which there exists a function $\phi : X^n \rightarrow [0, \infty)$ such that*

$$(3.2) \quad \tilde{\phi}(x_1, \dots, x_n) := \sum_{j=0}^{\infty} \left(\frac{1}{8}\right)^j \phi(2^j x_1, \dots, 2^j x_n) < \infty,$$

$$(3.3) \quad \|Df(x_1, \dots, x_n)\| \leq \phi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$(3.4) \quad \|f(x) - C(x)\| \leq \frac{1}{8m^2 s} \underbrace{\tilde{\phi}(x, \dots, x, 0, \dots, 0)}_{s\text{-terms}}$$

for all $x \in X$, and each integer $m \geq 1$ and $s \geq 1$.

Proof. By letting $x_j = x$ ($j = 1, \dots, s$) and $x_k = 0$ ($k = s + 1, \dots, n$) in the equation (3.3), we have

$$(3.5) \quad \left\| f(x) - \frac{1}{8}f(2x) \right\| \leq \frac{1}{8m^2 s} \underbrace{\phi(x, \dots, x, 0, \dots, 0)}_{s\text{-terms}}$$

for all $x \in X$. Replacing x by $2x$ in the equation (3.5), we have

$$(3.6) \quad \left\| f(2x) - \frac{1}{8}f(2^2x) \right\| \leq \frac{1}{8m^2 s} \underbrace{\phi(2x, \dots, 2x, 0, \dots, 0)}_{s\text{-terms}}$$

for all $x \in X$. Now, combining equations (3.5) and (3.6), we get

$$\| f(x) - \left(\frac{1}{8}\right)^2 f(2^2x) \| \leq \frac{1}{8m^2s} \left(\underbrace{\phi(x, \dots, x, 0, \dots, 0)}_{s\text{-terms}} + \frac{1}{8} \underbrace{\phi(2x, \dots, 2x, 0, \dots, 0)}_{s\text{-terms}} \right)$$

for all $x \in X$.

Continue this way, we may have

$$(3.7) \quad \| f(x) - \left(\frac{1}{8}\right)^t f(2^t x) \| \leq \frac{1}{8m^2s} \sum_{j=0}^{t-1} \left(\frac{1}{8}\right)^j \underbrace{\phi(2^j x, \dots, 2^j x, 0, \dots, 0)}_{s\text{-terms}}$$

for all positive integer t and all $x \in X$. For any positive integer r , dividing the equation (3.7) by 8^r and then substituting x by $2^r x$, we have

$$\begin{aligned} & \left(\frac{1}{8}\right)^r \| f(2^r x) - \left(\frac{1}{8}\right)^t f(2^{r+t} x) \| \\ & \leq \left(\frac{1}{8}\right)^r \cdot \frac{1}{8m^2s} \sum_{j=0}^{t-1} \left(\frac{1}{8}\right)^j \underbrace{\phi(2^{r+j} x, \dots, 2^{r+j} x, 0, \dots, 0)}_{s\text{-terms}} \end{aligned}$$

for all $x \in X$.

By taking $r \rightarrow \infty$, we may conclude that $\{(\frac{1}{8})^t f(2^t x)\}$ is a Cauchy sequence in a Banach space Y . This implies that the sequence $\{(\frac{1}{8})^t f(2^t x)\}$ converges. Hence we can define a function $C : X \rightarrow Y$ by

$$C(x) = \lim_{t \rightarrow \infty} \left(\frac{1}{8}\right)^t f(2^t x)$$

for all $x \in X$. Then

$$\begin{aligned} \| DC(x_1, \dots, x_n) \| &= \lim_{t \rightarrow \infty} \left(\frac{1}{8}\right)^t \| Df(2^t x_1, \dots, 2^t x_n) \| \\ &\leq \lim_{t \rightarrow \infty} \left(\frac{1}{8}\right)^t \phi(2^t x_1, \dots, 2^t x_n) \\ &= 0 \end{aligned}$$

for all $x_1, \dots, x_n \in X$. That is, $DC(x_1, \dots, x_n) = 0$. By Lemma 2.1, the function $C : X \rightarrow Y$ is cubic. It only remains to show that the function C is unique. Let $C' : X \rightarrow Y$ be another cubic function satisfying the equation (3.4). Then

$$\begin{aligned} \| C(x) - C'(x) \| &= \left(\frac{1}{8}\right)^t \| C(2^t x) - C'(2^t x) \| \\ &\leq \left(\frac{1}{8}\right)^t (\| C(2^t x) - f(2^t x) \| + \| f(2^t x) - C'(2^t x) \|) \\ &\leq \left(\frac{1}{8}\right)^t \frac{1}{8} \underbrace{\tilde{\phi}(2^t x, \dots, 2^t x, 0, \dots, 0)}_{s\text{-terms}} \end{aligned}$$

for all $x \in X$. As $t \rightarrow \infty$, we can conclude that $C(x) = C'(x)$ for all $x \in X$; that is, C is unique. □

Theorem 3.2. *Let $f : X \rightarrow Y$ be a mapping for which there exists a function $\phi : X^n \rightarrow [0, \infty)$ such that*

$$(3.8) \quad \tilde{\phi}(x_1, \dots, x_n) := \sum_{j=1}^{\infty} 8^j \phi(2^{-j}x_1, \dots, 2^{-j}x_n) < \infty,$$

$$(3.9) \quad \|Df(x_1, \dots, x_n)\| \leq \phi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$(3.10) \quad \|f(x) - C(x)\| \leq \frac{1}{m^2 s} \underbrace{\tilde{\phi}(x, \dots, x, 0, \dots, 0)}_{s\text{-terms}}$$

for all $x \in X$, and each integer $m \geq 1$ and $s \geq 1$.

Proof. If x is replaced by $\frac{1}{2}x$ in the equation (3.5) in the proof of Theorem 3.1, we have

$$\left|f(x) - 8f\left(\frac{1}{2}x\right)\right| \leq \frac{1}{m^2} \phi\left(\underbrace{\frac{1}{2}x, \dots, \frac{1}{2}x}_{s\text{-terms}}, 0, \dots, 0\right)$$

for all $x \in X$. The remains of the proof are similar to the proof of Theorem 3.1. □

4. Stability using alternative fixed point

In this section, we will investigate the stability of the given cubic functional equation (3.1) using the alternative fixed point. Before proceeding the proof, we will state the theorem, the alternative of fixed point.

Theorem 4.1 (The alternative of fixed point [4], [17]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \rightarrow \Omega$ with Lipschitz constant L . Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

- (1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) The sequence $(T^n x)$ is convergent to a fixed point y^* of T ;
- (3) y^* is the unique fixed point of T in the set

$$\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\};$$

- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Now, let $\phi : X^n \rightarrow [0, \infty)$ be a function such that

$$\lim_{r \rightarrow \infty} \frac{\phi(\lambda_i^r x_1, \dots, \lambda_i^r x_n)}{\lambda_i^{3r}} = 0$$

for all $x_1, \dots, x_n \in X$, where $\lambda_i = 2$ if $i = 0$ and $\lambda_i = \frac{1}{2}$ if $i = 1$.

Theorem 4.2. *Suppose that a function $f : X \rightarrow Y$ satisfies the functional inequality*

$$(4.1) \quad \|Df(x_1, \dots, x_n)\| \leq \phi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$. If there exists $L = L(i) < 1$ such that the function

$$(4.2) \quad x \mapsto \psi(x) = \frac{1}{m^2 s} \phi(\underbrace{\frac{1}{2}x, \dots, \frac{1}{2}x}_{s\text{-terms}}, 0, \dots, 0)$$

has the property

$$(4.3) \quad \psi(x) \leq L \cdot \lambda_i^3 \cdot \psi\left(\frac{x}{\lambda_i}\right)$$

for all $x \in X$, then there exists a unique cubic function $C : X \rightarrow Y$ such that the inequality

$$(4.4) \quad \|f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L} \psi(x)$$

holds for all $x \in X$.

Proof. Consider the set

$$\Omega = \{g \mid g : X \rightarrow Y\}$$

and introduce the generalized metric on Ω ,

$$d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) \mid \|g(x) - h(x)\| \leq K\psi(x), x \in X\}.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \rightarrow \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^3} g(\lambda_i x)$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K &\Rightarrow \|g(x) - h(x)\| \leq K\psi(x) \text{ for all } x \in X, \\ &\Rightarrow \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leq \frac{1}{\lambda_i^3} K\psi(\lambda_i x) \text{ for all } x \in X, \\ &\Rightarrow \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leq LK\psi(x) \text{ for all } x \in X, \\ &\Rightarrow d(Tg, Th) \leq LK. \end{aligned}$$

Hence we have that

$$d(Tg, Th) \leq Ld(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L . By setting $y = 0$, we have the equation (3.5) as in the proof of Theorem 3.1 and we use the equation (4.3) with the case where $i = 0$, which is reduced to

$$\|f(x) - \frac{1}{8}f(2x)\| \leq \frac{1}{m^2 s} \frac{1}{2^3} \psi(2x) \leq L\psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \leq L = L^1 < \infty$. Now, replacing x by $\frac{1}{2}x$ in the equation (3.5), multiplying 8, and using the equation (4.3) with the case where $i = 1$, we have that

$$\|f(x) - 2^3 f(\frac{x}{2})\| \leq \psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \leq 1 = L^0 < \infty$. In both cases we can apply the fixed point alternative and since $\lim_{r \rightarrow \infty} d(T^r f, C) = 0$, there exists a fixed point C of T in Ω such that

$$(4.5) \quad C(x) = \lim_{r \rightarrow \infty} \frac{f(\lambda_i^r x)}{\lambda_i^{3r}}$$

for all $x \in X$. Letting $x_j = \lambda_i^r x_j$ ($j = 1, \dots, n$) in the equation (4.1) and dividing by λ_i^{3r} ,

$$\begin{aligned} \|DC(x_1, \dots, x_n)\| &= \lim_{r \rightarrow \infty} \frac{\|Df(\lambda_i^r x_1, \dots, \lambda_i^r x_n)\|}{\lambda_i^{3r}} \\ &\leq \lim_{r \rightarrow \infty} \frac{\|\phi(\lambda_i^r x_1, \dots, \lambda_i^r x_n)\|}{\lambda_i^{3r}} = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in X$; that is, it satisfies the equation (1.3). By Lemma 2.1, the C is cubic. Also, the fixed point alternative guarantees that such a C is the unique function such that

$$\|f(x) - C(x)\| \leq K \psi(x)$$

for all $x \in X$ and some $K > 0$. Again using the fixed point alternative, we have

$$d(f, C) \leq \frac{1}{1-L} d(f, Tf).$$

Hence we may conclude that

$$d(f, C) \leq \frac{L^{1-i}}{1-L},$$

which implies the equation (4.4). \square

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