Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 709–715

# ON SOME RING CLASS FIELDS BY SHIMURA'S CANONICAL MODELS

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Abstract. We construct certain ring class fields over an imaginary quadratic field by making use of Shimura's canonical models and extend the result of Chen-Yui ([1] Theorem 3.7.5(2)) to the case where $(a, b, N) \neq N$ or $(a/N, N) \neq 1$ for a positive integer $N > 1$.

## 1. Introduction

When $\Gamma_0(N)$ is the Hecke subgroup of $SL_2(\mathbb{Z})$ for a positive integer $N$, Helling showed in [3] that the group $\Gamma_0(N)^*$ generated by $\{\Gamma_0(N), \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)\}$ has the genus zero exactly for $N = 1 \sim 21$, $23 \sim 27$, 29, 31, 32, 35, 36, 39, 41, 47, 49, 50, 59, 71. Moreover, for all such $N$ but 49 and 50, $\Gamma_0(N)^*$ has a fundamental Thompson series $T_N^*$ corresponding to itself ([2] Table 2). Throughout this paper, we denote $\alpha$ to be a Heegner point, that is, $\alpha$ is a root in $\mathfrak{H}$ of an integral equation $az^2 + bz + c = 0$ with $b^2 - 4ac < 0, (a, b, c) = 1$ and $a > 0$, and $K$ to be an imaginary quadratic field $\mathbb{Q}(\alpha)$. Chen and Yui showed in [1] Theorem 3.7.5(2) by using the Shimura reciprocity law that when $(a, N) = 1$ for a prime number $N$, $T_N^*(\alpha)$ generates a ring class field over $K$ of an imaginary quadratic order $\mathcal{O}'$ of discriminant $f^2 d_K$ where $f = mN$ and $b^2 - 4ac = m^2 d_K < 0$.

In this paper, over an imaginary quadratic field $K$ we study the class fields generated by the singular values of automorphic functions which give rise to Shimura's canonical models of $\Gamma_0(N)\backslash\mathfrak{H}^*$ or $\Gamma_0(N)^*\backslash\mathfrak{H}^*$ at imaginary quadratic arguments in the complex upper half plane $\mathfrak{H}$ (Theorem 2.4). Here, $N$ is any positive integer and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$, and our result is independent of the genus of curve under consideration. As a corollary (Corollary 2.5), we can extend Chen-Yui's result on a ring class field $K(T_N^*(\alpha))$ to the case where $(a, b, N) \neq N$ or $(\frac{a}{N}, N) \neq 1$ for a positive integer $N > 1$ by using the theory of complex multiplication, whose proof is different from their argument.

Throughout the article we adopt the following notations:

Received February 11, 2008.
2000 *Mathematics Subject Classification.* 11F06, 20H10, 22E40, 30F35.
*Key words and phrases.* class fields, Shimura's canonical models, Thompson series.
This work was supported by Korea Research Foundation Grant(KRF-2002-070-C00003).

- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) |\ c \equiv 0 \mod N\}$
- $\Gamma_0(N)^* = \left\langle \{\Gamma_0(N), \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)\}\right\rangle$
- $\mathbb{Z}_p$ the ring of $p$-adic integers
- $\mathbb{Q}_p$ the field of $p$-adic numbers
- $\mathfrak{H}$ the complex upper half plane
- $\zeta_N = e^{2\pi i/N}$
- $i = \sqrt{-1}$
- $T_N^*$ a fundamental Thompson series (that is, a normalized Hauptmodule) for a genus zero group $\Gamma_0(N)^*$
- $R^\times$ the group of units of a ring $R$

## 2. Class fields by Shimura's canonical models

Let $\Gamma$ be a Fuchsian group of the first kind. Then $X(\Gamma) = \Gamma\backslash\mathfrak{H}^*$ is a compact Riemann surface. Hence there exists a projective nonsingular algebraic curve $V_\Gamma$, defined over $\mathbb{C}$, biregularly isomorphic to $\Gamma\backslash\mathfrak{H}^*$. We specify a $\Gamma$-invariant holomorphic map $\varphi_\Gamma$ of $\mathfrak{H}^*$ to $V_\Gamma$ which gives a biregular isomorphism of $\Gamma\backslash\mathfrak{H}^*$ to $V_\Gamma$. In that situation, we call $(V_\Gamma, \varphi_\Gamma)$ a *model* of $\Gamma\backslash\mathfrak{H}^*$. For instance, if the genus of $\Gamma\backslash\mathfrak{H}^*$ is zero, then its function field $K(X(\Gamma))$ is equal to $\mathbb{C}(J')$ for some $J' \in K(X(\Gamma))$ and hence the pair $(\mathbb{P}^1(\mathbb{C}), J')$ gives a model of $\Gamma\backslash\mathfrak{H}^*$ ([4] Lemma 14).

Let $G_\mathbb{A}$ be the adelization of an algebraic group $G = GL_2$ defined over $\mathbb{Q}$. Put

$$\begin{aligned} G_p &= GL_2(\mathbb{Q}_p)\ (p: \text{ a rational prime}), \\ G_\infty &= GL_2(\mathbb{R}), \\ G_{\infty+} &= \{x \in G_\infty |\det(x) > 0\}, \\ G_{\mathbb{Q}_+} &= \{x \in GL_2(\mathbb{Q}) |\det(x) > 0\}, \\ U &= \prod_p GL_2(\mathbb{Z}_p) \times G_{\infty+}, \\ G_{\mathbb{A}_+} &= UG_{\mathbb{Q}_+}. \end{aligned}$$

We define the topology of $G_\mathbb{A}$ by taking $U$ to be an open subgroup of $G_\mathbb{A}$. Let $K$ be an imaginary quadratic field as described in the introduction and $\xi_\alpha$ be an embedding of $K$ into $M_2(\mathbb{Q})$. We call $\xi_\alpha$ *normalized* if it is defined by $a\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) = \xi_\alpha(a)\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right)$ for $a \in K$ where $\alpha$ is the fixed point of $\xi_\alpha(K^\times)\ (\subset G_{\mathbb{Q}_+})$ in $\mathfrak{H}$. Observe that the embedding $\xi_\alpha$ defines a continuous homomorphism of $K_\mathbb{A}^\times$ into $G_{\mathbb{A}+}$, which we denote again by $\xi_\alpha$. Indeed, $\xi_\alpha = (\xi_{\alpha,p})$ is defined by $x_p\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) = \xi_{\alpha,p}(x_p)\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right)$ for $x_p \in K \otimes_\mathbb{Q} \mathbb{Q}_p$ for all prime $p$. Here $G_{\mathbb{A}+}$ is the group $G_0 G_{\infty+}$ with $G_0$ the non-archimedean part of $G_\mathbb{A}$ and $K_\mathbb{A}^\times$ is the idele group of $K$. Let $\mathcal{Z}$ be the set of open subgroups $S$ of $G_{\mathbb{A}+}$ containing $\mathbb{Q}^\times G_{\infty+}$ such that $S/\mathbb{Q}^\times G_{\infty+}$ is compact. For $S \in \mathcal{Z}$, we see that $\det(S)$ is open in $\mathbb{Q}_\mathbb{A}^\times$. Therefore the subgroup $\mathbb{Q}^\times \det(S)$ of $\mathbb{Q}_\mathbb{A}^\times$ corresponds to a finite abelian extension of $\mathbb{Q}$, which we write $k_S$. Put $\Gamma_S = S \cap G_{\mathbb{Q}_+}$ for $S \in \mathcal{Z}$. As is well known ([6] Proposition 6.27), $\Gamma_S/\mathbb{Q}^\times$ is a Fuchsian group of the first kind commensurable with $PSL_2(\mathbb{Z})$. Let

$$\begin{aligned} U_p^0 &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) |\ c \equiv 0 \mod N\mathbb{Z}_p\}, \\ U^0 &= \{x = (x_p) \in U |\ x_p \in U_p^0 \text{ for all finite } p\}, \\ U_*^0 &= U^0 \cup U^0\Phi(N), \\ \Phi(N) &= (x_p) \in G_{\mathbb{A}+} \text{ with } x_p = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right). \end{aligned}$$

Then we have

**Lemma 2.1.** (i) *$\mathbb{Q}^\times U_*^0,\ \mathbb{Q}^\times U^0 \in \mathcal{Z}$,*

(ii) *$k_S = \mathbb{Q}$ for $S \in \{\mathbb{Q}^\times U_*^0,\ \mathbb{Q}^\times U^0\}$,*

(iii) *$\Gamma_S = \mathbb{Q}^\times \Gamma_0(N)^*$ (respectively, $\mathbb{Q}^\times\Gamma_0(N)$) if $S = \mathbb{Q}^\times U_*^0$ (respectively, $\mathbb{Q}^\times U^0$).*

*Proof.* It is well known for $S = \mathbb{Q}^\times U^0$. Since $\mathbb{Q}^\times U_*^0 = \mathbb{Q}^\times U^0 \cup \mathbb{Q}^\times U^0\Phi(N)$ and $\mathbb{Q}^\times U^0$ is an open subgroup in $G_{A+}$, $\mathbb{Q}^\times U_*^0$ is also an open subgroup in $G_{A+}$. Observing the fact that $\mathbb{Q}^\times U^0/\mathbb{Q}^\times G_{\infty+}$ is compact, we obtain $\mathbb{Q}^\times U_*^0 \in \mathcal{Z}$. As for (ii), we know that $\mathbb{Q}$ corresponds to the norm group $\mathbb{Q}^\times \mathbb{Q}_{\mathbb{A}}^{\times\infty}$ with $\mathbb{Q}_{\mathbb{A}}^{\times\infty} = \mathbb{R}^\times \prod_p \mathbb{Z}_p^\times$ and $\det(U_*^0) = N\det(U^0)$. But $\det(U^0) = \mathbb{Q}_{\mathbb{A}}^{\times\infty}$, and hence by the class field theory $k_S = \mathbb{Q}$. Indeed, $\det(U^0)$ is contained in $\mathbb{Q}_{\mathbb{A}}^{\times\infty}$. Conversely, for any element $(\alpha_p) \in \mathbb{Q}_{\mathbb{A}}^{\times\infty}$, take $y_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_p \end{smallmatrix}\right)$; then $(y_p) \in U_0$ and $\det(y_p) = (\det y_p) = (\alpha_p)$. Lastly, we readily get that $\Gamma_S = \mathbb{Q}^\times U_*^0 \cap G_{\mathbb{Q}+} = \mathbb{Q}^\times\left(U_*^0 \cap G_{\mathbb{Q}+}\right) = \mathbb{Q}^\times\Gamma_0(N)^*$. □

For two complex numbers $\omega_1$ and $\omega_2$ such that $\omega_1/\omega_2 \in \mathfrak{H}$, we have a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$. We then define a Fricke function

$$f_a(z) = \frac{g_2(\omega_1,\omega_2)g_3(\omega_1,\omega_2)}{\Delta(\omega_1,\omega_2)}\mathfrak{p}(a\left(\begin{smallmatrix} \omega_1 \\ \omega_2 \end{smallmatrix}\right); \omega_1,\omega_2) \qquad (a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2),$$

where

$$\begin{aligned} \mathfrak{p}(u;\omega_1,\omega_2) &= u^{-2} + \sum_{w \in L\setminus 0} [(u-w)^{-2} - w^{-2}], \\ g_2(\omega_1,\omega_2) &= 60 \sum_{w \in L\setminus 0} w^{-4}, \\ g_3(\omega_1,\omega_2) &= 140 \sum_{w \in L\setminus 0} w^{-6} \quad \text{and} \\ \Delta(\omega_1,\omega_2) &= g_2(\omega_1,\omega_2)^3 - 27g_3(\omega_1,\omega_2)^2. \end{aligned}$$

Now let us put, for a positive integer $N$,

$$\mathfrak{F}_N = \mathbb{Q}(j, f_a |\ a \in N^{-1}\mathbb{Z}^2, \notin \mathbb{Z}^2) \qquad \text{and} \qquad \mathfrak{F} = \bigcup_{N=1}^{\infty} \mathfrak{F}_N,$$

where $j$ is the elliptic modular function. Then $\mathfrak{F}$ is a Galois extension of $\mathfrak{F}_1$ and $\mathbb{C}\mathfrak{F}$ is the field of modular functions of all levels.

**Proposition 2.2.** *For any $u \in U$, one can define an element $\tau(u)$ of* $\mathrm{Gal}(\mathfrak{F}/\mathfrak{F}_1)$ *by $f_a^{\tau(u)} = f_{au}$ for all $a \in (\mathbb{Q}\backslash\mathbb{Z})^2$. Moreover, $\tau(u)$ has the following properties*:

(1) *The sequence $1 \to \{\pm 1\} \cdot G_{\infty+} \to U \to \mathrm{Gal}(\mathfrak{F}/\mathfrak{F}_1) \to 1$ is exact,*
(2) $\tau(u) = [\det(u)^{-1}, \mathbb{Q}]$ *on $\mathbb{Q}_{ab}$,*
(3) $h^{\tau(\gamma)} = h \circ \gamma$ *for all $h \in \mathfrak{F}$ and $\gamma \in SL_2(\mathbb{Z})$.*

*Proof.* See [6, Proposition 6.21]. □

Here, $[\ ,\mathbb{Q}]$ is the reciprocity map and $\mathbb{Q}_{\mathrm{ab}}$ is the maximal abelian extension of $\mathbb{Q}$.

We shall now define a homomorphism $\tau :\ G_{\mathbb{A}_+} \longrightarrow \mathrm{Aut}(\mathfrak{F})$ as follows. Since $G_{\mathbb{A}_+} = UG_{\mathbb{Q}_+}$ for $\beta \in G_{\mathbb{Q}_+}$, we define $\tau(\beta)$ by

$$h^{\tau(\beta)} = h \circ \beta \qquad \text{for all } h \in \mathfrak{F}.$$

And, for $x = u\beta \in G_{\mathbb{A}_+}$ with $u \in U$ and $\beta \in G_{\mathbb{Q}_+}$ we put

$$\tau(x) = \tau(u)\tau(\beta)$$

so that $j^{\tau(x)} = j \circ \beta$ and $f_a^{\tau(x)} = f_{au} \circ \beta$. Indeed, the map $\tau$ is defined independently of the choice of $u$ and $\beta$ by virtue of Proposition 2.2.

Next, for $S = \mathbb{Q}^\times U_*^0$ or $\mathbb{Q}^\times U^0$ we can find a model $(V_{\Gamma_S}, \varphi_{\Gamma_S})$ of the curve $\Gamma_S\backslash\mathfrak{H}^*$, which is characterized by the following properties:

(i) $V_S$ is defined over $k_S = \mathbb{Q}$ (Lemma 2.1),
(ii) $\mathfrak{F}_S = \{f \circ \varphi_{\Gamma_S} \mid f \in \mathbb{Q}(V_{\Gamma_S})\}$,

where $\mathfrak{F}_S = \{h \in \mathfrak{F} \mid h^{\tau(x)} = h \text{ for all } x \in S\}$ and $\mathbb{Q}(V_{\Gamma_S})$ denotes the field of functions on $V_{\Gamma_S}$ rational over $\mathbb{Q}$.

**Example 2.3.** Let $T_N^*$ be a fundamental Thompson series for a genus zero group $\Gamma_0(N)^*$ and $S = \mathbb{Q}^\times U_*^0$. Then $\mathfrak{F}_S = \mathbb{Q}(T_N^*)$ because $\mathbb{C}$ is linearly disjoint with $\mathfrak{F}_S$ over $k_S(=\mathbb{Q})$.

**Theorem 2.4.** *Let $\alpha$ be a root in $\mathfrak{H}$ of a primitive integral equation $az^2 + bz + c = 0$ with $a > 0$ such that $b^2 - 4ac = m^2 d_K (< 0)$. Put $K = \mathbb{Q}(\alpha)$ and $\mathcal{O}(= \mathbb{Z}[a\alpha])$ be an order in $K$ of discriminant $m^2 d_K$, where $d_K$ is the discriminant of $K$.*

(1) *If $N \geq 1$, $S = \mathbb{Q}^\times U^0$ and $(V_{\Gamma_S}, \varphi_{\Gamma_S})$ is Shimura's canonical model as in the above, then $\varphi_{\Gamma_S}(\alpha)$ generates the ring class field of an order $\mathcal{O}'$ of discriminant $f^2 d_K$ where the conductor $f$ of $\mathcal{O}'$ is $mN/(a, N)$.*
(2) *Assume that $(a, b, N) \neq N$ or $(\frac{a}{N}, N) \neq 1$ for $N \geq 2$. If $S = \mathbb{Q}^\times U_*^0$ and $(V_{\Gamma_S}, \varphi_{\Gamma_S})$ is a model, then $\varphi_{\Gamma_S}(\alpha)$ generates the ring class field of an order $\mathcal{O}'$ of discriminant $f^2 d_K$ where its conductor $f$ equals $mN/(a, N)$.*

*Proof.* Let $L$ be a lattice $= \mathbb{Z}N\alpha + \mathbb{Z}$. Then $\mathcal{O}' = \mathbb{Z} + \frac{mN}{(a,N)}\mathcal{O}_K$ is an order of $L$ in $K$ ([5] Ch.8). Let us consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Q}^2 & \xrightarrow{\iota_\alpha} & K \\ {\scriptstyle \xi_\alpha(\mu)}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbb{Q}^2 & \xrightarrow{\iota_\alpha} & K \end{array}$$

where $\iota_\alpha(x,y) = x\alpha + y$ and $\iota_\alpha((x,y)\xi_\alpha(\mu)) = \mu\iota_\alpha(x,y)$.

We may consider the mapping $\iota_\alpha$ (respectively, $\xi_\alpha : K^\times \to GL_2(\mathbb{Q})$) as an isomorphism of affine varieties (respectively, a homomorphism of algebraic groups) over $\mathbb{Q}$. Thus, taking the $\mathbb{Q}_\mathbb{A}$-valued points, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_\mathbb{A}^2 & \xrightarrow{\iota_\alpha} & K_\mathbb{A} \\ {\scriptstyle \xi_\alpha(s)}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{Q}_\mathbb{A}^2 & \xrightarrow{\iota_\alpha} & K_\mathbb{A} \end{array}$$

for any idele $s \in K_\mathbb{A}^\times$.

Let $s = s_\infty s_0 \in K_\mathbb{A}^\times$ with infinite part $s_\infty$ and finite part $s_0 = (s_p)_p$ of $s$. We then have the following statements:

$\xi_\alpha(s) \in U$ if and only if $\xi_{\alpha,p}(s_p) \in GL_2(\mathbb{Z}_p)$ for all finite $p$.

if and only if $\xi_\alpha(s_0) \in \prod_p GL_2(\mathbb{Z}_p)$.

if and only if $\xi_\alpha(s_0)$ induces an automorphism of $(\prod_p \mathbb{Z}_p)^2$.

if and only if the multiplication by $s_0$ induces an automorphism of $(\mathbb{Z}\alpha + \mathbb{Z}) \otimes_\mathbb{Z} \hat{\mathbb{Z}}$

(because $\iota_\alpha[(\prod_p \mathbb{Z}_p)^2] = (\mathbb{Z}\alpha + \mathbb{Z}) \otimes_\mathbb{Z} \hat{\mathbb{Z}}$).

if and only if $s_0 \in (\mathcal{O}_1 \otimes_\mathbb{Z} \hat{\mathbb{Z}})^\times$

where $\mathcal{O}_1$ is the order of $\mathbb{Z}\alpha + \mathbb{Z}$ in $K$.

Similarly, when $\xi_\alpha(s) \in U$, we have that

$\xi_\alpha(s) \in U^0$ if and only if $\xi_\alpha(s_0)$ induces an automorphism of $(N\prod_p \mathbb{Z}_p) \oplus \prod_p \mathbb{Z}_p$.

if and only if the multiplication by $s_0$ induces an automorphism of $L \otimes_\mathbb{Z} \hat{\mathbb{Z}}$.

if and only if $s_0 \in (\mathcal{O}' \otimes_\mathbb{Z} \hat{\mathbb{Z}})^\times$.

Then the assertion (1) follows from this, Lemma 2.1 and [6, Proposition 6.33] that $K \cdot k_S((\varphi_{\Gamma_S}(\alpha)) = K(\varphi_{\Gamma_S}(\alpha))$ is the ring class field of $\mathcal{O}'$.

In order to verify (2), we have only to show that $\xi_\alpha(K_{\mathbb{A}}^\times) \cap U^0\Phi(N)$ is an empty set under our assumptions. For $x_p \in K_p^\times$, let

$$\xi_{\alpha,p}(x_p) = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$$

with $a_p,\ b_p,\ c_p,\ d_p \in \mathbb{Q}_p$.

Then

$$\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = x_p \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$$

implies

$$\xi_{\alpha,p}(x_p) = \begin{pmatrix} d_p - \frac{b}{a}c_p & \frac{-c}{a}c_p \\ c_p & d_p \end{pmatrix}.$$

Now suppose that $\xi_\alpha((x_p)) = (\xi_{\alpha,p}(x_p))_p \in U^0\Phi(N)$ for some $(x_p)_p \in K_{\mathbb{A}}^\times$. For each finite prime $p$, we can write

$$\begin{pmatrix} d_p - \frac{b}{a}c_p & \frac{-c}{a}c_p \\ c_p & d_p \end{pmatrix} = \begin{pmatrix} N\beta_p & -\alpha_p \\ Nw_p & -\gamma_p \end{pmatrix}$$

with some $\begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & w_p \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ satisfying $\gamma_p \equiv 0 \bmod N\mathbb{Z}_p$.

Then $\alpha_p w_p - \gamma_p \beta_p \in \mathbb{Z}_p^\times$ says that $\alpha_p,\ w_p \in \mathbb{Z}_p^\times$ for all prime $p|N$. Notice that $c_p = Nw_p$ and $d_p = -\gamma_p \in \mathbb{Z}_p$ for all prime $p|N$. Since $\frac{c}{a}c_p = \frac{c}{a}Nw_p = \alpha_p \in \mathbb{Z}_p^\times$ for all prime $p|N$, $(a,N) = N$. Moreover, $\frac{b}{a}w_p \in \mathbb{Z}_p$ and hence $(b,N) = N$ because $d_p - \frac{b}{a}c_p = -\gamma_p - \frac{b}{a}Nw_p = N\beta_p$ and $N|\gamma_P$ for all prime $p|N$. If $(\frac{a}{N}, N) \neq 1$, then we can take a prime factor $p$ of $(\frac{a}{N}, N)$. Our factor $p$ divides $c$ because $\frac{c}{a}Nw_p = \alpha_p \in \mathbb{Z}_p^\times$ and $\frac{c}{a}N \in \mathbb{Z}_p^\times$. Therefore, $p$ divides $(a,b,c)$, which is a contradiction. □

**Corollary 2.5.** *Notations and assumptions being the same as in Theorem 2.4, we further suppose that* $(a,b,N) \neq N$ *or* $(\frac{a}{N}, N) \neq 1$ *for* $N \geq 2$*. Then* $T_N^*(\alpha)$ *generates the ring class field of an order* $\mathcal{O}'$ *of discriminant* $f^2 d_K$ *whose conductor* $f$ *is* $mN/(a,N)$.

*Proof.* It is immediate from Lemma 2.1 and Theorem 2.4(2). □

Note that this corollary can also be proved by Chen-Yui's method.

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