Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 701–707

ON f-DERIVATIONS OF LATTICES

Yilmaz Çeven and Mehmet Ali Öztürk

Abstract. In this paper, as a generalization of derivation on a lattice, the notion of f-derivation for a lattice is introduced and some related properties are investigated.

1. Introduction and preliminaries

Lattices play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis [2, 6, 13, 8]. Recently the properties of lattices were widely researched [1, 2, 5, 7, 8, 9, 13, 14].

In the theory of rings and near rings, the properties of derivations are an important topic to study [3, 4, 11, 12]. Y. B. Jun and X. L. Xin [10] applied the notion of derivation in ring and near ring theory to BCI-algebras. In [15], J. Zhan and Y. L. Liu introduced the notion of left-right (or right-left) f-derivation of a BCI algebra and investigated some properties. In [14], X. L. Xin, T. Y. Li, and J. H. Lu introduced the notion of derivation on a lattice and discussed some related properties.

In this paper, as a generalization of derivation on a lattice, the notion of f-derivation of a lattice is introduced and some related properties, which are discussed in [14] for a derivation on a lattice, are investigated for the f-derivation on a lattice. As important results of f-derivations on lattices, distributive and modular lattices are characterized by f-derivations under some conditions.

Definition 1 ([5]). Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a lattice if it satisfies the following conditions for all $x, y, z \in L$:

(1) $x \wedge x = x$, $x \vee x = x$,

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$.

Definition 2 ([5]). A lattice L is called a distributive lattice if it satisfies the identity (5) or (6) for all $x, y, z \in L$:

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

Received January 30, 2008.
2000 *Mathematics Subject Classification.* 06B35, 06B99.
Key words and phrases. lattice, derivation, f-derivation.

(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 3 ([1])**.** A lattice L is called a modular lattice if it satisfies the following condition for all $x, y, z \in L$:
(7) If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 4 ([5])**.** Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 5 ([5])**.** Let L and M be two lattices. The function $g : L \longrightarrow M$ is called a lattice homomorphism if it satisfies the following conditions for all $x, y \in L$:
(8) $g(x \wedge y) = g(x) \wedge g(y)$,
(9) $g(x \vee y) = g(x) \vee g(y)$.

It is known that a homomorphism is called an epimorphism if it is onto.

Lemma 1 ([14])**.** *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as the Definition 4. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the g.l.b of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

Definition 6 ([14])**.** A function $d : L \longrightarrow L$ on a lattice L is called a derivation on L if it satisfies the following condition

$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy).$$

The abbreviation dx is used for $d(x)$ in the above definition.

Definition 7 ([14])**.** Let L be a lattice and d be a derivation on L.
(1) If $x \leq y$ implies $dx \leq dy, d$ is called an isotone derivation,
(2) If d is one-to-one, d is called a monomorphic derivation,
(3) If d is onto, d is called an epimorphic derivation.

2. The f-derivations in lattices

The following definition introduces the notion of f-derivations on lattices.

Definition 8. Let L be a lattice. A function $d : L \longrightarrow L$ is called an f-derivation on L if there exists a function $f : L \longrightarrow L$ such that

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \tag{2.1}$$

for all $x, y \in L$.

It is obvious in the Definition 8 that if f is an identity function then d is a derivation on L. Furthermore, according to Definition 8, a function d on L can be an f-derivation only when a function f satisfying equation (2.1) exists. But, to obtain some results, d or f must satisfy some additional conditions as in the following propositions and theorems.

In this paper, we'll abbreviate $d(x)$ as dx and $f(x)$ as fx.

Example 1. Let L be the lattice of Figure 1 and define a function d by $d0 = 0$, $da = a$, $db = a$, $dc = c$, $d1 = c$.

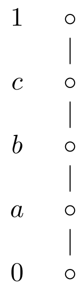

Figure 1

Then d is not a derivation on L since $a = d\,(b \wedge 1) \neq (db \wedge 1) \vee (b \wedge d1) = (a \wedge 1) \vee (b \wedge c) = a \vee b = b$. If we define a function f by $f0 = 0, fa = a, fb = a, fc = 1, f1 = 1$, then d satisfies the equation (2.1) for all $x, y \in L$ and so d is an f-derivation on L.

Example 2. Let L be a lattice and $a \in L$. Define a function $d : L \longrightarrow L$ by $dx = fx \wedge a$ for all $x \in L$ where $f : L \longrightarrow L$ satisfies $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$. Then d is an f-derivation. In addition, if f is an increasing function then d is an isotone derivation.

Proposition 1. *Let L be a lattice and d be an f-derivation on L. Then the following identities hold for all $x, y \in L$.*

a) $dx \leq fx$,

b) $dx \wedge dy \leq d\,(x \wedge y) \leq dx \vee dy$,

c) $d\,(x \wedge y) \leq fx \vee fy$,

d) *If L has a least element* 0*, then* $f0 = 0$ *implies* $d0 = 0$.

Proof. a) Since $dx = d(x \wedge x) = dx \wedge fx$, we have $dx \leq fx$.

b) We have $dx \wedge fy \leq d(x \wedge y)$ and $fx \wedge dy \leq d\,(x \wedge y)$ from the equation (2.1). Since $dx \leq fx$, we obtain $dx \wedge dy \leq fx \wedge dy$ and hence we have $dx \wedge dy \leq d\,(x \wedge y)$. We know that $dx \wedge fy \leq dx$ and $fx \wedge dy \leq dy$. Then we obtain $d\,(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) \leq dx \vee dy$.

c) Since $dx \wedge fy \leq fy$ and $fx \wedge dy \leq fx$, we obtain $d(x \wedge y) \leq fx \vee fy$.

d) Since $dx \leq fx$ for all $x \in L$, $f0 = 0$ and 0 is the least element of L, we have $0 \leq d0 \leq f0 = 0$. □

Proposition 2. *Let L be a lattice and d be an f-derivation and* 1 *be the greatest element of L and $f1 = 1$. Then the following identities hold;*

a) *If $fx \leq d1$, then $dx = fx$,*

b) *If $fx \geq d1$, then $dx \geq d1$.*

Proof. a) Since $dx = d\,(x \wedge 1) = (dx \wedge f1) \vee (fx \wedge d1) = dx \vee fx$, we have $fx \leq dx$. From Proposition 1 a), we obtain $dx = fx$.

b) Since $dx = (dx \wedge f1) \vee (fx \wedge d1) = dx \vee d1$, we have $dx \geq d1$. □

Remark 1. Note that if $d1 = 1$, since $d1 \leq f1$, we have $f1 = 1$ where 1 the greatest element of L. In this case, from Proposition 2 a), we get $d = f$.

Let L be a lattice and d be an f-derivation of L. Define a set $F = \{x \in L : dx = fx\}$.

Proposition 3. *Let L be a lattice and d be an f-derivation. If f is an increasing function, then $y \leq x$ and $x \in F$ implies $y \in F$.*

Proof. Note that, since $x \in F$ and $dy \leq fy \leq fx = dx$, we get $dy = d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) = (fx \wedge fy) \vee dy = fy \vee dy = fy$. □

Proposition 4. *Let L be a lattice and d be an isotone f-derivation on L. Then for any $x, y \in L$, $dx = dx \vee (fx \wedge d(x \vee y))$.*

Proof. Since d is an isotone f-derivation, we know that for all $x, y \in L$, $dx \leq d(x \vee y) \leq f(x \vee y)$. Hence we have $dx = d((x \vee y) \wedge x) = (d(x \vee y) \wedge fx) \vee (f(x \vee y) \wedge dx) = dx \vee (fx \wedge d(x \vee y))$. □

Proposition 5. *Let L be a lattice and d be an isotone f-derivation. If $x, y \in F$ and f is a decreasing function, then $x \vee y \in F$.*

Proof. Since $x \leq x \vee y$ and $y \leq x \vee y$, we have $f(x \vee y) \leq fx$ and $f(x \vee y) \leq fy$ respectively. Then we obtain $f(x \vee y) \leq fx \vee fy = dx \vee dy \leq d(x \vee y)$ since d is an isotone f-derivation. It is known that $d(x \vee y) \leq f(x \vee y)$, hence we get $x \vee y \in F$. □

Theorem 1. *Let L be a lattice with greatest element 1 and d be an f-derivation on L. Let $f1 = 1$ and $f(x \wedge y) = fx \wedge fy$ for all $x, y \in L$. Then the following conditions are equivalent*:

(1) *d is an isotone f-derivation,*
(2) *$dx \vee dy \leq d(x \vee y)$,*
(3) *$dx = fx \wedge d1$,*
(4) *$d(x \wedge y) = dx \wedge dy$.*

Proof. (1) $\Longrightarrow$ (2) : Suppose that d is an isotone f-derivation. We know that $x \leq x \vee y$ and $y \leq x \vee y$. Since d is isotone, $dx \leq d(x \vee y)$ and $dy \leq d(x \vee y)$. Hence we obtain $dx \vee dy \leq d(x \vee y)$.

(2) $\Longrightarrow$ (1) : Suppose that $dx \vee dy \leq d(x \vee y)$ and $x \leq y$. Then we have $dx \leq dx \vee dy \leq d(x \vee y) = dy$.

(1) $\Longrightarrow$ (3) : Suppose that d is an isotone f-derivation. We have $dx \leq d1$. It is known that $dx \leq fx$ from Proposition 1 a). Then we get $dx \leq fx \wedge d1$. From Proposition 4, for $y = 1$, we have $dx = dx \vee (fx \wedge d1) = fx \wedge d1$.

(3) $\Longrightarrow$ (4) : Assume that (3) holds. Then $d(x \wedge y) = f(x \wedge y) \wedge d1 = fx \wedge fy \wedge d1 = (fx \wedge d1) \wedge (fy \wedge d1) = dx \wedge dy$.

(4) $\Longrightarrow$ (1) : Let $d(x \wedge y) = dx \wedge dy$ and $x \leq y$. Since $dx = d(x \wedge y) = dx \wedge dy$, we get $dx \leq dy$. □

Theorem 2. *Let L be a modular lattice and d be an f-derivation on L.*

(a) *d is an isotone f-derivation if and only if $d(x \wedge y) = dx \wedge dy$,*

(b) *If d is an isotone f-derivation where $f(x \vee y) = fx \vee fy$, then $dx = fx$ implies $d(x \vee y) = dx \vee dy$.*

Proof. (a) Suppose that d is an isotone f-derivation. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, we get $d(x \wedge y) \leq dx$ and $d(x \wedge y) \leq dy$. Hence $d(x \wedge y) \leq dx \wedge dy$. Also using Proposition 1 a) and the fact $dx \wedge fy \leq dx \leq fx$, we get

$$\begin{aligned} dx \wedge dy &= (dx \wedge dy) \wedge (fx \wedge fy) \\ &\leq (dx \vee dy) \wedge (fx \wedge fy) \\ &= ((dy \vee dx) \wedge fy) \wedge fx \\ &= (dy \vee (dx \wedge fy)) \wedge fx \\ &= ((dx \wedge fy) \vee dy) \wedge fx \\ &= (dx \wedge fy) \vee (fx \wedge dy) \\ &= d(x \wedge y). \end{aligned}$$

Conversely, let $d(x \wedge y) = dx \wedge dy$ and $x \leq y$. Since $dx = d(x \wedge y) = dx \wedge dy$, we have $dx \leq dy$.

(b) Suppose that d is an isotone f-derivation and $dx = fx$. Using Proposition 4 and since L is modular lattice, we have

$$\begin{aligned} dy &= dy \vee (fy \wedge d(x \vee y)) \\ &= (dy \vee fy) \wedge d(x \vee y) \\ &= fy \wedge d(x \vee y). \end{aligned}$$

Hence using the hypothesis, we obtain

$$\begin{aligned} dx \vee dy &= dx \vee (fy \wedge d(x \vee y)) \\ &= (dx \vee fy) \wedge d(x \vee y) \\ &= (fx \vee fy) \wedge d(x \vee y) \\ &= f(x \vee y) \wedge d(x \vee y) \\ &= d(x \vee y). \end{aligned}$$

□

Theorem 3. *Let L be a distributive lattice and d be an f-derivation on L where $f(x \vee y) = fx \vee fy$. Then the following hold:*

(1) *d is an isotone f-derivation implies $d(x \wedge y) = dx \wedge dy$,*

(2) *d is an isotone f-derivation if and only if $d(x \vee y) = dx \vee dy$.*

Proof. (1) Since d is isotone f-derivation, we know that $d(x \wedge y) \leq dx \wedge dy$. From Proposition 1 a), we get

$$\begin{aligned} dx \wedge dy &= (dx \wedge fx) \wedge (dy \wedge fy) \\ &= (dx \wedge fy) \wedge (fx \wedge dy) \end{aligned}$$

$$\begin{aligned} &\leq (dx \wedge fy) \vee (fx \wedge dy) \\ &= d(x \wedge y). \end{aligned}$$

Hence we have $d(x \wedge y) = dx \wedge dy$.

(2) Let d is an isotone f-derivation. From (1), we have $d(x \wedge y) = dx \wedge dy$. Then, from Proposition 1 a) and Proposition 4, we get $dy = dy \vee (fy \wedge d(x \vee y)) = (dy \vee fy) \wedge (dy \vee d(x \vee y)) = fy \wedge d(x \vee y)$ and similarly $dx = fx \wedge d(x \vee y)$. Then we obtain

$$\begin{aligned} dx \vee dy &= (fx \wedge d(x \vee y)) \vee (fy \wedge d(x \vee y)) \\ &= (fx \vee fy) \wedge d(x \vee y) \\ &= f(x \vee y) \wedge d(x \vee y) \\ &= d(x \vee y). \end{aligned}$$

Conversely, suppose that $d(x \vee y) = dx \vee dy$ and $x \leq y$. Then since $dy = d(x \vee y) = dx \vee dy$, we have $dx \leq dy$. □

Theorem 4. *Let L be a lattice. If there exists an f-derivation d on L such that $d(x \vee y) = dx \vee dy$ for all $x, y \in L$ and f is an epimorphism, then L is a distributive lattice.*

Proof. We know from Example 2 that the function d defined by $dx = fx \wedge c$ for $c \in L$ where f is a homomorphism is an f-derivation on L. Also suppose that f is onto and $d(x \vee y) = dx \vee dy$ for all $x, y \in L$. Then, for all $a, b \in L$ there exist $u, v \in L$ such that $fu = a$ and $fv = b$. Hence

$$\begin{aligned} (a \vee b) \wedge c &= (fu \vee fv) \wedge c \\ &= f(u \vee v) \wedge c \\ &= d(u \vee v) \\ &= du \vee dv \\ &= (fu \wedge c) \vee (fv \wedge c) \\ &= (a \wedge c) \vee (b \wedge c). \end{aligned}$$

□

Since every distributive lattice is a modular lattice, we have the following corollary.

Corollary 1. *Let L be a lattice. If there exists an f-derivation d on L such that $d(x \vee y) = dx \vee dy$ for all $x, y \in L$ and f is an epimorphism, then L is a modular lattice.*

Acknowledgements. The authors are highly grateful to the referees for their valuable comments and suggestions for the paper.

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Mo., 1974.

[2] A. J. Bell, *The co-information lattice*, 4^{th} Int. Symposium on Independent Component Analysis and Blind Signal Seperation (ICA2003), Nara, Japan, 2003, 921–926.

[3] H. E. Bell and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar. **53** (1989), no. 3-4, 339–346.

[4] H. E. Bell and G. Mason, *On derivations in near-rings*, Near-rings and near-fields (Tubingen, 1985), 31–35, North-Holland Math. Stud., 137, North-Holland, Amsterdam, 1987.

[5] G. Birkhoof, *Lattice Theory*, American Mathematical Society, New York, 1940.

[6] C. Carpineto and G. Romano, *Information retrieval through hybrid navigation of lattice representations*, International Journal of Human-Computers Studies **45** (1996), 553–578.

[7] C. Degang, Z. Wenxiu, D. Yeung, and E. C. C. Tsang, *Rough approximations on a complete completely distributive lattice with applications to generalized rough sets*, Inform. Sci. **176** (2006), no. 13, 1829–1848.

[8] G. Durfee, *Cryptanalysis of RSA using algebraic and lattice methods*, A dissertation submitted to the department of computer sciences and the committe on graduate studies of Stanford University (2002), 1–114.

[9] A. Honda and M. Grabisch, *Entropy of capacities on lattices and set systems*, Inform. Sci. **176** (2006), no. 23, 3472–3489.

[10] Y. B. Jun and X. L. Xin, *On derivations of BCI-algebras*, Inform. Sci. **159** (2004), no. 3-4, 167–176.

[11] K. Kaya, *Prime rings with α-derivations*, Hacettepe Bull. Natural. Sci. and Eng. **16-17** (1987/1988), 63–71.

[12] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[13] R. S. Sandhu, *Role hierarchies and constraints for lattice-based access controls*, Proceedings of the 4^{th} Europan Symposium on Research in Computer Security, Rome, Italy, 1996, 65–79.

[14] X. L. Xin, T. Y. Li, and J. H. Lu, *On derivations of lattices*, Inform. Sci. **178** (2008), no. 2, 307–316.

[15] J. Zhan and Y. L. Liu, *On f-derivations of BCI-algebras*, Int. J. Math. Math. Sci. (2005), no. 11, 1675–1684.

Yilmaz Çeven
Department of Mathematics
Faculty of Arts and Sciences
Süleyman Demirel University
32260 Isparta, Turkey
E-mail address: yceven@fef.sdu.edu.tr

Mehmet Ali Öztürk
Department of Mathematics
Faculty of Arts and Sciences
Adiyaman University
02040 Adiyaman, Turkey
E-mail address: mehaliozturk@gmail.com