COMMON FIXED POINT AND INVARIANT APPROXIMATION IN MENGER CONVEX METRIC SPACES

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ABSTRACT. Necessary conditions for the existence of common fixed points for noncommuting mappings satisfying generalized contractive conditions in a Menger convex metric space are obtained. As an application, related results on best approximation are derived. Our results generalize various well known results.

1. Introduction and preliminaries

Geometric conditions on a mappings and(or) underlying spaces play a crucial role in metric fixed point theory. Although, it has purely metric facet, it is a major branch of a nonlinear functional analysis with close ties to Banach space geometry (see for example, [10]). Several results concerning existence and approximation of fixed point of a mapping rely on convexity hypothesis and geometric properties of Banach spaces. Aronszajn and Panitchapakdi [3] and Menger [19] defined a convexity structure on metric spaces through closed balls and investigated their properties. Khalil [17] further studied existence of fixed points and best approximation in this space.

Recently, Beg and Abbas [4] obtained fixed point and approximate fixed point of nonexpansive and quasi nonexpansive mappings defined on compact convex subsets of uniformly convex metric spaces (see also, [5]). On the other hand, Shahzad [21], O'Regan and Hussain [20] introduced a class of noncommuting mappings called R-subweakly commuting mappings, pointwise R-subweakly commuting mappings and applied it to S-nonexpansive mappings in normed spaces.

This paper deals with the study of common fixed points for C_q-commuting and uniformly C_q-commuting mappings in a Menger convex metric space which extends several results in the literature. We also establish results on invariant approximation for these mappings.

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel.

Definition 1.1 ([19])**.** Let (X, d) be a metric space. It is said to be (*Menger*) *convex metric space* if for every x, y in X, $x \neq y$, $0 \leq r \leq d(x, y)$,

$$B[x, r] \cap B[y, d(x, y) - r] \neq \phi,$$

where $B[x, r] = \{y \in X : d(x, y) \leq r\}$.

Some of the properties of these spaces were introduced by Blumenthal [8]. For further study, we refer to Khalil [16], Berard [7] and references mentioned therein.

Let X be a convex metric space. A nonempty subset E of X is said to be *convex,* if for all x, y in E,

$$B[x, r] \cap B[y, d(x, y) - r] \subseteq E,$$

where $0 \leq r \leq d(x, y)$. We note that in general, $B[x, r]$ need not be a convex set in a convex metric space.

Definition 1.2. A convex metric space is said to have property (A), if for every x, y in X,

$$B[x, (1 - t)d(x, y)] \cap B[y, td(x, y)]$$

is a singleton set for $t \in [0, 1]$. We denote this singleton set by $m(x, y, t)$. If $x = y$, then obviously $m(x, y, t) = \{x\}$. The set $B[x, r]$ is convex in a convex metric space with property (A).

A subset E of a convex metric space is said to be *q-starshaped,* if there exists q in E such that for all x in E and $0 \leq r \leq d(x, q)$,

$$B[x, r] \cap B[y, d(x, q) - r] \subseteq E.$$

Obviously q-starshaped subsets of X contain all convex subsets of X as a proper subclass.

A convex metric space X is said to have property (B) if

$$d(m(x, y, \lambda), m(x, z, \lambda)) \leq \lambda d(y, z)$$

and

$$d(m(y, x, \lambda), m(z, x, \lambda)) \leq (1 - \lambda)d(y, z)$$

for all $x, y, z \in X$ and $\lambda \in (0, 1)$. Property (B) is a convex metric space analogue of condition (I) for starshaped metric space of Guay, Singh and Whitfield (see, Definition 3.2 [11]).

Throughout this paper, a metric space X has properties (A) and (B).

Definition 1.3. Let $f, g : X \to X$. A point $x \in X$ is called;
  (1) *fixed point of f* if $f(x) = x$;
  (2) *coincidence point of a pair (f, g)* if $fx = gx$;
  (3) *common fixed point of a pair (f, g)* if $x = fx = gx$.
$F(f), C(f, g)$ and $F(f, g)$ denote the set of all fixed points of f, coincidence points of the pair (f, g) and common fixed points of the pair (f, g), respectively.

Definition 1.4. Let E be q-starshaped subset of a convex metric space X with properties (A) and (B), $q \in F(S)$ and E be both T and S invariant where $T, S : X \to X$. Put

$$Y_q^{Tx} = \{y_\lambda : y_\lambda = m(q, Tx, \lambda) \text{ and } \lambda \in (0, 1]\}.$$

Now, for each x in X, $d(Sx, Y_q^{Tx}) = \inf_{\lambda \in [0,1]} d(Sx, y_\lambda)$. The map T is said to be:

(1) an *S-contraction,* if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(Sx, Sy).$$

(2) an *asymptotically S-nonexpansive,* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy)$$

for each x, y in E and each $n \in N$. If $k_n = 1$, for all $n \in N$, then T is known as a S-nonexpansive mapping. If $S = I$ (identity map), then T is asymptotically nonexpansive.

(3) *R-weakly commuting*, if there exists a real number $R > 0$ such that

$$d(TSx, STx) \leq Rd(Tx, Sx),$$

for all x in E.

(4) *R-subweakly commuting*, if there exists a real number $R > 0$ such that

$$d(TSx, STx) \leq Rd(Sx, Y_q^{Tx}),$$

for all $x \in E$.

(5) *uniformly R-subweakly commuting*, if there exists a real number $R > 0$ such that

$$d(T^n Sx, ST^n x) \leq Rd(Sx, Y_q^{T^n x});$$

for all $x \in E$.

(6) C_q-*commuting,* if $STx = TSx$ for all $x \in C_q(S, T)$, where $C_q(S, T) = U\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k x = m(q, Tx, k)$.

Clearly C_q-commuting maps are weakly compatible but the converse is not true (see for example [2]).

(7) A self mapping T on a convex metric space X is said to be *affine* on E, if

$$T(m(x, y, \lambda)) = m(Tx, Ty, \lambda),$$

for all $x, y \in E$ and $\lambda \in (0, 1)$.

(8) *uniformly asymptotically regular* on E, if for each $\varepsilon > 0$ there exists a positive integer N such that $d(T^n x, T^n y) < \varepsilon$ for all $n \geq N$ and for all x in E.

Definition 1.5. Let X be a metric space and M be a closed subset of X. If there exists a $y_0 \in M$ such that $d(x, y_0) = d(x, M) = \inf_{y \in M} d(x, y)$, then y_0 is called a *best approximation to x out of M*. We denote by $P_M(x)$, the set of all best approximation to x out of M.

Let M be a closed convex subset of a convex metric space with properties (A) and (B) and $x \in X$. Then the set $P_M(x)$ is closed and convex [17]. It can also be shown that $P_M(x) \subset \partial M$, where ∂M stands for the boundary of M.

2. Common fixed point theorems

We begin with the definition of uniformly C_q-commuting mappings.

Definition 2.1 ([12]). Let E be a q-starshaped subset of a convex metric space X. Let $S, T : E \to E$ be maps with $q \in F(S)$. Then S and T are said to be *uniformly C_q-commuting* on E, if $ST^n x = T^n S x$ for all $x \in C_q(S, T^n)$ and $n \in N$.

Clearly, uniformly C_q-commuting maps on E are C_q-commuting but the converse is not true.

Example 2.2. Let X be a set of all real numbers with usual metric and $E = [1, \infty)$. Let $Tx = 2x - 1$ and $Sx = x^2$, for all $x \in E$. Let $q = 1$. Then E is q-starshaped with $Sq = q$ and $C_q(S, T) = \{1\}$ and $C_q(S, T^2) = [1, 3]$. Note that S and T are C_q-commuting maps but not uniformly C_q-commuting, because $ST^2 x \neq T^2 S x$, for all $x \in (1, 3] \subset C_q(S, T^2)$.

And also, we know that uniformly R-subweakly commuting maps are uniformly C_q-commuting but the converse does not hold in general, to see this we consider the following example.

Example 2.3. Let X be a set of all real numbers with usual metric and $E = [0, \infty)$. Let $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x \geq 1$, and $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$. Then E is 1-starshaped with $S1 = 1$, $C_q(S, T) = [1, \infty]$ and $C_q(S, T^n) \subseteq [1, \infty]$ for each $n > 1$. Note that S and T are uniformly C_q-commuting but not R-weakly commuting for all $R > 0$. Thus S and T are neither R-subweakly commuting nor uniformly $R-$subweakly commuting mappings.

In this section, existence of common fixed points of C_q-commuting and uniformly C_q-commuting, and uniformly R-subweakly commuting mappings is established in a Menger convex metric space.

Theorem 2.4. *Let E be a nonempty q-starshaped complete subset of a convex metric space, and T, f and g be self mappings on X. Suppose $q \in F(f) \cap F(g)$, T is continuous, f and g are continuous and affine on E, $cl(T(E))$ is compact and $T(E) \subset f(E) \cap g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy the inequality*

$$(1) \qquad d(Tx, Ty) \leq \max\{d(fx, gy), d(fx, Y_q^{Tx}), d(gy, Y_q^{Ty}),$$
$$\frac{1}{2}[d(fx, Y_q^{Ty}) + d(gy, Y_q^{Tx})]\}$$

for all $x, y \in E$, then T, f and g have a common fixed point in E.

Proof. Define $T_n : E \to E$ by

$$T_n x = m(q, Tx, \lambda_n),$$

where $\lambda_n \in (0, 1)$ with $\lim_{n \to \infty} \lambda_n = 1$. Since E is q-starshaped, T_n is the self mapping on E for each $n \geq 1$. Since f and T are C_q-commuting and g is affine on E with $fq = q$, for each $x \in C_n(f, T) \subseteq C_q(f, T)$, we have

$$f T_n x = f(m(q, Tx, \lambda_n)) = m(q, fTx, \lambda_n) = m(q, Tfx, \lambda_n) = T_n f x.$$

Thus f and T_n are weakly compatible for all n. Also since g and T are C_q-commuting and g is affine on E with $gq = q$, g and T_n are weakly compatible for all n. Also, we have

$$
\begin{aligned}
d(T_n x, T_n y) &= d(m(q, Tx, \lambda_n), m(q, Ty, \lambda_n)) \\
&\leq \lambda_n d(Tx, Ty) \\
&\leq \lambda_n \max\{d(fx, gy), d(fx, Y_q^{Tx}), \\
&\qquad d(gy, Y_q^{Ty}), \tfrac{1}{2}[d(fx, Y_q^{Ty}) + d(gy, Y_q^{Tx})]\} \\
&\leq \lambda_n \max\{d(fx, gy), d(fx, T_n x), \\
&\qquad d(gy, T_n y), \tfrac{1}{2}[d(fx, T_n y) + d(gy, T_n x)]\}.
\end{aligned}
$$

By Corollary 3.1 of [9], for each $n \geq 1$, there exists x_n in E such that x_n is a common fixed point of f, g, and T_n. The compactness of $cl(T(E))$ implies that there exists a subsequence $\{Tx_k\}$ of $\{Tx_n\}$ such that $Tx_k \to y$ as $k \to \infty$. From the definition of $T_k x_k$ and convexity of a metric space, $x_k \to y$. Hence we have $y \in F(T) \cap F(f) \cap F(g)$, by the continuity of T, f and g. $\square$

Corollary 2.5. *Let E be a nonempty q-starshaped complete subset of a convex metric space X, and T, f and g be self mappings on X. Suppose $q \in F(f) \cap F(g)$, T is continuous, f and g are continuous and affine on E, $cl(T(E))$ is compact and $T(E) \subset f(E) \cap g(E)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are R-subweakly commuting mappings satisfying (1), then T, f and g have a common fixed point in E.*

Corollary 2.6. *Let E be a nonempty closed q-star shaped subset of a convex metric space X, T and S be two R-subweakly commuting mappings on E such that $T(E) \subset S(E)$, and $cl(T(E))$ is compact where $q \in F(S)$. If T is continuous and S-nonexpansive and S is affine on E, then $F(T) \cap F(S)$ is nonempty.*

Theorem 2.7 ([15], [13])**.** *Let E be a subset of a metric space (X, d) and S and T be weakly compatible self-maps of E. Assume that $clT(E) \subset S(E)$, $clT(E)$ is complete, and T and S satisfy the inequality*

$$d(Tx, Ty) \leq h \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$$

for all $x, y \in E$ and $0 \leq h < 1$. Then $E \cap F(f) \cap F(T)$ is a singleton.

Theorem 2.8. *Let E be a nonempty closed q-starshaped subset of a convex complete metric space X, and T and S be two uniformly C_q-commuting mappings on $E - \{q\}$ such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, where $q \in F(S)$. Suppose that T is continuous and asymptotically S-nonexpansive with sequence $\{k_n\}$ and S is affine on E. And for each $n \geq 1$, define a mapping T_n on E by*

$$T_n x = m(q, T^n x, \alpha_n),$$

where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\lim_{n \to \infty} \lambda_n = 1$. Then for each $n \in N$, $F(T_n) \cap F(S)$ is a singleton.

Proof. For all $x, y \in E$, we have

$$
\begin{aligned}
d(T_n(x), T_n(y)) &= d(m(q, T^n x, \alpha_n), m(q, T^n y, \alpha_n)) \\
&\leq \alpha_n d(T^n x, T^n y) \\
&\leq \lambda_n d(Sx, Sy).
\end{aligned}
$$

Moreover, since T and S are uniformly C_q-commuting and S is affine on E with $Sq = q$, for each $x \in C_n(S, T) \subseteq C_q(S, T)$, we have

$$ST_n x = S(m(q, T^n x, \lambda_n)) = m(q, ST^n x, \lambda_n) = m(q, T^n Sx, \lambda_n) = T_n Sx.$$

Thus S and T_n are weakly compatible for all n. Now, the result follows from Theorem 2.7. □

Corollary 2.9. *Let E be a nonempty closed q-starshaped subset of a convex complete metric space X, and T and S be two uniformly R-subweakly commuting mappings on $E - \{q\}$ such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, where $q \in F(S)$. Suppose that T is continuous and asymptotically S-nonexpansive with sequence $\{k_n\}$ and S is affine on E. And for each $n \geq 1$, define a mapping T_n on E by*

$$T_n x = m(q, T^n x, \alpha_n),$$

where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\lim_{n \to \infty} \lambda_n = 1$. Then for each $n \in N$, $F(T_n) \cap F(S)$ is a singleton.

Theorem 2.10. *Let E be a nonempty closed q-starshaped subset of a convex metric space X, and T and S be two self mappings on E such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, $q \in F(S)$. Suppose T is continuous, uniformly asymptotically regular and asymptotically S-nonexpansive and S is affine on E. If $cl(E - \{q\})$ is compact and S and T are uniformly C_q-commuting mappings on $E - \{q\}$, then $F(T) \cap F(S)$ is a singleton.*

Proof. From Theorem 2.8, for each $n \in N$, $F(T_n) \cap F(S)$ is a singleton in E. Thus,

$$Sx_n = x_n = m(q, T^n x_n, \alpha_n).$$

Also,

$$
\begin{aligned}
d(x_n, T^n x_n) &= d(m(q, T^n x_n, \alpha_n), T^n x_n)) \\
&\leq (1 - \alpha_n) d(q, T^n x_n) \\
&\leq (1 - \alpha_n) d(q, T^n x_n).
\end{aligned}
$$

Since $T(E - \{q\})$ is bounded, $d(x_n, T^n x_n) \to 0$ as $n \to \infty$. Now, since S commutes with T^n on $C_q(S, T^n)$, $x_n \in C_q(S, T^n)$ and $x_n = Sx_n$, we have,

$$
\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(ST^n x_n, Sx_n) \\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n Sx_n, Sx_n), \alpha_n)) \\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n, \alpha_n)),
\end{aligned}
$$

which implies that, $d(x_n, Tx_n) \to 0$, as $n \to \infty$. As $cl(E - \{q\})$ is compact and E is closed, therefore there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x_0 \in E$ as $i \to \infty$. By the continuity of T, we have $T(x_0) = x_0$. Since $T(E - \{q\}) \subset S(E - \{q\})$, it follows that $x_0 = T(x_0) = Sy$, for some $y \in E$. Moreover

$$
d(Tx_{n_i}, Ty) \leq k_1 d(Sx_{n_i}, Sy) = k_1 d(x_{n_i}, x_0).
$$

Taking the limit as $i \to \infty$, we get $Tx_0 = Ty$. Thus, $Tx_0 = Sy = Ty = x_0$. Since S and T are uniformly C_q-commuting on $E - \{q\}$, and $y \in C(S, T)$, we have

$$
d(Tx_0, Sx_0) = d(TSy, STy) = 0.
$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.11. *Let E be a nonempty closed q-starshaped subset of a convex metric space X, and T and S be two self mappings on E such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, $q \in F(S)$. Suppose T is continuous, uniformly asymptotically regular and asymptotically S-nonexpansive and S is affine on E. If $cl(E - \{q\})$ is compact and S and T are uniformly R-subweakly commuting mappings on $E - \{q\}$, then $F(T) \cap F(S)$ is a singleton.*

Corollary 2.12. *Let E be a nonempty closed q-starshaped subset of a convex metric space X, and T and S be two self mappings on E such that $S(E) = E$ and $T(E - \{q\}) \subset S(E - \{q\})$, $q \in F(S)$. Suppose T is continuous, uniformly asymptotically regular and asymptotically S-nonexpansive and S is affine on E. If $cl(E - \{q\})$ is compact and S and T are C_q-commuting mappings on $E - \{q\}$, then $F(T) \cap F(S)$ is a singleton.*

3. Invariant approximation results

Meinardus [18] was the first to employ fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results appeared in the literature of approximation

theory ([1], [21] and [22]). In this section, we obtain results on best approxima-
tion as a fixed point of C_q-commuting and uniformly C_q-commuting mappings
in the setting of a Menger convex metric space.

Theorem 3.1. *Let M be a nonempty subset of a convex metric space X, and
T, f and g be self maps on X such that u is common fixed point of f, g and
T and $T(\partial M \cap M) \subset M$. Suppose f and g are affine and continuous on
$P_M(u)$, where $P_M(u)$ is q-starshaped with $f(P_M(u)) = P_M(u) = g(P_M(u))$ and
$q \in F(f) \cap F(g)$. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy
for all $x \in P_M(u) \cup \{u\}$*

$$d(Tx, Ty) \leq \begin{cases} d(fx, gu), ify = u, \\ \max\{d(fx, gy), d(fx, Y_q^{Tx}), d(gy, Y_q^{Ty}), \\ \frac{1}{2}[d(fx, Y_q^{Ty}) + d(gy, Y_q^{Tx})]\}, \ if \ y \in P_M(u). \end{cases}$$

*If $cl(P_M(u))$ is compact and $P_M(u)$ is complete, then $P_M(u) \cap F(T) \cap F(f) \cap F(g)$
is nonempty.*

Proof. Let $x \in P_M(u)$. Then $d(x, u) = d(x, M)$. Note that for any $\lambda \in (0, 1)$

$$\begin{aligned} d(y_\lambda, u) &= d(m(u, x, \lambda), u) \\ &\leq \lambda d(x, u) < d(x, u) = d(x, M), \end{aligned}$$

which shows that, $Y_u^x = \{y_\lambda : y_\lambda = m(u, x, \lambda)\} \cap M$ is empty so $x \in \partial M \cap M$ and
$Tx \in M$. Since $fx \in P_M(u)$, u is common fixed point of f, g and T. Therefore,
from the contractive condition, we obtain

$$\begin{aligned} d(Tx, u) &= d(Tx, Tu) \\ &\leq d(fx, gu) = d(fx, u) = d(u, M). \end{aligned}$$

Thus $P_M(u)$ is T-invariant. Hence,

$$T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u)).$$

The result follows from Theorem 2.4. $\qquad\qquad\square$

Theorem 3.2. *Let M be a nonempty subset of a convex metric space X,
and T and S be two self mappings on X such that and $T(\partial M \cap M) \subset M$,
$u \in F(S) \cap F(T)$ for some u in X. Suppose T is continuous, uniformly asymp-
totically regular and asymptotically S-nonexpansive and S is affine on $P_M(u)$
with $S(P_M(u)) = P_M(u)$, $q \in F(S)$ and $P_M(u)$ is q-starshaped. If $cl(P_M(u))$
is compact and $P_M(u)$ is complete and S and T are uniformly C_q-commuting
mappings on $P_M(u) \cup \{u\}$ satisfying $d(Tx, Tu) \leq d(Sx, Su)$, then $P_M(u) \cap
F(T) \cap F(S)$ is nonempty.*

Proof. Let $x \in P_M(u)$. Then $d(x, u) = d(x, M)$. Note that for any $\lambda \in (0, 1)$

$$\begin{aligned} d(y_\lambda, u) &= d(m(u, x, \lambda), u) \\ &\leq \lambda d(x, u) < d(x, u) = d(x, M), \end{aligned}$$

which shows that, $Y_u^x = \{y_\lambda : y_\lambda = m(u, x, \lambda)\} \cap M$ is empty so $x \in \partial M \cap M$ and $Tx \in M$. Since $Sx \in P_M(u)$, u is common fixed point of S and T. Therefore, from the contractive condition, we obtain

$$
\begin{aligned}
d(Tx, u) &= d(Tx, Tu) \\
&\leq d(Sx, Su) = d(Sx, u) = d(u, M).
\end{aligned}
$$

Thus $P_M(u)$ is T-invariant. Hence,

$$
T(P_M(u)) \subset P_M(u) = S(P_M(u)).
$$

Now the result follows from Theorem 2.10.   □

Remark 3.3.   (1) Theorem 2.4 extends and improves Theorem 2.2 of Al-Thagafi [1], Theorem 2.2(i) of Hussain and Jungck [13] and Lemma 2.2 of Shahzad [21].
    (2) Theorems 2.8–2.10 extend the results of Beg, Sahu and Dewan [6] to more general classes of mappings.
    (3) Following the arguments in [13], we may prove Theorem 2.2-Theorem 2.12 and Theorem 2.14-Corollary 2.16 of Hussain and Jungck [13] for asymptotically (f, g)-nonexpansive maps by using Theorem 2.10.
    (4) As an application of Theorem 2.10, we can prove the recent invariant approximation results in [14, 15], for uniformly C_q-commuting, asymptotically I-nonexpansive map T.

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