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ORBITAL SHADOWING PROPERTY

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Abstract. Let M be a generalized homogeneous compact space, and let $Z(M)$ denotes the space of homeomorphisms of M with the C^0 topology. In this paper, we show that if the interior of the set of weak stable homeomorphisms on M is not empty then for any open subset W of $Z(M)$ containing only weak stable homeomorphisms the orbital shadowing property is generic in W.

1. Introduction

The concept of shadowing is investigated by many authors (see e.g. [2, 4, 8]). In [3] Corless and Pilyugin proved that weak shadowing is a C^0 generic property for discrete dynamical systems of a compact smooth manifold M. Subsequently, Pilyugin and Plamenevskaya [9] improved this result by showing C^0 genericity of the shadowing property. Both proofs given in [3] and [9] required that M be a C^∞ smooth manifold. Mazur in [7] showed that for C^0 genericity of weak shadowing neither the differential structure on M, nor even being a manifold is a crucial assumption, but what matters is a generalized version of a topological property called homogeneity. Koscielniak and Mazur in [5] have given a proof for C^0 genericity of periodic orbital shadowing on a compact topological manifold of dimension at least 2. In this note we show that if the space M is generalized homogeneous and has no isolated points, then in an open subset of $Z(M)$ the orbital shadowing property is generic.

2. Notations

Let (M, d) be a compact metric space and let $f : M \to M$ be a homeomorphism (a discrete dynamical system on M). A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an orbit of f, denote by $o(x, f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and we call it a δ-pseudo-orbit of f if,

$$d(f(x_n), x_{n+1}) \leq \delta,\ \forall n \in \mathbb{Z}.$$

The homeomorphism f is said to have the weak shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ of f we

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can find a point $y \in M$ such that $\{x_n\}_{n\in\mathbb{Z}} \subset N_\epsilon(o(y,f))$, where $N_\epsilon(S)$ is the ϵ-neighborhood of the set $S \subset M$.

A system f is said to have the orbital shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ of f we can find a point $y \in M$ with the property that $\{x_n\}_{n\in\mathbb{Z}} \subset N_\epsilon(o(y,f))$ and $o(y,f) \subset N_\epsilon(\{x_n\}_{n\in\mathbb{Z}})$. We denoted the set of all homeomorphisms of M by $Z(M)$. Introduce in $Z(M)$ the complete metric

$$d_0(f,g) = \max\{\max_{x\in M} d(f(x),g(x)), \max_{x\in M} d(f^{-1}(x),g^{-1}(x))\},$$

which generates the C^0 topology.

The space M is said to be generalized homogeneous if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\} \subset M$ is a pair of sets of mutually disjoint elements satisfying $d(x_i,y_i) \leqslant \delta$, $i \in \{1,\ldots,n\}$, then there exists $h \in Z(M)$ satisfying $d_0(h,id_M) \leqslant \epsilon$ and $h(x_i) = y_i, i \in \{1,\ldots,n\}$. Such a δ is called an ϵ-modulus of homogeneity of M. We say that $x \in M$ is a weak stable point for f if for any $\epsilon > 0$ there is $\delta > 0$ and positive integer N such that $o(z,f) \subset N_\epsilon(\{f^i(z) : i = -N,\ldots,N\})$ for every $z \in M$ with $d(x,z) < \delta$. We say that f is weak stable if every point of M is a weak stable point for f.

A property P is said to be generic for elements of a topological space M if the set of all $x \in M$ satisfying P is residual, i.e., it includes a countable intersection of open and dense subsets of M.

3. Results

Proposition 1. *Let f be a homeomorphism on a compact metric space M. Then the set of weak stable points is residual in M.*

Proof. Fix an $\epsilon > 0$. Let $U = \{U_i : i = 1,\ldots,k\}$ be a finite covering of M by open sets with diameter less than $\frac{\epsilon}{2}$. Put $K = \{1,2,\ldots,k\}$. For each $x \in M$, choose a subset L_x of K satisfying the following conditions:

$$o(x,f) \subset \cup\{U_i : i \in L_x\}$$

$$o(x,f) \cap U_i \neq \phi \;\text{ for all } i \in L_x.$$

Let A_ϵ be the set of all $x \in M$ such that there is δ_x and positive integer N_x such that $o(z,f) \subset N_\epsilon(\{f^i(z) : i = -N_x,\ldots,N_x\})$ for every $z \in M$ with $d(x,z) < \delta_x$. We claim that A_ϵ is open and dense in M. Clearly A_ϵ is open. To see that A_ϵ is dense, let $x \in M$ be arbitrary. We can find a positive integer T such that

$$\{f^i(x) : i = -T,\ldots,T\} \cap U_j \neq \phi \;\text{ for all } j \in L_x.$$

Hence $o(x,f) \subset N_\epsilon(\{f^i(x) : i = -T,\ldots,T\})$. Choose $\delta > 0$ such that

$$d(f^i(x),f^i(z)) < \frac{\epsilon}{2} \;\text{ for all } i = -T,-T+1,\ldots,T$$

for every $z \in M$ with $d(x,z) < \delta$. Suppose that $x \notin A_\epsilon$. Given any ζ with $0 < \zeta < \delta$ there is $x_1 \in N_\zeta(x)$ such that $f^{T_1}(x_1) \notin N_\epsilon(\{f^i(x) : i = -T,\ldots,T\})$

for some T_1 with $|T_1| > T$. If $f^{T_1}(x_1) \in N_{\frac{\epsilon}{2}}(\{f^i(x) : i = -T, \ldots, T\})$, then we have

$$d(f^{T_1}(x_1), f^i(x_1)) \leq d(f^{T_1}(x_1), f^i(x)) + d(f^i(x), f^i(x_1)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for some $i \in \{-T, \ldots, T\}$. This is a contradiction. Thus

$$f^{T_1}(x_1) \notin N_{\frac{\epsilon}{2}}(\{f^i(x) : i = -T, \ldots, T\}).$$

Since $f_1^T(x_1) \notin U_j$ for all $j \in L_x$, there is $j \in K - L_x$ such that $f^{T_1}(x_1) \in U_j$. Thus L_x is a proper subset of L_{x_1}. We can find $T_2 \geq |T_1|$ such that

$$o(x_1, f) \subset N_\epsilon(\{f^i(x_1) : i = -T_2, \ldots, T_2\}).$$

Choose $\delta_1 > 0$ such that $d(f^i(x_1), f^i(z)) < \frac{\epsilon}{2}$ for all $i \in \{-T_2, -T_2 + 1, \ldots, T_2\}$ for every $z \in M$ with $d(x_1, z) < \delta_1$. If $x_1 \in A_\epsilon$ we are done, otherwise there is $x_2 \in N_\xi(X_1) \subset N_\zeta(x)$ such that

$$f^{T_3}(x_2) \notin N_\epsilon(\{f^i(x_2) : i = -T_2, \ldots, T_2\})$$

for some T_3 with $|T_3| > T_2$, where $\xi = \min(\zeta, \delta_1)$. Since

$$f^{T_3}(x_2) \notin N_{\frac{\epsilon}{2}}(\{f^i(x_1) : i = -T_2, \ldots, T_2\}),$$

$f_3^T(x_2) \notin U_j$ for all $j \in L_x$. There is $j \in K - L_{x_1}$ such that $f^{T_3}(x_2) \in U_j$. Thus L_{x_1} is a proper subset of L_{x_2}. By continuing this process, since K is finite, we can find $x' \in N_\xi(x)$ such that $L_{x'} = K$. Then $x' \in A_\epsilon$. Thus A_ϵ is dense in M. Now let $R = \bigcap_{n=1}^{\infty} A_{\frac{1}{n}}$ then R is a residual subset of M consisting of weak stable points. □

A homeomorphism $f : M \longrightarrow M$ is called minimal if $f(A) = A$, A closed, implies either $A = M$ or $A = \phi$. It is easy to see that f is minimal if and only if $\overline{o(x, f)} = M$ for each $x \in M$.

Proposition 2. *Let f be a homeomorphism on a compact metric space M. If f is minimal, then f is weak stable.*

Proof. Let $x \in M$ and $\epsilon > 0$ be arbitrary. Let $U = \{U_i : i = 1, \ldots, k\}$ be a finite covering of M by open sets with diameter less than $\frac{\epsilon}{2}$. Since $\overline{o(x, f)} = M$, there are $n_1, n_2, \ldots, n_k \in \mathbb{Z}$, such that $f^{n_i}(x) \in U_i$ for $i = 1, 2, \ldots, k$. Let

$$N = \max\{|n_i| : 1 \leq i \leq k\}.$$

There exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^i(x), f^i(y)) < \frac{\epsilon}{2}$ for all $-N \leq i \leq N$. Let $y \in N_\delta(x)$. Given any $n \in \mathbb{Z}$, since $f^n(y) \in M = \bigcup_{i=1}^k U_i$, $f^n(y) \in U_i$ for some $i = 1, \ldots, k$. But $f^{n_i}(x) \in U_i$ so $d(f^n(y), f^{n_i}(x)) \leq \text{diam}(U_i) < \frac{\epsilon}{2}$. Since $d(x, y) < \delta$ and $-N \leq n_i \leq N$ we have $d(f^{n_i}(x), f^{n_i}(y)) < \frac{\epsilon}{2}$. Thus we have

$$d(f^n(y), f^{n_i}(y)) \leq d(f^n(y), f^{n_i}(x)) + d(f^{n_i}(x), f^{n_i}(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $o(y, f) \subset N_\epsilon(\{f^i(y) : i = -N, \ldots, N)$. This shows that x is an weak stable point and f is an weak stable homeomorphism. □

We denote the set of all weak stable homeomorphisms by WSH.

Theorem 1. *Let* $\mathrm{int}(WSH) \neq \phi$ *and let* W *be an open subset of* $Z(M)$ *containing only weak stable homeomorphisms. Then there is a residual subset* R_1 *of* W *such that for each* $f \in R_1$ *and* $\epsilon > 0$*, there is a neighborhood* U_f *of* f *and positive integer* N_f *such that* $o(x, g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N_\epsilon(g^i(x))}$ *for every* $g \in U_f$ *and* $x \in M$*.*

To prove this theorem we need the following two lemmas.

Lemma 1. *Let* $\epsilon > 0$ *be arbitrary. Then the function* $\psi_\epsilon : W \longrightarrow \mathbb{N}$*; defined by*

$$\psi_\epsilon(f) = N_f,$$

where

$$N_f = \min\{N \in \mathbb{N} : o(x, f) \subset \bigcup_{i=-N}^{N} \overline{N_\epsilon(f^i(x))} \forall x \in M\},$$

is lower semi-continuous.

Proof. For $f \in W$ there is $x_0 \in M$ such that $o(x_0, f) \not\subseteq \bigcup_{i=-N_f+1}^{N_f-1} \overline{N_\epsilon(f^i(x_0))}$. So $f^k(x_0) \notin \bigcup_{i=-N_f+1}^{N_f-1} \overline{N_\epsilon(f^i(x_0))}$ for some $k \in \mathbb{Z}$ with $|k| \geq N_f$. Choose $\epsilon' > 0$ such that

$$(*) \qquad d(f^{N_f}(x_0), f^l(x_0)) \geq \epsilon + \epsilon', \ -N_f + 1 \leq l \leq N_f - 1.$$

Choose a neighborhood U_f of f such that $d(f^i(x), g^i(x)) < \frac{\epsilon'}{2}$, $|i| \leq k + 1$ for each $x \in M$ and $g \in U_f$. If $N_g < N_f$, then $d(g^k(x_0), g^l(x_0)) < \epsilon$ for some $-N_f + 1 \leq l \leq N_f - 1$. So

$$\begin{aligned} d(f^k(x_0), f^l(x_0)) &\leq d(f^k(x_0), g^k(x_0)) + d(g^k(x_0), g^l(x_0)) + d(g^l(x_0), f^l(x_0)) \\ &< \frac{\epsilon'}{2} + \epsilon + \frac{\epsilon'}{2} = \epsilon + \epsilon', \end{aligned}$$

which contradicts $(*)$. Hence $N_g \geq N_f$. This complete the proof of the lemma. □

Now, we recall a topology lemma, for the proof see [6].

Lemma 2. *Let* X *be a Bair topological space and* $\Gamma : X \longrightarrow \mathbb{N}$ *be a lower semi-continuous map. Then there exists a residual subset* R *of* X *such that* $\Gamma |_R$ *is locally constant on each point of* R*.*

Proof of Theorem 1. Using lemmas 1 and 2, for any $\epsilon > 0$ let R_ϵ be a residual subset of W such that ψ_ϵ is locally constant on R_ϵ. Then $R_1 = \cap\{R_{\frac{1}{n}} : n = 1, 2, \ldots\}$ is the required residual set. □

Theorem 2. *Let* M *be a generalized homogeneous space with no isolated point. Then either* $\mathrm{int}(WSH) = \phi$*, or for every* $f \in \mathrm{int}(WSH)$ *and every open neighborhood* W *of* f *in* $\mathrm{int}(WSH)$ *the orbital shadowing property is generic in* W*.*

For the proof we need the following lemma from [10].

Lemma 3. *If h is upper semi-continuous, then for any $\epsilon > 0$ the set of all $x \in X$ such that there exists a neighborhood U of x with the property that $d_H(h(x), h(y)) \leq \epsilon$ for all $y \in U$, is open and dense in X. Here d_H is the Hausdorff metric.*

Proof of Theorem 2. Assuming $\text{int}(WSH) \neq \phi$. Let $f \in \text{int}(WSH)$ and W be an open neighborhood of f in $\text{int}(WSH)$. Let $\epsilon > 0$ be arbitrary, and $A = \{U_1, \dots, U_k\}$ be a finite covering of M by closed sets with diameter less than $\frac{\epsilon}{2}$. Consider the set $K = \{1, 2, \dots, k\}$ as a compact metric space with discrete metric. Let $C(K)$ be the set of all subset of K, then define the map $\varphi_\epsilon : Z(M) \longrightarrow C(C(K))$ by

$$\varphi_\epsilon(f) = \{L \subset K : \exists x \in M \text{ such that } o(x, f) \subset \bigcup\{U_i : i \in L\},$$
$$o(x, f) \cap U_i \neq \phi \forall i \in L\}.$$

The map φ_ϵ is upper semi-continuous [1]. Let R_ϵ be the set of all $f \in Z(M)$ such that there exists a neighborhood U_f of f with the property that $d_H(\varphi_\epsilon(f), \varphi_\epsilon(g)) \leq \epsilon$ for every $g \in U_f$. The set R_ϵ is open and dense in $Z(M)$ by Lemma 3. Moreover, it is easy to see that the map φ_ϵ is locally constant on the set R_ϵ if $\epsilon < 1$ that is for any $f \in R_\epsilon$ there is a neighborhood U_f of f satisfying $\varphi_\epsilon(f) = \varphi_\epsilon(g)$ for all $g \in U_f$. Consider $R_2 = \bigcap_{n=1}^{\infty} R_{\frac{1}{n}}$ and $R = R_2 \bigcap R_1 \bigcap W$, where R_1 is as in Theorem 1. To complete the proof , it remains to show that the set R has orbital shadowing property. Let $0 < \epsilon < 1$ be arbitrary and $f \in R$. There is a neighborhood U_f of f satisfying $\varphi_\epsilon(f) = \varphi_\epsilon(g)$ and $\psi_\epsilon(f) = \psi_\epsilon(g)$ for all $g \in U_f$. Choose $\beta > 0$ such that $N_\beta(f) \subset U_f$. Let $\gamma > 0$ be a β-modulus of homogeneity of M, and put $0 < \delta < \min\{\frac{\gamma}{2}, \frac{\epsilon}{2}\}$. Fix any δ-pseudo-orbit $y = \{y_n\}_{n \in \mathbb{Z}}$. There exists a positive integer $l \geq N_f$ such that $y \subset N_\epsilon(y_t)$ where $y_t = \{y_n\}_{n=-l}^{l}$. Since M has no isolated point we can easily find a finite 2δ-pseudo-orbit $y'_t = \{y'_n\}_{n=-l}^{l}$ such that $y_t \subseteq N_\epsilon(y'_t)$ and $y'_t \subseteq N_\epsilon(y_t)$ and $y_i \neq y_j$ for $i \neq j$ ([11]). Since $d(f(y'_i), y'_{i+1}) < 2\delta < \gamma$ there exists $h \in Z(M)$ such that $d_0(h, id_M) \leq \beta$ and $h(f(y'_i)) = y'_{i+1}$ for all $i = -l, \dots, l$. Set $g = hof$. Then the sequence

$$o(y'_0, g) = \{\dots, g^{-2}(y'_{-l}), g^{-1}(y'_{-l}), y'_{-l}, y'_{-l+1}, \dots, y'_l, g(y'_{-l}), g^2(y'_{-l}), \dots\}$$

is an orbit of g. Since $g \in N_\beta(f)$ we have $\varphi_\epsilon(f) = \varphi_\epsilon(g)$ and $\psi_\epsilon(f) = \psi_\epsilon(g)$. Choose $L \in \varphi_\epsilon(g)$ such that $o(y'_0, g) \subset \bigcup_{i \in L} U_i$ and $o(y'_0, g) \cap U_i \neq \phi$ for all $i \in L$. But $L \in \varphi_\epsilon(f)$, thus there exists $x \in M$ satisfying $o(x, f) \subset \bigcup_{i \in L} U_i$ and $o(x, f) \cap U_i \neq \phi$ for all $i \in L$. This implies that $y \subseteq N_{3\epsilon}(o(x, f)), o(x, f) \subset N_\epsilon(o(y'_0, g))$. Since $\psi_\epsilon(f) = \psi_\epsilon(g)$ we have $o(y'_0, g) \subset \bigcup_{n=-N_f}^{N_f} \overline{N_\epsilon(y'_n)} \subset \bigcup_{n=-l}^{l} \overline{N_\epsilon(y'_n)}$ and $y'_t \subset N_\epsilon(y_t) \subset N_\epsilon(y)$. Hence we get $o(x, f) \subset N_{3\epsilon}(y)$. This complete the proof of Theorem 2. □

As Mazur has shown in [7] the spaces (i), (ii) and (iii) in the following corollary are homogeneous. Thus using Theorem 2 we have:

Corollary. *If the space M is one of the followings*:

(i) *a topological manifold with boundary* ($\dim M \geq 2$ *If* $\partial M \neq \phi$).

(ii) *a cartesian product of a countably infinite number of manifolds with nonempty boundary.*

(iii) *a cantor set.*

Then orbital shadowing is generic property in W.

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