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COINCIDENCE POINTS OF WEAKLY COMPATIBLE MAPPINGS

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Abstract. We present coincidence points and common fixed point results for (f,g)-contractive mapping and (f,g)-nonexpansive mappings. Our results generalize and complement various known results existing in the literature.

1. Introduction and preliminaries

Let K be a nonempty subset of a metric space E and f, g, and T three selfmaps of K, and $C(f,g,T)$ the set of coincidence points of f, g, and T (i.e., $C(f,g,T)=\{x\in K; fx=Tx=gx\}$), and $F(T)$ the set of fixed points of T, $F(T)=\{x\in K; x=Tx\}$. We shall denote the closure of K by $\overline{K}$, the boundary of K by ∂K and all positive integer by $\mathbb{N}$. When $\{x_n\}$ is a sequence in E, then $x_n\to x$ (respectively, $x_n\rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

The set K is called (1) *q-starshaped* with $q\in K$ if $kx+(1-k)q\in K$ for all $x\in K$ and all $k\in[0,1]$; (2) *convex* if $kx+(1-k)y\in K$ for all $x,y\in K$ and all $k\in[0,1]$.

The selfmap f on K is called (3) *affine* if K is convex and $f(kx+(1-k)y)=kfx+(1-k)fy$ for all $x,y\in K$ and all $k\in[0,1]$; (4) *q-affine* if K is q-starshaped and $f(kx+(1-k)q)=kfx+(1-k)q$ for all $x\in K$ and all $k\in[0,1]$. Note that $fq=q$ whenever f is a q-affine selfmap of a q-starshaped set K [1].

(5) The selfmap T on K is called (f,g)*-contraction* if, there exists $0\le k<1$ such that $d(Tx,Ty)\le kd(fx,gy)$ for any $x,y\in K$. If $k=1$, then T is called (f,g)*-nonexpansive*. If $f=g$, then T is called f*-contraction* (or f-nonexpansive). If $f=g=I$, an identity operator, then T is called *contraction* (or *nonexpansive*).

A mapping $T:K\to K$ is called (6) *continuous* if for all $\{x_n\}$ such that $\{x_n\}$ converges to x implies that $\{Tx_n\}$ converges strongly to Tx; *strongly continuous* if for all $\{x_n\}$ such that $\{x_n\}$ converges weakly to x implies that

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$\{Tx_n\}$ converges strongly to Tx; *weakly continuous* if for all $\{x_n\}$ such that $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges weakly to Tx. Clearly the strong continuity of T implies both continuity and weakly continuity of T but not conversely [11]; (7) demiclosed at 0 if for every sequence $\{x_n\} \subset K$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

The map pair (T, f) is called (8) *commuting* if $Tfx = fTx$ for all $x \in K$; (9) *R-weakly commuting* [6] if for all $x \in K$ there exists $R > 0$ such that $d(fTx, Tfx) \leq Rd(fx, Tx)$. If $R = 1$, then the map pair are called *weakly commuting*; (10) *compatible* [2] if $\lim_{n\to\infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$ for some t in K; (11) *weakly compatible* [1] if they commute at their coincidence points, i.e., $fTx = Tfx$ whenever $fx = Tx$. Suppose that E is compact metric space and both T and f are continuous, then (f, T) compatible equivalent to (f, T) weakly compatible [2, Theorem 2.2, Corollary 2.3].

Suppose that K is q-starshaped with $q \in F(f)$ and is both T-and f-invariant. Then (T, f) are called (12) *R-subweakly commuting* on K (see [1, 8, 9]) if for all $x \in K$, there exists a real number $R > 0$ such that $d(fTx, Tfx) \leq R\delta(fx, [Tx, q])$, where $[Tx, q] = \{kTx + (1-k)q; x \in K, k \in (0, 1]\}$ and $\delta(p, K) = \inf_{z\in K} d(z, p)$ for $p \in E$; (13) *R-subcommuting* [10] on K if for all $x \in K$, there exists a real number $R > 0$ such that $d(fTx, Tfx) \leq \frac{R}{k} d(kTx + (1-k)q, fx)$ for all $k \in (0, 1]$; (14) *C_q-commuting* [1] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \bigcup\{C(f, T_k); 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$. Clearly, C_q-commuting maps are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general [1]. (15) A normed space E is said to be satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \ \forall y \in E \text{ with } x \neq y.$$

Hilbert space and the space l_p $(1 \leq p < \infty)$ satisfy Opials condition.

During the last decades, the hybrid mapping results have been obtained by many mathematicians (for example, see reference [1, 2, 3, 4, 6, 7, 8, 9, 10]).

The aim of this paper is to prove results extending the above mentioned invariant approximation results. In particular, we establish general common fixed point theorems for (f, g)-nonexpansive weakly compatible mappings. Our results, on the one hand, extend and unify the work of Al-Thagafi and Shahzad [1], Jungck [3], and on the other hand, provide generalizations and complementarities of the recent work of Jungck and Sessa [4] and Shahzad [7, 8, 9, 10].

2. Coincidence and common fixed point theorems

Theorem 2.1. *Let K be a subset of a metric space (E, d), and $T, f, g : K \to K$ be three mappings and T be a (f, g)-contraction with a constant $k \in (0, 1)$ which* *satisfies* $\overline{T(K)} \subset f(K) \bigcap g(K)$*. Suppose that either* $\overline{T(K)}$ *or* $f(K)$ *or* $g(K)$ *is complete, then*

(i) *there exist* $z, u, v \in K$ *such that* $fu = Tu = z = Tv = gv$*, that is,* $u \in C(T, f)$ *and* $v \in C(T, g)$*;*

If, in addition, (T, f) *and* (T, g) *are weakly compatible, then*

(ii) $F(T) \bigcap F(f) \bigcap F(g)$ *is singleton.*

Proof. Take $x_0 \in K$. As $\overline{T(K)} \subset f(K) \bigcap g(K)$, we can choose a sequence $\{x_n\}$ in K such that $Tx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = gx_{2n+2}$ for all $n \geq 0$. It follows that

$$d(Tx_{2n+1}, Tx_{2n}) \leq kd(fx_{2n+1}, gx_{2n}) = kd(Tx_{2n}, Tx_{2n-1}).$$

Similarly, we also have that

$$d(Tx_{2n-1}, Tx_{2n}) \leq kd(fx_{2n-1}, gx_{2n}) = kd(Tx_{2n-2}, Tx_{2n-1}).$$

Therefore, for all $n \geq 0$,

$$d(Tx_{n+1}, Tx_n) \leq kd(Tx_{n-1}, Tx_n) \leq k^n d(Tx_1, Tx_0).$$

Thus,

$$d(Tx_{n+p}, Tx_n) \leq \sum_{i=0}^{p} d(Tx_{n+i}, Tx_{n+i+1}) \leq \sum_{i=0}^{p} k^{n+i} d(Tx_1, Tx_0).$$

Hence, $\{Tx_n\}$ is a Cauchy sequence. By the definition of $\{Tx_n\}$, then the sequence $\{fx_{2n+1}\}$ and $\{gx_{2n+2}\}$ are also Cauchy sequences.

Since either $\overline{T(K)}$ or $f(K)$ or $g(K)$ is complete, suppose $f(K)$ is complete. Then $fx_{2n+1} \to z \in K$, and by the definition of $\{Tx_n\}$, we obtain that

$$gx_{2n}, fx_{2n+1}, Tx_n \to z \in \overline{T(K)} \subset f(K) \bigcap g(K).$$

Hence there exist $u, v \in K$ such that $fu = z = gv$. Then as $n \to \infty$,

$$d(Tx_{2n+1}Tv) \leq kd(fx_{2n+1}, gv) = kd(fx_{2n+1}, z) \to 0.$$

Thus $Tx_n \to Tv = z = gv$. Similarly, we also can show that $Tu = z = fu$. (i) is proved.

Finally we prove (ii). As (T, f) and (T, g) are weakly compatible and $gv = Tv = z = Tu = fu$, then

$$gz = gTv = Tgv = Tz = Tfu = fTu = fz.$$

We claim that z is a common fixed point of T, f, g. Since

$$d(z, Tz) = d(Tu, Tz) \leq kd(fu, gz) = kd(z, Tz),$$

then $z = Tz$, i.e., $z \in F(T) \bigcap F(f) \bigcap F(g)$. If there exists another point $v \in K$ such that $v = Tv = gv = fv$, then

$$d(z, v) = d(Tz, Tv) \leq kd(fz, gv) = kd(z, v).$$

Hence $z = v$. The proof is complete. □

Theorem 2.1 contains the Banach Contraction Principle as a special case ($f = g = I$). It generalizes Al-Thagafi and Shahzad [1, Theorem 2.1]. It also extends Shahzad [8, Lemma 2.1] and Pant [6, Theorem 1].

Theorem 2.2. *Let K be a nonempty q-starshaped subset of a normed space E, and $T, f, g : K \to K$ three mappings. Assumed that T is a (f, g)-nonexpansive mapping, and f and g are q-affine and $\overline{T(K)} \subset f(K) \cap g(K)$. Suppose that either $\overline{T(K)}$ or $f(K)$ or $g(K)$ is compact, then*

(i) *there exist $z, u, v \in K$ such that $fu = Tu = z = Tv = gv$;*

If, in addition, (T, f) and (T, g) are weakly compatible and $ffx = fx$ for all $x \in C(T, f)$, then

(ii) $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Choose a sequence $\{k_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} k_n = 1$. For each n, define T_n by

$$T_n x = (1 - k_n)q + k_n T x, \ \forall x \in K.$$

Then, for each n, $\overline{T_n(K)} \subset f(K) \cap g(K)$ by q-starshapedness of K and q-affiness of f and g. And also for all $x, y \in K$,

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\| \leq k_n \|fx - gy\|,$$

so T_n is (f, g)-contractive mapping with contractive coefficient $k_n \in (0, 1)$. As either $\overline{T(K)}$ or $f(K)$ or $g(K)$ is compact, then $\overline{T(K)}$ or $f(K)$ or $g(K)$ is complete [11]. It follows from Theorem 2.1(i) that for each n, there exist $x_{m(n)}, x_{t(n)} \in K$ such that

$$fx_{m(n)} = k_n T x_{m(n)} + (1 - k_n)q = y_n = gx_{t(n)} = k_n T x_{t(n)} + (1 - k_n)q. \tag{2.1}$$

It follows from the compactness of either $\overline{T(K)}$ or $f(K)$ or $g(K)$ that there exist $\{y_{n_i}\} \subset \{y_n\}$ and $z \in K$ such that

$$y_{n_i} = fx_{m(n_i)} = gx_{t(n_i)} \to z\,(i \to \infty),$$

$$Tx_{m(n_i)} = Tx_{t(n_i)} = \frac{y_{n_i} - (1 - k_{n_i})q}{k_{n_i}} \to z \in \overline{T(K)}.$$

And also $z \in f(K) \cap g(K)$ by $\overline{T(K)} \subset f(K) \cap g(K)$. Hence there exist $u, v \in K$ such that $z = fu = gv$. As $i \to \infty$,

$$\|Tu - Tx_{t(n_i)}\| \leq \|fu - gx_{t(n_i)}\| = \|z - gx_{t(n_i)}\| \to 0,$$

therefore, $Tx_{t(n_i)} \to Tu = z$, i.e., $z = Tu = fu$. Similarly, we also can show that $Tv = z = gv$. (i) is proved.

Subsequently, we show (ii). It follows from (i) that there exist $z, u, v \in K$ such that $fu = Tu = z = gv = Tv$. Since (T, f) and (T, g) are weakly compatible and $ffx = fx$ for all $x \in C(T, f)$, then

$$fz = fTu = Tfu = Tz = Tgv = gTv = gz$$

and

$$fz = ffu = fu = z.$$

Thus $z = fz = gz = Tz$, which proves (ii). □

Theorem 2.3. *Let K be a nonempty q-starshaped subset of a Banach space E, and $T, f, g : K \to K$ three mappings. Assumed that T is a (f, g)-nonexpansive mapping, and f and g are q-affine and $\overline{T(K)} \subset f(K) \cap g(K)$. Suppose that one of the following conditions is satisfied:*

(a) *T is strongly continuous and K is weakly compact;*

(b) *f or g is strongly continuous and K is weakly compact;*

(c) *$\overline{T(K)}$ is weakly compact and E satisfies Opial's condition.*

Then (i) $C(T, f, g) \neq \emptyset$;

If, in addition, (T, f) and (T, g) are weakly compatible and $ffx = fx$ for all $x \in C(T, f)$, then

(ii) $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Let $\{k_n\}$ and $\{T_n\}$ be defined as in Theorem 2.2. Then a similar argument shows that there exist $x_{m(n)}, x_{t(n)} \in K$ such that

$$fx_{m(n)} = k_n Tx_{m(n)} + (1 - k_n)q = y_n = gx_{t(n)} = k_n Tx_{t(n)} + (1 - k_n)q.$$

Suppose the condition (a) holds. Since $\{x_{m(n)}\} \subset K$ together with the weak compactness of K, then there exists $y \in K$ and $\{x_{m(n_i)}\} \subset \{x_{m(n)}\}$ such that $x_{m(n_i)} \rightharpoonup y$ $(i \to \infty)$. It follows from the strong continuity of T that

$$Tx_{m(n_i)} \to Ty \in \overline{T(K)} \subset f(K) \cap g(K).$$

Thus there exist $u, v \in K$ such that $Ty = fu = gv$, and noticing $k_n \to 1$,

$$fx_{m(n_i)} = gx_{t(n_i)} = y_{n_i} = k_{n_i} Tx_{m(n_i)} + (1 - k_{n_i})q \to Ty.$$

We claim that $Tu = Ty = fu$. Indeed, since as $i \to \infty$

$$\|Tu - Tx_{t(n_i)}\| \leq \|fu - gx_{t(n_i)}\| = \|Ty - gx_{t(n_i)}\| \to 0,$$

then $Tx_{t(n_i)} \to Tu = Ty$. Similarly, we also can show that $Tv = Ty = gv$. (i) is proved.

Suppose the condition (b) holds. Assumed that f is strongly continuous, then $gx_{t(n_i)} = fx_{m(n_i)} \to fy$. Since as $i \to \infty$

$$\|Ty - Tx_{t(n_i)}\| \leq \|fy - gx_{t(n_i)}\| = \|fy - fx_{m(n_i)}\| \to 0,$$

then $Tx_{t(n_i)} \to Ty$, that is, T is strongly continuous at y. It follows from (a) that we also reach our objective.

Suppose the condition (c) holds. By the weak compactness of $\overline{T(K)}$, there exists $y \in K$ and $\{Tx_{m(n_i)}\} \subset \{Tx_m(n)\}$ such that $Tx_{m(n_i)} \rightharpoonup z$ $(i \to \infty)$. Therefore by $k_n \to 1$, we have

$$fx_{m(n_i)} = gx_{t(n_i)} = k_{n_i} Tx_{m(n_i)} + (1 - k_{n_i})q \rightharpoonup z.$$

Since weak closedness of subset of E implies closedness in Banach space E [11, 5], then $z \in \overline{T(K)} \subset f(K) \cap g(K)$. Thus $\exists u, v \in K$ such that $z = fu = gv$. As $\{Tx_n\}$ is bounded by the weak compactness of $\overline{T(K)}$, then

$$\|fx_m(n) - Tx_m(n)\| = (1 - k_n)\|Tx_m(n) - q\| \to 0 \ (n \to \infty).$$

We claim that $Tv = z$. Suppose not, By E satisfying Opial's condition, we get

$$\begin{aligned}\limsup_{i\to\infty}\|Tx_{m(n_i)} - z\| &< \limsup_{i\to\infty}\|Tx_{m(n_i)} - Tv\| \\ &\le \limsup_{i\to\infty}\|fx_{m(n_i)} - gv\| = \limsup_{i\to\infty}\|fx_{m(n_i)} - z\| \\ &\le \limsup_{i\to\infty}\|fx_{m(n_i)} - Tx_{m(n_i)}\| + \limsup_{i\to\infty}\|Tx_{m(n_i)} - z\| \\ &= \limsup_{i\to\infty}\|Tx_{m(n_i)} - z\|.\end{aligned}$$

Which is a contradiction. Hence, $z = Tv = gv$. Similarly, we also can show that $z = Tu = fu$. (i) is proved.

It follows from the similar argumentation of Theorem 2.3(ii) that $Tu \in F(T) \cap F(f) \cap F(g)$. Which finishes the proof. □

Theorem 2.4. *Let K be a nonempty q-starshaped subset of a normed space E, and $T, f, g : K \to K$ three mappings which satisfy that T is a (f, g)-nonexpansive mapping and $\overline{T(K)} \subset f(K) \cap g(K)$. Suppose that (T, f) and (T, g) are C_q-commuting, and f and g are q-affine, and one of the three mappings T, f, g is continuous. If either $\overline{T(K)}$ or $f(K)$ or $g(K)$ is compact, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. Let $\{k_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} k_n = 1$. For each n, define T_n by

$$T_n x = (1 - k_n)q + k_n Tx, \ \forall x \in K.$$

It follows from the similar argumentation of Theorem 2.2 that $\overline{T_n(K)} \subset f(K) \cap g(K)$ for each n and T_n is (f, g)-contractive mapping with contractive coefficient $k_n \in (0, 1)$. Since (T, f) and (T, g) are C_q-commuting, and f and g are q-affine, then $q \in F(f) \cap F(g)$ [1], and furthermore, for all $T_n x = fx = gx$, we have

$$T_n fx = (1-k_n)q+k_n Tfx = (1-k_n)fq+k_n fTx = f((1-k_n)q+k_n Tx) = fT_n x.$$

Namely, (T_n, f) is weakly compatible. Similarly, (T_n, g) is weakly compatible also. It follows from Theorem 2.1(ii) that for each n, there exists unique $x_n \in K$ such that

$$x_n = fx_n = gx_n = k_n Tx_n + (1 - k_n)q. \tag{2.2}$$

The alike argument as Theorem 2.2(i) implies that there exist $z, u, v \in K$ and $\{x_{n_i}\} \subset \{x_n\}$ such that $Tu = fu = z = Tv = gv$, and $x_{n_i} = fx_{n_i} = gx_{n_i} \to z$ and $Tx_{n_i} \to z$ as $i \to \infty$. As C_q-commuting of (T, f) and (T, g) implies weakly compatible [1], then

$$fz = fTu = Tfu = Tz = Tgv = gTv = gz.$$

It follows from continuity of either T or f or g that either $Tx_{n_i} \to Tz$ or $fx_{n_i} \to fz$ or $gx_{n_i} \to gz$. Hence

$$z = Tz = fz = gz.$$

This completes the proof. □

Corollary 2.5. *Let K be a nonempty q-starshaped subset of a normed space E, and $T : K \to K$ a nonexpansive mapping and $\overline{T(K)} \subset K$. If $\overline{T(K)}$ is a compact subset of E, then $F(T) \neq \emptyset$.*

Theorem 2.6. *Let K be a nonempty q-starshaped subset of a Banach space E, and $T, f, g : K \to K$ three mappings which satisfy that T is a (f, g)-nonexpansive mapping and $\overline{T(K)} \subset f(K) \cap g(K)$. Suppose that (T, f) and (T, g) are C_q-commuting and f and g are q-affine, and T is strongly continuous, and either K or $\overline{T(K)}$ or $f(K)$ or $g(K)$ is weakly compact. Then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. Let $\{k_n\}$ and $\{T_n\}$ be defined as in Theorem 2.4. Then a similar argument shows that for each n, there exists unique $x_n \in K$ such that

$$x_n = fx_n = gx_n = k_n T x_n + (1 - k_n)q.$$

The alike argument as Theorem 2.3(i) implies that there exist $z, u, v \in K$ and $\{x_{n_i}\} \subset \{x_n\}$ such that $Tu = fu = z = Tv = gv$, and $x_{n_i} = fx_{n_i} = gx_{n_i} \rightharpoonup z$ and $Tx_{n_i} \rightharpoonup z$ as $k \to \infty$. Since C_q-commuting of (T, f) and (T, g) implies weakly compatible [1], then

$$fz = fTu = Tfu = Tz = Tgv = gTv = gz.$$

As T is strongly continuous together with $x_{n_i} \rightharpoonup z$, then $Tx_{n_i} \to Tz$. By $Tx_{n_i} \rightharpoonup z$, we have $z = Tz = fz = gz$. The proof is completed. □

Remark. 1. Both Theorem 2.2 and Theorem 2.4 generalize Al-Thagafi and Shahzad [1, Theorem 2.2], Jungck [3, Theorem 3.1] and Shahzad [8, Lemma 2.2].

2. Both Theorem 2.3 and Theorem 2.6 extend and improve Al-Thagafi and Shahzad [1, Theorem 2.4] and Shahzad [7, Theorem 3].

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