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텐세그리티 구조물 설계를 위한 다목적 최적화 기법에 관한 연구

Multi-objective Optimization for Force Design of Tensegrity Structures

Makoto Ohsaki* Jingyao Zhang* Jae-Yeol Kim**

요 약

텐세그리티 구조물의 설계를 위한 다목적 최적화 기법이 제시되었다. 구조물의 기하가 먼저 주어지며, 설계변수는 부재력이다. 목적함수는 최대 강성매트릭스에 대한 최저 고유치와 찾고자 하는 목표값으로부터 가장 근접하게 일치하는 부재력이다. 복수의 목적함수 문제가 구속조건을 도입하여 일련의 단일 목적함수 문제로 전환되었다. 본 논문의 타당성을 알아보기 위해 텐세그리티 그리드에 대한 최적해를 구해 보았다.

Abstract

A multi-objective optimization approach is presented for force design of tensegrity structures. The geometry of the structure is given a priori. The design variables are the member forces, and the objective functions are the lowest eigenvalue of the tangent stiffness matrix that is to be maximized, and the deviation of the member forces from the target values that is to be minimized. The multi-objective programming problem is converted to a series of single-objective programming problems by using the constraint approach. A set of Pareto optimal solutions are generated for a tensegrity grid to demonstrate the validity of the proposed method.

키워드 : 텐세그리티 구조물, 부재력 설계, 다목적 최적화, 강성 최적화
Keywords : Tensegrity structure, Force design, multi-objective optimization, Stiffness optimization

1. Introduction

The design process of tensegrity structure is divided to *geometry* (*shape*) design and *force design*. In the latter process, the member forces can be effectively designed through the process of force design to increase the stability and stiffness of tensegrity structures, since the member forces can be arbitrarily determined by designers as long as the self-equilibrium equations are satisfied. The purpose of this paper is to present a multi-objective optimization approach to the force design of tensegrity structures.

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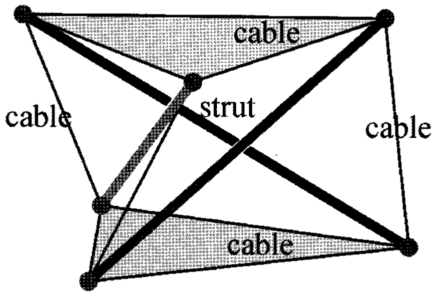
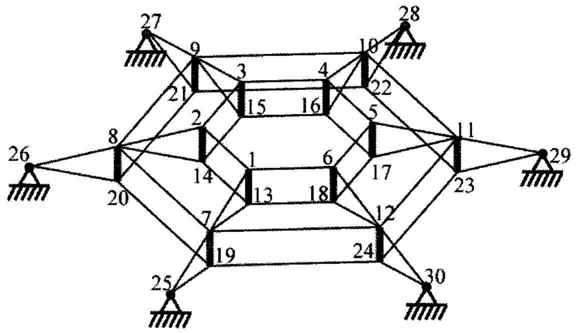
Tensegrity is a contraction of *tensional integrity*, that is named by Fuller[1]. It is classified as a prestressed pin-jointed structure according to its property in stability[2,3]. Tensegrity refers to the integrity of stable structures based on the balance between the members in compressive and tensile states. The members carrying compressive forces are called *struts*, and those carrying tensile forces are *cables*. For example, Fig. 1 shows the tensegrity structures consisting of these two types of members, where thick and thin lines denote struts and cables, respectively. Tensegrity structures can be free-standing, e.g., Fig. 1(a), or have supports, e.g., Fig. 1(b), to maintain their self-equilibrium states and stability.

Tensegrity structures are first inspired by artists, and then attracted engineers and researchers in various disciplines. Civil and architectural engineers use them



(a) a prismatic structure



(b) a cable dome model

〈Fig. 1〉 Examples of tensegrity structures consisting of struts and cables. (a) free standing structure with only one force mode; (b) structure supported at its boundary and has more than one independent force modes.

as a novel light-weight structural system to cover large space, e.g., a model of cable dome as shown in Fig. 1.(b). Aeronautical engineers use them as a new concept for deployable structure that can be easily conveyed and deployed in space. Biomedical researchers use them to predict response of cells subject to environmental changes. Mathematicians use the principle of them to solve many challenging mathematical problems, such as packing problems.

The interaction between configuration and member forces is the fundamental aspect of tensegrity structures that attracts so many attentions, but also leads to the main difficulty in the design of this kind of structures[4,5]. When geometry of the structure is determined *a priori*, there exist several independent modes of member forces that can be combined to maintain the structure in a self-equilibrium state. It is therefore possible, in the process of force design, to increase the stiffness of tensegrity structures, and to have the member forces as close as possible to the desired distribution, by optimizing the coefficients for each force mode in the linear combination of member forces. However, these objectives usually conflict with each other, and there is no single optimal solution that simultaneously optimizes them. Therefore, the force design problem turns out to be a *multi-objective programming problem.*

Multi-objective programming approach has been extensively studied in the field of structural optimization. Ohsaki et al.[6] presented a tradeoff design approach between static deformation and the deviation of the shape of an arch-type truss from the configuration desired by the designer. Ohsaki and Fujiwara[7] presented a developability condition of membrane structures, and applied it to multi-objective shape design. Ohsaki *et al.*[8] formulated a multi-objective programming problem for improving the energy dissipation performance of the frame structures.

In this paper, we formulate a multi-objective programming problem for force design of tensegrity structures, and present a method for generating a set of compromise solutions called Pareto optimal solutions. Designers can select the most preferred solution from the Pareto optimal solutions in view of the trade-off between the two objectives and other performance measures.

2. Member Forces and Stiffness

We consider the force design problem of a tensegrity structure under following assumptions:

(a) The members are straight and are connected by pin-joints.
(b) Forces are to be determined for self-equilibrium state without self-weight or external load.
(c) Members are in elastic state without yielding or buckling.
(d) Geometry of the structure is given.

From (a) and (b), the members transmit only axial forces.

Consider a tensegrity structure consisting of n free nodes and m members in d-dimensional space. Let $s \in \Re^m$ denote the vector of member forces. The self-equilibrium equation of the structure can be written as[4)]

$$Ds=0 \tag{1}$$

where the trivial vector 0 in the right-hand side of (1) indicates that there is no external load applied to the structure, and the equilibrium matrix $D \in \Re^{dn \times m}$ is determined by the geometry of the structure, i.e., connectivity of members and nodes, and the nodal coordinates. Hence, D is a constant matrix from assumption (d).

Let R denote the rank of D. If $R < m$, then there exist m-R independent modes f_i of member forces satisfying the self-equilibrium equation as

$$Df_i = 0, \ (i=1,...,m-R) \tag{2}$$

Then the member forces of the structure can be written as a linear combination of these modes through the coefficients α_i as

$$s = \sum_{i=1}^{m-R} \alpha_i f_i = Fa \tag{3}$$

The tangent stiffness matrix K is written as the sum of the linear stiffness matrix K^E and the geometrical stiffness matrix K^G as[9)]

$$K = K^E + K^G \tag{4}$$

where K^E is always positive semi-definite for structures made of conventional materials that have positive axial stiffness; and the positive definiteness of K^G depends on the distribution of member forces.

A non-trivial displacement is called *mechanism* if it does not change the member lengths. Let M denote the mechanism matrix for which the ith column corresponds to the ith independent mechanism. The quadratic form Q of K with respect to M turns out to be that of K^G as follows, because $M^TK^EM = O$ holds[5,10)]:

$$Q = M^TKM = M^TK^GM \tag{5}$$

Q is positive definite if the structure is stable when the terms higher than the second-order of the expanded form of the total potential energy are not considered. Therefore it is sufficient to investigate the positive definiteness of Q in the stability analysis of tensegrity structures, if the forces are small enough compared to the member stiffness[2)]. Since M is constant when the geometry of the structure is determined, stability and stiffness of the structure is directly related to the distribution of member forces, which determines the geometrical stiffness.

3. Multi-objective Optimization Problem

We formulate an optimization problem with two objective functions: maximization of the stiffness, and minimization of the differences of the member forces from the target values. The signs of the member forces; i.e., tension for cables and compression for struts, are also incorporated as constraints.

3.1 Overview of Multi-objective Programming Approach

In the process of structural design, it is natural to consider multiple objective functions to be minimized or maximized. The optimization problem with multiple objective functions is called *multi-objective programming problem*, where generally there exists no solution that is optimal for all objectives. In this case, we can select the most preferable solution from the set of compromise solutions called Pareto optimal solutions.

The process of selecting the most preferred solution is categorized to *approach with a priori information* and *approach with posteriori information.* In the former approach, the Pareto optimal solutions are searched interactively, or determined by transforming the multi-objective problem to a single-objective problem, based on the *a priori* information of preference. In this paper, we use the latter approach, and the most preferred solution will be selected by the designer after generating the set of Pareto optimal solutions by the method called *constraint approach.*

3.2 Maximum Stiffness

From the stability criterion based on (5), the stability and stiffness of tensegrity structures can be defined by the lowest eigenvalue λ of the matrix Q, when the member stiffness is large enough compared to the level of member forces. This is because λ also turns out to be the lowest eigenvalue of the tangent stiffness matrix K in this case. To simplify the problem, we assume that the stiffness of all members is infinite so that stability of the structure can be verified by the sign of λ: when λ is positive, the structure is stable; when it is negative, then the structure is unstable. Note that stability of the structure needs further investigation based on the higher-order terms of the total potential energy if $\lambda=0$. Furthermore, the stiffness of the structure against external loads can be evaluated by the magnitude of λ; i.e., lager λ corresponds to stronger structure. Hence, to increase the stiffness, a distribution of member forces resulting in a larger value of λ is to be found.

Therefore, for the given design conditions, such as material properties and loading conditions, we maximize the lowest eigenvalue λ of the quadratic form Q as one of the objective functions of the multi-objective optimization problem.

3.3 Uniform Member Forces

Uniform distribution of member forces can have many advantages in design, construction and even in maintenance of tensegrity structures. For example, fabrication costs and complexity of construction process can be significantly reduced, if the member cross-sectional areas are same for the same type of members; moreover, we will have the same safety factor against yielding or buckling of members.

The vector of target member forces is denoted by $\bar{s}$, where the target values may be same for each type of members, or can be specified arbitrary by the designers. The difference $\| s-\bar{s} \|$ between s and $\bar{s}$ is to be minimized as the other objective function. The least square method can simply give the optimal solution for this problem as

$$s = FF^{-}\bar{s} \tag{6}$$

where F^{-} denotes the Moore-Penrose generalized inverse of F.

Note that both of the two objectives mentioned above are described in terms of member forces. However, they do not have global optimal solutions at the same time. Therefore, a trade-off between the two objective functions is generally required in the force design of tensegrity structures.

3.4 Constraints

Suppose that the member force vector s is scaled to ks ($k>0$). The equilibrium state of the structure is retained after scaling, and the stability does not depend on the value of k when the member stiffness is assumed to be infinite. However, λ is also modified to $k\lambda$. Therefore, the problem of maximizing the stiffness becomes that of searching the maximum scalar for the member forces if we have no further constraints on the member forces. Hence, the value of strain energy provided by the member forces to pretension the structure is specified as a normalization condition for the member forces.

The strain energy Π of the structure can be written as

$$\Pi = \sum_{i=1}^{m} \frac{1}{2} s_i^2 l_i A_i E_i \tag{7}$$

where l_i, E_i and A_i are the length, Young modulus and the cross-sectional area of member i, respectively. For simplicity, they are assumed to be same for all members, and therefore, the strain energy of the structure is rewritten as

$$\Pi = a\sum_{i=1}^{m} s_i^2 = a s^T s \tag{8}$$

where a is a constant denoting $l_i A_i E_i / 2$ that has the same value for all members. The specified value of strain energy $\overline{\Pi}$ can be further simplified by ignoring the constant a as

$$s^T s = \overline{\Pi} \tag{9}$$

Moreover, it is also expected that the member forces conform to the types of the members; i.e., tension for cables and compression for struts, because cables do not have compressive stiffness. Let s^c and s^s denote the member force vectors of cables and struts, respectively. Then we have $s^c \geq 0$ and $s^s \leq 0$ for the constraints on the signs of the member forces.

3.5 Formulation of Multi-objective Optimization Problem

The multiobjective optimization problem is formulated as

$$\text{Minimize} \quad -\lambda \ \text{ and } \ \| s - \bar{s} \| \tag{10}$$

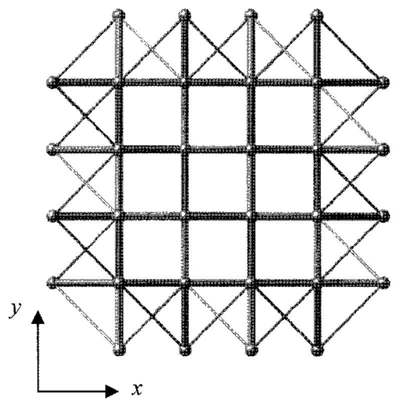
$$\text{subject to} \quad s^c \geq 0, \quad s^s \leq 0, \quad s^T s = \overline{\Pi}$$

Many methods have been developed to solve the multiobjective programming problem, among which we adopt the *constraint approach* in the next section to generate the set of Pareto optimal solutions as candidates for the decision making based on *posteriori* information of preference.

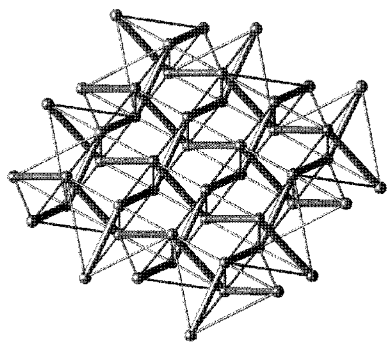
4. Examples

In this section, we consider the force design of a special tensegrity structure, called tensegrity grid, proposed by Motro[11] as shown in Fig. 2. It is constructed by assembling the unit cells as shown in Fig. 3 in x- and y-directions.

Let r and c denote the numbers of rows and columns of the struts, respectively. Hence, there are $r+1$ struts in each column and $c+1$ struts in each row. Thus, the structure has $2rc+r+c$ struts and $n = 2(rc+r+c)$ nodes. The total number of members is $m = 7rc + 5r + 5c - 4$. The structure has only one infinitesimal mechanism irrespective of r and c. Thus, the quadratic form Q turns out to be a scalar,

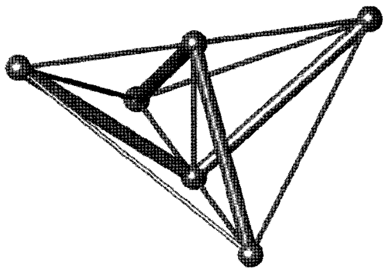


(a) Top view



(b) Bird-eye view

⟨Fig. 2⟩ An example of tensegrity grid with four rows (r=4) and four columns (c=4) of struts, which are connected to each other at their ends in each row and column.



〈Fig. 3〉 Unit cell of tensegrity grid, which is to be connected consecutively in *x*- and *y*-directions to assemble a tensegrity grid as shown in Fig. 2.

which is equal to λ. Moreover, there are at most $rc-r-c+3$ independent modes of member forces. Note that the members, which are parallel to the xy-plane and connected to the boundary nodes, are bars carrying no force. The bars have both compressive and tensile stiffness so as to maintain stability of the structure.

The structure as shown in Fig. 2, which consists of 48 nodes and 148 members ($r=4$ and $c=4$) is used for demonstrating the process of multiobjective programming approach for the force design. The height of the structure is 4.0, and the projection of each strut on xy-plane has length of 5.0 as well. The number of struts is 40, and statical indeterminacy is 11. Hence, the member forces are a linear combination of 11 independent modes as formulated in (2), where the coefficients are to be determined by solving the optimization problem (10). Note again that the structure has only one mechanism.

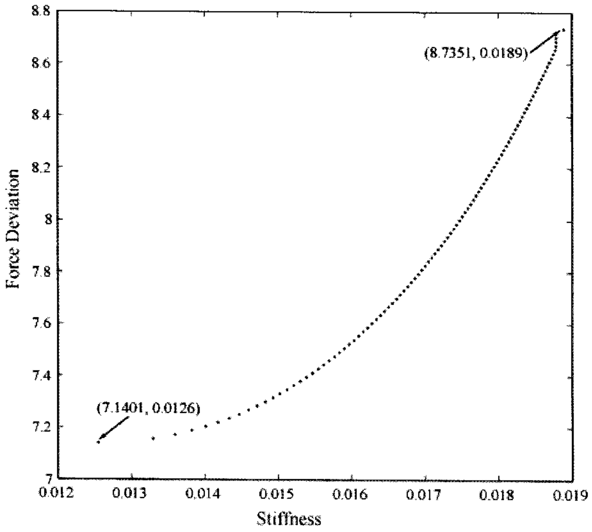
In order to find the set of Pareto optimal solutions, we adopt the constraint approach where the second objective function of the optimization problem (10) is incorporated in the constraints. Hence, problem (10) is transformed to a single-objective optimization problem as

$$\begin{aligned} \text{Minimize} \quad & -\lambda && (11) \\ \text{subject to} \quad & \| s-\bar{s} \| \le \epsilon \\ & s^c \ge 0 \\ & s^s \le 0 \\ & s^T s = \overline{\Pi} \end{aligned}$$

where ε is the upper bound for the norm of difference between the member forces from their target values. The set of Pareto optimal solutions for the original bi-objective optimization problem (10) can be derived by solving the revised single-objective optimization problem (11), where the upper bound ε for $\| s-\bar{s} \|$ is varied gradually. The smallest value of ε can be determined by solving problem (10) ignoring the objective function $-\lambda$, or it can also be easily found as the least square solution in (6). The largest value of ε can be derived by solving problem (10) to minimize $-\lambda$ only. We use the function *fmincon* ( ) in the Optimization Toolbox in MATLAB[12] for the single-objective problem (11). *fmincon*( ) is a nonlinear programming routine, which attempts to find a constrained minimum of a scalar function of multiple variables starting from an initial estimate.

The target member forces of struts and cables are set to -1 and 1, respectively. If the member forces exactly agree with the target values, the revised strain energy introduced to the structure is $s^T s = 132$ because there are 132 struts and cables in addition to 16 bars carrying no force on the boundary. Hence, we set $\overline{\Pi} = 132$ for this problem. Note that these values are purely numerical without explicit physical meaning, and are used to demonstrate the validity of the proposed method for presenting the set of Pareto optimal solutions. The coefficients α_i of the force modes f_i are the variables in the optimization problem. The initial solution to start the fmincon( ) is determined by the least square method given in (6).

It has been found by solving the single-objective optimization problem for minimizing each objective function that the difference between the member forces and the target values is distributed in the range [7.1401, 8.7351], and the range of the eigenvalue is [0.0126,



〈Fig. 4〉 Pareto optimal solutions of problem (10) for maximizing stiffness and minimizing the deviation of member forces from uniform distribution.

0.0189]. The upper bound ε is varied in this range to find λ by solving the problem (11). The generated Pareto optimal solutions are plotted in Fig. 4.

A trade-off relation between the two objective functions is clearly observed in Fig. 4, which conforms to the definition of the Pareto optimal solutions. Basically, larger difference between the member forces and the target forces leads to higher stiffness, but they do not have linear relation. In the force design process, a compromise between the two objectives should be made. The curve of the Pareto optimal solutions can provide direct information to assist designers in the further understanding of the structure and decision making in the force design process.

5. Conclusions and Discussions

In the structural design, it is always desirable that the structure has moderately large stiffness against possible external loads under the given design conditions. Moreover, uniform distribution of member forces has many advantages, such as reduction of fabrication costs and complexity of construction process, and the same safety factor for the failure of members.

Since tensegrity structures usually have multiple independent force modes, we have the freedom to choose the member forces to control the mechanical properties of the structures. A multiobjective optimization problem is presented here to maximize the stiffness and to minimize the norm of difference between the member forces and their target values, subject to the constraints of given strain energy and types of members.

It has been clearly observed from the numerical example that the distribution of member forces has significant influence on the stiffness of the structure. Presentation of the curve of the Pareto optimal solutions enables designers to select a solution from the candidate solutions according to their preferences, although it is not possible to have maximum stiffness and uniform distribution of member forces at the same time.

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