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A Cognitive Structure Theory and its Positive Researches in Mathematics Learning

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The concept field is defined as the schema of all equivalent definitions of a mathematics concept. Concept system is defined as the schema of a group concept network where there are abstract mathematics relations. Proposition field is defined as the schema of all equivalent proposition sets. Proposition system is defined as a schema of proposition sets where one mathematics proposition at least is "derived" from the other proposition. CPFS structure that consists of concept field, concept system proposition field, proposition system describes more precisely mathematics cognitive structure, and reveals the unique psychological phenomena and laws in mathematics learning.

Keywords: mathematics learning, cognitive structure, concept field, concept system, proposition field, proposition system, the Growth Teaching Strategy (GTS)
ZDM Classification: C30, C70, D50
MSC2000 Classification: 97C50

1. INTRODUCTION

There is a phenomenon in students' mathematics learning: though some students have learnt a mathematics concept. They always make various mistakes when they apply the mathematics concept. They cannot understand the connotation of the mathematics concept and cannot identify a counter-example of the mathematics concept, and cannot understand various forms of the same concept. Similarly, though some students have learnt a mathematics proposition, especially a group of propositions, they cannot neatly apply the propositions to mathematics problem solving. Obviously, there are many reasons for the phenomena. In Mathematics Education, scholars try to explain these phenomena through analyzing the way of mathematics knowledge construction.

Combined mathematics and individual learning psychology, we put forward the theory of CPFS structure.

2. NOTIONS OF CPFS THEORY

2.1. Concept field

Generally, the mathematics concepts are described as the characteristics of abstraction, formalization, logicalization from a static perspective and simplification.

However, if we analyze a concept or conceptual system from a dynamic perspective, the characteristics of mathematics concepts can be described more accurately.

First, we analyze the following two examples.

Example 1 (About the definition of arithmetic progression).

The sequence $\{a_n\}$ is an arithmetic progression if and only if

$$a_{n+1} - a_n = d,$$

where d is a constant and $n \in N,\ n \geq 1$.

$\Leftrightarrow$ The sequence $\{a_n\}$ is an arithmetic progression if and only if

$$a_{n+1} - a_n = a_n - a_{n-1},\quad n \in N,\ n \geq 2.$$

$\Leftrightarrow$ The sequence $\{a_n\}$ is an arithmetic progression if and only if

$$a_n = a_1 + (n-1)d,$$

where d is a constant and $n \in N,\ n \geq 2$.

$\Leftrightarrow$ The sequence $\{a_n\}$ is an arithmetic progression if and only if

$$a_n = a_m + (n-m)d,$$

where d is a constant and $n,\ m \in N$.

$\Leftrightarrow$ ...

Example 2 (About the definition of distance). A, B indicate respectively the points; l, m indicate respectively the straight lines; N, P indicate respectively the planes; $\rho(A, N)$ indicates the distance from point A to plane N; $\rho(A,\ l)$ indicates the distance from point A to line l; inf indicates the infimum; and S indicates point sets in the space. Then there is the concept expansion chain as follows:

$$\rho(A,\ l) = \inf\{\rho(A,\ B) \mid B \in l\}$$
$$\rightarrow \rho(A,\ N) = \inf\{\rho(A,\ B) \mid B \in N\}$$
$$\rightarrow \rho(A,\ m) = \inf\{\rho(A,\ B) \mid B \in m,\ m \subset S\}$$
$$\rightarrow \rho(l,\ m) = \inf\{\rho(A,\ B) \mid A \in l,\ B \in m\}$$

$\rightarrow\ \rho(l,\ N) = \inf\ \{\rho(A,\ B)\,|\,A \in l,\ B \in N\}$
$\rightarrow\ \rho(N,\ P) = \inf\ \{\rho(A,\ B)\,|\,A \in N,\ B \in P\}$
$\rightarrow\ \rho(N_1,\ M_2) = \inf\{\rho(A,\ B)\,|\,A \in M_1,\ B \in M_2\}$

where M_1, M_2 indicates respectively point set.

Through analyzing the examples above, we can see three features of mathematics concepts:

(1) The same concept can be described with different forms or different perspectives. Namely, a mathematics concept can be described with a group of mutually equivalent definitions.
(2) Mathematics concept is developing and it can be made new sense under different contexts.
(3) Mathematics concept is not isolated. A new concept is often defined with many old concepts. There are some mathematical abstract relations between the new concept and old concepts. All these concepts and the relations compose a concept system where concepts are called nodes and the relations are called links.

The schema is defined as the stored way of knowledge according to its properties in modern cognitive psychology. The schema is a common coding way of the same kind proposition or general characteristic of perception.

Let's apply schema to define the concept field.[1] The schema of all equivalent definition of a mathematical concept C is defined as the *concept field* of concept C (Yu & Ma, 2002). Specifically, it means as follows:

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[1] From: http://rama.poly.edu/~marc/field.htm
The Field Concept: In natural field phenomena, one firstly searches for sources that excite the field and then seeks for observables (properties) that describe the field. A quantitative description of the variation of these field observables in space and time is usually phrased in mathematical terms. Such efforts have been remarkably successful in describing many natural phenomena. Although human phenomena do not fit readily into such a field mold, there appears to be sufficient analogy to explore a field approach in seeking a physical description of thought and memory processes.
An example of a field with some similarity to the mind field may clarify the field concept. For the case of an electromagnetic field, a radio station containing a local electrical network plus an auditory microphone, visual camera, and an antenna constitutes a source that radiates electromagnetic waves into space *where* they can be measured by a receiving antenna. The distinction between the radio station and the radiated waves is analogous to that between computer hardware and software. The radiated electromagnetic waves, which changing content mirrors the input from the source, are representative of the mind whereas the electrical networks etc. represent the physical brain structure. The mind field view suggests that cortical locations within the brain play the role of transmitting and receiving antennas and thereby suggest the need for developing a physical means to measure wave type excitations within the brain.

(1) The concept field is an important kind of mental representation that individual represents mathematical concept. If a group of equivalent definition of concept represents an external form, the concept field is an internal form of this group of equivalent definition of concept. We say that a learner has formed a concept field of some concept. In other words, the equivalent definition of concept has been internalized by the learner, and has been stored in his/her long-term memory.

(2) The concept field is individual's proposition network and mental image of equivalent definitions (knowledge) of some mathematical concept. For example, the concept of the "odd function" can be defined as: " $y = f(x)$ in the domain of function, if for each x, $f(-x) = -f(x)$, This function is called an odd function. We also can define it as: "If the function image is symmetrical about the origin. This function is called an odd function." The first definition is represented by proposition; the second definition is represented by concept image. The two definitions are equivalent. Therefore, to form the concept field of the "odd function," the learner must establish two kinds of representation in his brain. In fact, there is a corresponding relation between point set in the plane and the ordered real number pair. Therefore, there is also a corresponding relation between the analytical expression and the image of the function. So, function question which is usually defined by the algebra way can be defined by geometry method. On the other hand, function question which is usually studied by the algebra way can be solved by the geometry method. A concept network is formed through this corresponding relation between "number" and "shape," then the concept field is internalized.

(3) Relations between nodes in proposition network are equivalent. "Equivalence" means equivalent in logical sense, that is to say, two propositions have the same true value, or one concept can be derived from another concept. Thus, we may reveal the unique psychological process in mathematics learning through more precise description about relations between nodes.

(4) There exists a typical proposition in the proposition network of the concept field. In general, there is one basic definition in equivalent definitions. This basic definition is often selected as a definition of the concept in the textbook. The typical definition of concept C is called the basic definition.

For example, "the parallelogram" has some equivalent definitions as follows:

① The quadrangle with its opposite sides parallel is called a parallelogram.

② The quadrangle with a pair of opposite sides parallel and equal is called a parallelogram.

③ The quadrangle with two pairs of opposite side equal, respectively, is called a parallelogram.

④ The quadrangle with two pairs of opposite angle equal, respectively, is called a parallelogram.

....

Among equivalent definitions, definition ① is the typical definition of the parallelogram concept.

A typical definition is similar to "prototype theory" of concepts (Medin, 1989). It is much easier to learn for students; and it is not lack of rigorous mathematical definition. That is to say, the schema of the typical definition is one example of the concept field.

Example 3. The concept field of "Isosceles triangle" is the schema of some equivalent definitions as follows:

A triangle that has two equal sides is called an isosceles triangle.
A triangle that has two equal angles is called an isosceles triangle.
A triangle in which the bisectors of an angle equally divide opposite side is called an isosceles triangle.
A triangle that has two equal altitudes on two sides is called an isosceles triangle.
A triangle that has two equal mid-lines is called an isosceles triangle. ...

Example 4. The concept field of "limit" is the schema of equivalent definitions as follows:

Regarding the infinite series $\{a_n\}$, if any given a positive number $\varepsilon > 0$, there exits a number a, such that $|a_n - a| < \varepsilon$ is always satisfied when there is an $N > 0$ and $n > N$, a is called the limit of $\{a_n\}$, recording it as:

$$\lim_{n\to\infty} a_n = a.$$

Regarding the infinite series $\{a_n\}$, if there exits a number a and an infinitesimal series $\{\alpha_n\}$, which satisfy $a_n = a + \alpha_n$, the limit of $\{a_n\}$ is called a, recording it as:

$$\lim_{n\to\infty} a_n = a.$$

We can also understand more extensively the concept field in two mathematical structures. If there exits an isomorphism relation between two structures, there exits a corresponding equivalent form between two structures, the schema of the concept field with the isomorphism relation is called the *generalized concept field.*

Example 5. The generalized concept field of "straight line" is the schema of the equivalent definitions in the different structures as follows:

The general equation in a plane rectangular coordinate system:

$$Ax + By + C = 0,$$

where not all of A and B is 0. The parametric equation in a plane rectangular coordinate system:

$$x = x_1 + \rho \cos\alpha, \quad y = y_1 + \rho \sin\alpha,$$

in a polar coordinate system: $\rho \cos(\theta - \omega) = p$.

Example 6. The generalized concept field of a "plural number" is the schema of the equivalent definitions in the different structures as follows:

The algebraic expression of a plural number: $Z = a + bi$, where $a, b \in R$.

The trigonometric expression of a plural number: $Z = r\,(\cos\alpha + i\sin\alpha)$.

The vector expression of a plural number: $\overrightarrow{OZ}$.

2.2. Concept System

Xu & Zhang (1986) proposed the mathematical abstract concept and the abstract analytical method; they thought that mathematical objects can be depicted with three kinds of abstract relations as follows:

(1) **Weak abstraction**. If structure B is abstracted from some characteristic (side face) of primary structure A, structure B is broader than structure A, so the primary structure A become an exceptional case of the structure B. The abstraction from A to B weak abstraction is called weak abstraction, or there is a weak abstraction relation between A and B. For example,

Euclidean space → inner product space → metric space → topological space,

which forms a weak abstraction chain.

(2) **Strong abstraction**. If some new characteristics are introduced into the primary structure A and a new concept or theory B is obtained, the abstraction from A to B strong abstraction is called strong abstraction, or there is a strong abstraction relation between A and B. For example,

Function → continuous function → differentiable function → analytic function,

which forms a strong abstraction chain.

(3) **Generalized abstraction**. If concept A is used when we define concept B, or proposition A is used when we prove proposition B, B is called the *generalized abstraction* from A, namely, B is more abstract than A.

Obviously, three kinds of abstraction relations as mentioned above reflect unique relations between mathematical objects. Based on these three kinds of relation, we can define the concept system as follows:

In a group of concept $C_1, C_2, \Lambda\ , C_n$, if there exists a sequence of relations:

$$C_1\ R_1\ C_2\ R_2\ \Lambda\ R_{n-1}\ C_n \qquad (*)$$

where R_i $(i = 1, 2, \Lambda\ , n-1)$ represents one of the three mathematics relations as mentioned above: strong abstraction, weak abstraction and generalized abstraction, (*) is called a *concept chain*. Recording it as:

$$\lambda = \{C_1, C_2, \Lambda\ , C_n\}.$$

If the intersection of two concept chains is not empty, the two concept chains intersect. If one of m concept chains intersects another of $m-1$ concept chains, the schema of the concept network consisting of n chains is called concept system.

In short, concept system is the concept network where exits some specific mathematics relations and it is internalized by individual. The concept field is the sub-schema of concept system. The concept system can be generalized to isomorphism structure, and then the generalized concept system is formed.

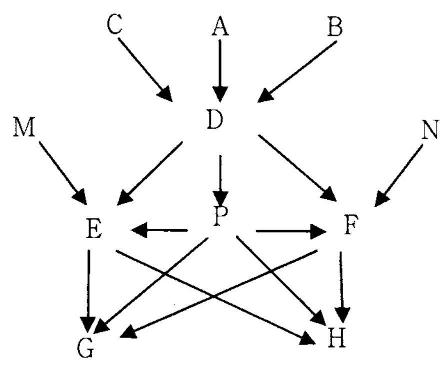


Figure 1. The concept network of "sequence"

Example 7. The concept system of "sequence" is the schema of concept network as follows:

Assume A: a number; B: a function; C: ordinal relations; D: a sequence; E: a arithmetic sequence; F: a geometric sequence; G: a general term formula; H: a formula of the sum of from 1 to n, M: the difference of the two numbers; N: a quotient of the two numbers; and P: a sequence $\{a_n\}$ satisfy

$$a_{n+1} = Aa_n + B,$$

where A, B are constants, $n \in N$, $a_1 = a$. These concepts form the concept network of Figure 1.

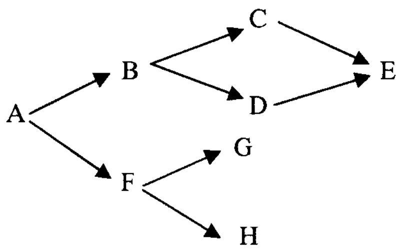


Figure 2. The concept network of "quadratic polynomial"

Example 8. The concept system of "quadratic polynomial" is the schema of concept network as follows:

Assume A: a quadrangle; B: a parallelogram; C: a rectangle; D a rhombus; E: a square; F: a trapezoid; G: an isosceles trapezoid; H: a right trapezoid. These concepts form the concept network of Figure 2.

3. PROPOSITION FIELD AND PROPOSITION SYSTEM

3.1. Proposition Field

The theory of proposition field is a natural extension of the concept field. If proposition A establishes, if and only if proposition B establishes, A is equivalent to B, recording it as A $\Leftrightarrow$ B. Sets of all equivalent proposition of proposition A is defined as equivalent proposition class of proposition A, written as {A}, and A is called the typical proposition of {A}. The equivalent propositions which consist of {A} and the structure which is formed by the relations between these propositions are called the network of equivalent propositions. The schema of network of equivalent propositions is defined as the *proposition field* of proposition A. Specifically, it means as follows (Yu, 1999):

The proposition field is

(1) Proposition field that is individual's proposition network is a component of cognitive structure of mathematics.
(2) All propositions in proposition network are equivalent in logic.
(3) Proposition field is a way how proposition network stores in the individual's brain, thus it is relevant to organization form of proposition network.

(4) The typical proposition in proposition field often composes a core of proposition field. The typical proposition is easier to extract when the individual applies propositions.

We also can understand more extensively the concept field in two mathematical structures. If there exits an isomorphism relation between two structures, there exits a corresponding equivalent form between two structures, the schema of proposition network with isomorphism relation is called the generalized proposition field.

Example 9. The proposition field "the sum of two non-negative real numbers equals to 0 if and only if these two real numbers themselves is 0" is the schema of some equivalent proposition as follows:

$$\text{As } a, b \in R, a \geq 0, b \geq 0, \text{ then}$$
$$a^2 + b^2 = 0 \Leftrightarrow a = b = 0,$$
$$\sqrt{a} + \sqrt{b} = 0 \Leftrightarrow a = b = 0,$$
$$|a| + |b| = 0 \Leftrightarrow a = b = 0,$$
$$a^n + b^n = 0\,(n \in N) \Leftrightarrow a = b = 0,$$
$$\cdots$$

Example 10. The proposition field "sufficient and necessary condition for parallel lines" is the schema of some equivalent propositions as follows:

Corresponding angles are equal $\Leftrightarrow$ Two lines are parallel;
Alternate angles are equal $\Leftrightarrow$ Two lines are parallel;
Anterior angles on the same side of the transversal are supplementary
$\Leftrightarrow$ Two lines are parallel.

The propositions as mentioned above and the proposition relevant to in a plane rectangular coordinate system compose the generalized proposition field "sufficient and necessary condition for parallel lines."

Assume two lines

$$l_1 : A_1x + B_1y + C_1 = 0, \quad l_2 : A_2x + B_2y + C_2 = 0,$$

then

$$l_1 \,/\!/\, l_2 \Leftrightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}.$$

Example 11. The generalized proposition field "circle equation in which center of circle is the origin in a plane rectangular coordinate system" is the schema of some equivalent propositions as follows:

In the rectangular coordinate
- Circle' parametric equation $x^2 + y^2 = r^2$
- Circle's ordinary equation: $\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$

In the polar coordinate: $x = \rho\cos\theta,\ y = \rho\sin\theta$

3.2. Proposition System

If a group of propositions $A_1, A_2, \Lambda\ , A_n$ exists a producing relation (generalized abstraction):

$$A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_n,$$

it is called a *proposition chain*, written as

$$\lambda = \{A_1, A_2, \Lambda\ , A_n\}.$$

If the intersection of two proposition chains is not empty, two proposition chains intersect.

If one of m proposition chains intersects another of $m-1$ proposition chains, the system composed by these m pieces of the chain is called network of the half equivalent proposition. The schema of network of the half equivalence proposition is called a *proposition system.*

In short, concept system is the concept network where exits some specific mathematics relations and it is internalized by individual. The concept field is the sub-schema of concept system form which the generalized concept system with isomorphism structure also can be generalized.

Obviously, proposition system is the extension of proposition field. A proposition field often acts as a sub-schema of proposition system which the generalized proposition system with isomorphism structure also can be generalized. To some propositions, there is not any direct relation between them, but there is certain similar or latent relation. The schema of proposition network with some latent relations is called latent proposition system. For instance, there are latent relations between propositions relevant to inequality and equality, and there are latent relations between propositions in two-dimensional space and in three-dimensional space, too.

Example 12. A group of weak abstraction propositions about "the mean value inequality" are as follows, the schema composes a proposition system:

$$\sqrt[n]{a_1 a_2 a_3 \Lambda\ a_n} \le \frac{1}{n}(a_1 + a_2 + \Lambda\ \Lambda\ + a_n)$$

$$\sqrt[3]{a_1 a_2 a_3} \le \frac{1}{3}(a_1 + a_2 + a_3), \quad \sqrt{a_1 a_2} \le \frac{1}{2}(a_1 + a_2),$$

where $a_i > 0$, $i = 1, 2, 3, \Lambda, n$.

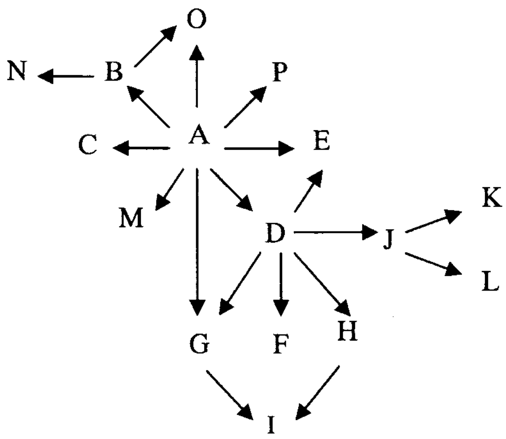


Figure 3. Proposition Network of Propositions about Line Segment

Example 13. A group of propositions are as follows, which composes a proposition network:

A: Determination theorems about congruent triangles (SSS, SAS, ASA, AAS, HL);

B: If a triangle has two equal angles, then opposite sides of the two angles are always equal;

C: The distances from a point of perpendicular bisector of the line segment to two endpoints of the line segment are equal;

D: The opposite sides of a parallelogram are equal;

E: The diagonals of a parallelogram are bisected mutually;

F: The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

G: The diagonals of a rectangle are equal;

H: Four sides of a lozenge are all equal;

I: Four sides of a square are all equal;

J: If transversals segment coming from one straight line cut by parallel lines is equal, then the transversal segments coming from other straight lines cut by parallel lines is also equal;

K: The straight line passing through the midpoint of one side and parallel to the other side must bisect the third side of a triangle.

L: The straight line passing through the midpoint of one waist and parallel to the base must bisect the other waist of a trapezoid;

M: A trapezoid with two equal base angles is an isosceles trapezoid;

N: A triangle with three equal angles is an equilateral triangle;

O: A diameter that is perpendicular to a chord bisects it; and

P: Two tangent lines drawn from a point outside a circle are equal.

These concepts form the concept network of Figure 3.

The proposition network is relevant to propositions that prove two lines are equal in the plane. If learners can internalize it and form a proposition system, they can activate knowledge relevant to equal line segment in the long-term memory rapidly, and flexibility extract information to apply the propositions to solve them when they can solve and prove some problems relevant to equal line segment.

4. CPFS STRUCTURE

CPFS structure consists of the concept field, the concept system, the proposition field and the proposition system (Yu, 2003, 2004a, 2004b; Bao, 2006). Specifically, it means as follows:

(1) Mathematics knowledge network consists of mathematics knowledge internalized by the individual, where exits some specific mathematics relations (strong abstraction, weak abstraction or generalized abstraction relations) between all sorts of mathematics knowledge (concept, proposition).

(2) Just because there is certain abstraction relation between mathematical knowledge, and these abstraction relations themselves implies some mathematical thinking method, namely "the links sets" between mathematical knowledge implies a "method system."

When students study mathematics concept, if they can not deeply understand the concept from multi-perspectives or multi-backgrounds, they can not form a concept system. They would not know what once confronted with mathematics concept interpreted by an alternative concept. Similarly, if students had not form a perfect proposition field and proposition system, they would not have enough propositions and would not apply different propositions timely to solve problems in an appropriate and effective way. In fact, some propositions are equivalent theoretically, but actually they play the role in supporting problem solving widely different.

5. POSITIVE RESEARCHES ABOUT CPFS STRUCTURE

Based on the theory frame of CPFS structure, we have been carrying on the following positive research of CPFS structure:

(1) Assessment of CPFS structure;
(2) The existence of CPFS structure;
(3) The relations between the CPFS structure and mathematics learning;
(4) The teaching strategy to improve students' CPFS structure.

5.1. Assessment of CPFS structure

The use of concept maps as an indication of the connectedness of knowledge is largely based on the work of Novak & Gowin (1985) in the area of science education. As graphical representations linking related concepts to form chains of relationships, concept maps have been "developed specifically to tap into a learner's cognitive structure and to externalize, for both the learner and the teacher to see, what the learner already knows" (Novak & Musonda, 1991). In a concept map, related concepts are represented as nodes, and the specific relationship between two concepts is indicated by linking words that are written along the line connecting the nodes. Scholars have proposed various scoring schemes for assessing concept maps (Goldsmith & Johnson, 1992; Goldsmith, Johnson & Acton, 1991; Ruiz-Primo & Schavelson, 1996).

The use of concept maps to assess the organization of mathematical knowledge (Hasemann & Mansfield, 1995) is closely tied to a model developed by Hiebert & Carpenter (1992) for analyzing the issues of learning and teaching mathematics with understanding.

Based on scoring schemes for assessing concept maps, we suppose there are some methods to assess CPFS structure as follows.

(1) Goal recalling. Give a goal; let students try to recall the knowledge or the method which is relevant to the goal.
(2) Nodes linking. Give a group of concepts or propositions, let students produce the link between nodes (concepts or propositions) according to abstraction relation between concepts or deducing relation between propositions.
(3) Identification and inference. Give conditions of a problem, let students infer possible conclusion (conclusion is not only one).
(4) Equivalent inference. Give a concept or a proposition; let students write as many propositions equivalent to the concept or the proposition as possible by themselves.

(5) Proposition adapting. Give a concept or a proposition; let students adapt it to a problem to solve with it.

5.2. Partial researches and results

Based on positive researches of CPFS structure, we have carried out some studies as follows:

5.2.1. Study on the relations between individual CPFS structure and mathematical problem representation

In this study, we tried to answer if individual' CPFS structure effected mathematical problem representation (*cf.* Yu, 2003).

There were two parts in these test materials. The first part of test materials included two mathematics questions:

Question 1. Prove: If the sum of the reciprocal of three real numbers is equal to the reciprocal of the sum of these real numbers, there must be two mutually reversal numbers in these three real numbers.

Question 2. For what values of m = ?, the equation $x^2 - 2mx + m + 1 = 0$ has two roots, where one root is bigger than 5, the other root is smaller than 5.

The solution of Question 1 and Question 2 were relied on appropriate problem representation. That is to say, if students use appropriate problem representation, they can simplify the solution process and reduce the questions' difficulty. 15 minutes was established through pretest.

The second part of test materials also included two mathematics questions. The aim of these questions was to examine if subjects had integrated CPFS structure. For example, Question 1 included eight questions, such as

① If $(a-b)(b-c)(c-a)=0$, what conclusions can you get?

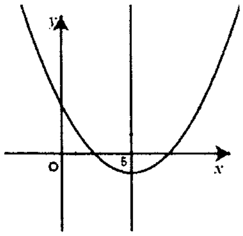
② If at least one of a, b, c equals 5, how do you express this sentences?

And so on.

The questions ① and ② correspond to Question 1 in the first part of test materials. Question 2 was an open question:

What conclusions can you get from Graph 1?

Subjects were asked to give conclusions as many as possible, and it corresponded to Question 2 in the first part of test materials.



Graph 1

We selected 151 students in high school to take this test. The test process was as follows: Subjects first answered questions in the first part of test materials in 15 minutes, and then testers took back papers and asked subjects to answer questions in the second part of test materials in 15 minutes. The results showed that there existed an obvious correlation between individual's CPFS structure and mathematical problem representation.

Furthermore, we respectively divided 151 students into three groups (high scores group, middle scores group and low scores group) according to their scores in the two test materials. Through T-test, the results showed that integrated CPFS helped students reach correct problem representation and problem solving. If individual could represent problem correctly, they would have integrated CPFS structure.

The results showed: There exist an obvious correlation between the individual's CPFS structure and mathematical problem representation. Integrated CPFS structure help to reach the correct problem representation and problem solving. If individual could represent problem correctly, them would have excellent CPFS structure.

5.2.2. Effect of individual CPFS structure to transfer on mathematical problem solving

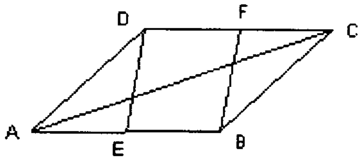
In this study, we tried to answer if individual' CPFS structure effected transfer on mathematical problem solving (*cf.* Yu, 2004b).

The test process was as follows: First, we carried out pretest, namely, asked 316 students to answer 3 questions.

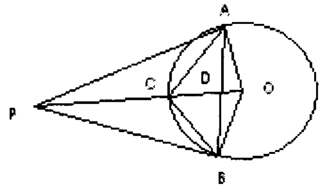
Question 1 (Goal recalling). If you wanted to prove two lines equaled, what theorems or methods would you use? Please give methods as more as possible. This question should be finished in 10 minutes.

Question 2 (Identification and inference). What conclusions could you get from following graph and some conditions?

① (Graph 2) ABCD is a parallelogram, E is a midpoint of line AB, F is a midpoint of line CD.



Graph 2



Graph 3

② (Graph 3) PA, PB is respectively the tangent line of circle O, and A, B is respectively the tangent point. C is the intersection of line OP and circle O, D is the intersection of line OP and chord AB. Connect AC and BC.

This question should be finished in 10 minutes.

Question 3 (Proposition application). Please make mathematics questions that fit following conditions as many as possible.

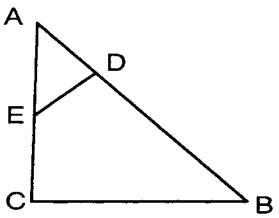
① The conclusion is to prove two lines equal and the solving process must be involved in properties of an isosceles triangle.
② The conclusion is to prove two lines equal and the solving process must be involved in following theorem:

Theorem A. The mid-line of oblique line in a right triangle is a half of the oblique line.

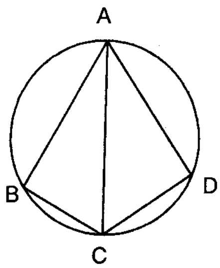
Theorem B. A tangent line of a circle is perpendicular to the radius which the tangent point cross.

In the process, if students could not organize corresponding mathematics questions, he/she could ask testers to give him/her prearranged questions (The questions were organized before the test.) This question should be finished in 20 minutes.

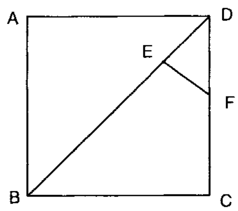
Second, we divided subjects into three groups according to their scores in the first test, where high scores group is called group A, low scores group is called group B.



Graph 4



Graph 5



Graph 6

Third, we trained group B with two questions:

Question 1 (Graph 4). $\triangle ABC$ is an isosceles right triangle, $\angle C = 90°$. Get point D on line AB so that BD = BC, then draw DE from point D so that it is perpendicular to AB. Prove: AD = EC.

Question 2 (Graph 5). AC is a diameter of a circle O, AB = AD. Prove: BC = CD.

Students supposed to think about the two questions by themselves at first, and then the teacher guided them to prove the two questions with various methods. However, the teacher did not give the relations between the two questions.

The next day, group A and group B were asked to solve the following question:

Question 3 (Graph 6). There was a point E in the diagonal line of square ABCD, BE = BC, drew EF from point E so that it was perpendicular to BD. Prove: ED = FC (Please give conclusions as more as possible.)

There were inherent relations between Question 3 and forenamed trained Questions 1 and 2. Question 3 can be solved with the method of forenamed trained Question 1 or Question 2. But the insight about discerning correlated relations was needed.

The results showed that there were significant differences between group A and group B. And even though group B was trained by some questions, their scores were lower than the scores of subjects in group A. That is to say, there was a close correlation between individual's CPFS structure and far transfers. Integrated CPFS structure helped individual to bring about far transfer.

5.2.3. Influences of Self-controlling Ability and CPFS Structure on the Mathematical Achievement

In this study, we tried to answer if individual CPFS structure had influences on mathematical achievement (*cf.* Yu, 2004a)

On the one hand, we absorbed the ideas of concept map and organized two research materials to test if subjects had integrated CPFS structure, where one was used by students in middle school ant the other was used by students in high school.

For example, there were some questions in test materials for students in middle school:

Question 1. What does $|x-5| \leq 2$ mean? Please express it with language and with graph.

The aim of Question 1 was to see if subjects had the concept field of "absolute value." If they could not recognize "absolute value" from "number" and "shape," they had not integrated the concept field of “absolute value.”

Question 2. We have the proposition “If the sum of two nonnegative real numbers equals 0, the two real number must be 0.”

① Use mathematics language to express the proposition.
② Please give its equivalent propositions as more as possible, such as “$a = 0$, $b = 0$ if a, b is respectively a real number, and $a^2 + b^2 = 0$.”

The aim of Question 2 was to see if subjects had correlative proposition field.

Question 3. What theorems did you usually use when you wanted to prove one lines was equal to the other line?

The aim of Question 3 was to see if subjects had the proposition system of “one lines is equal to the other line.”

On the other hand, we selected students’ (90 students in middle school and 53 students in high school) scores of two quiz and their scores of final examination in 2000–2001 school term. Furthermore, we computed students average score as his/her mathematics achievement.

The result showed:

(1) CPFS structure had an important effect on the mathematical achievement.
(2) Comparing high-achievement students with low-achievement students, there was distinct difference between CPFS structures.

5.2.4. Study on the relations between individual’ CPFS structure and their flexibility and profoundness of mathematics thinking

In this study, we tried to answer if individual’s CPFS structure affected the flexibility and profoundness of individual’s mathematics thinking (*cf.* Qin & Yu, 2005).

The test material consisted of two parts: The aim of Part 1 was to test subjects’ CPFS structure. There were 4 questions in the Part 1. For examples,

Question 2.

① Suppose A: point slope form of line; B: two-point form of line; C: gradient intercept form of line; D: intercept of line; E: general form of line. Please connected two points using directed line segment (for example, draw $A \to B$) if the point slope form of linear equation could transform into the two-point form of linear equation, and draw $A \leftrightarrow B$ if the two-point form of linear equation could also be transform into the point slope form of linear equation.
② How to judge the dubiety between a line and a circle? Please give methods as

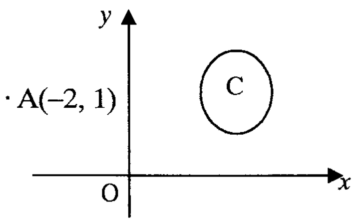
many as possible.

Question 3. What theorems did you usually use to judge one line is perpendicular to other line? Please give methods as many as possible.

The aim of Part 2 was to test subjects' flexibility and profoundness of mathematics thinking. There were 9 questions in Part 2. For example,

Question 4. Supposed point A (2, –3), B (–3, –2), what is the scope of gradient k if point P(1, 1) is on the line l and line AB intersected line l ?

Question 5 (Graph 7). Supposed circle C: $(x-3)^2+(y-2)^2=1$ and point A (–2, 1).



Graph 7

What was the maximum and the minimum of $x^2 + y^2$?

What was the maximum and the minimum of $\frac{|x+y-1|}{\sqrt{2}}$?

Found point P on the x axis so that the distance of $|PA|+|PC|$ was the minimum.

Found point Q on the y axis so that the distance between point A and point C was the maximum.

Question 8. Given line $l: y = k(x-2)+4$ and curve

$$C: y = 1+\sqrt{4-x^2},$$

please discuss their dubiety.

The results showed

(1) There were remarkable correlations between individual's CPFS structure and their flexibility and profoundness of mathematics thinking.

(2) High CPFS structure (HC) students had better flexibility and profoundness of mathematics thinking than low CPFS structure (LC).

(3) Those students with better flexibility and profoundness of mathematics thinking

had more integrated CPFS structure than other students.

5.2.5. Study on the relation between individual's CPFS structure and problem inquiry ability

In this study, we tried to answer (*cf.* Yu & Li, 2006):

(1) Did individual's CPFS structure affect individual's problem inquiry ability?
(2) Did individual's CPFS structure affect the properties of posing questions?
(3) Had individual's CPFS structure influences on questions which had different degree of difficulty?

Thus, we carried out two researches.

Research 1, Part 1

We selected "function" as test materials, and organized two test parts:

Part 1 includes 12 questions dealing with function's defining field, function's value field, the relations between a function and its graph, an odd function, an even function, an increasing function, a decreasing function, function's periodicity, the change of the graph of function, and so on. The aim of Part 1 was to test if students had integrated CPFS structure of function. For example, Question 3 is as follows:

Question 3.

① Which knowledge will you use when you discuss properties of the quadratic function $y = ax^2 + bx + c$? Please write them.

② What conclusions (include equality relations and inequality relations) about coefficient a, b, c can you get if the graph of $y = ax^2 + bx + c$ is given, and $x = 1/3$ is its symmetric axis? Please write them as many as possible.

Research 1, Part 2

At the same time, the aim of Part 2 was to test students' ability of posing question. The Part included 8 questions dealing with function, which consisted of 4 facets: problem posing by generalization and induction, problem posing by analogy reasoning, problem posing by intuition reasoning, problem posing by logic reasoning. Meanwhile, there were not only familiar questions but also unfamiliar question in Part 2.

For example, Question 1 observes following function expression:

Question 1.

(1) $f(x) = x^3 = \frac{1}{2}[x^3 + (-x^3)] + \frac{1}{2}[x^3 - (-x^3)] = P(x) + Q(x)$;

(2) $f(x) = x^2 + x = \frac{1}{2}[(x^2 + x) + [(-x)^2 - x]] + \frac{1}{2}[(x^2 + x) - [(-x)^2 - x]]$

$= p(x) + Q(x);$

(3) $f(x) = a^2 = \frac{1}{2}(a^x + a^{-x}) + \frac{1}{2}(a^x + a^{-x}) = P(x) + Q(x)$;

(4) $f(x) = \sin x = \frac{1}{2}[\sin x + \sin(-x)] + \frac{1}{2}[\sin x + \sin(-x)] = P(x) + Q(x)$.

It could be seen that each of these functions were respectively divided into the sum of two functions $P(x)$, $Q(x)$, please answered following question:

① Is P(x), Q(x) respectively an odd function or an even function?

② Can you discover any regularity? Can you prove this regularity?

The aim of this question was to test if students could pose some questions by generalization and induction.

Question 2. We all know the sum of two even numbers is even number, the sum of two odd numbers is even number, the sum of an odd number and an even number is odd number. Now, change "number" into "function." What conclusions could you get? Please prove your guess.

The aim of this question was to test if students could pose some questions by analogy reasoning.

The test process was as follows: First, we asked 95 students in high school to answer questions in Part 1 in 50 minutes. The next day, we asked them to answer questions in Part 2 in 90 minutes. At the same time, the test design is 2×2 design, where factor A is CPFS structure and factor B is the degree that subjects were familiar with the part. So, there were two levels in factor A (high CPFS structure (HC) and low CPFS structure (LC)) and two levels in factor B (questions that subjects were familiar with and questions that subjects were unfamiliar with).

The results showed

(1) There were remarkable correlations between individual's CPFS structure and problem posing. Particularly, the relations between individual's CPFS structure and problem posing by intuition reasoning and the relations between individual's CPFS structure and posing questions by logic reasoning were stronger than the relations between individual's CPFS structure and problem posing by generalization and induction, problem posing by analogy reasoning.

(2) The problem inquiry ability differentiated between HC and LC students on problem inquiry performance, compared with LC group, HC students had more problem inquiry ability. Particularly, individual's CPFS structure were more influential to problem solvers putting forward problem by their insight.

(3) The problem whether was familiar to subjects or the problems whether was

difficult to subjects would affect problem inquiry.

Research 2, Part 1

We gave LC some hints, and saw if they could attain the same level as HC. Thus, we would know the function that individual's CPFS structure effected problem posing from a deep-level perspective.

Part 1 was the same as Part 1 in Research 1, which was aimed to test individual's CPFS structure.

Research 2, Part 2

Part 2 was the same as Part 2 in Research 1, which aim was to test subjects' ability of problems posing, but we added some hints in Part 2 in Research 1 and used these hints as external control factor about subjects posing problems.

The test design was as follows: First, we asked 116 students to answer questions in Part 1. Then, we in divided them into three groups according to their scores: high CPFS structure (HC), middle CPFS structure (MC) and low CPFS structure (LC). Two days later, we asked HC students to answer questions in Part 2 in Research 1 and we asked LC students to answer questions in Part 2 in Research 2 (where questions were the same as questions in Part 2 in Research 1, but were added some hints.) After these tests, we interviewed with subjects.

The results showed that the achievement were remarkable difference between HC group and LC group when problems were easier or less difficult. However, when problems were more difficult to problem solver, the achievement was not difference between HC group and LC group, HC group were more difficult to problem solver, and the achievement was not difference between HC and LC group.

5.2.6. Study on the Teaching Strategy of Improving Individual CPFS Structure

Our correlative researches have indicated that individual's CPFS structure is a knowledge base of solving mathematics problems. Integrated CPFS structure is an essential condition for perfecting cognitive structure (*cf.* Bao & Yu, 2006). Therefore, one of our teaching tasks is to improve individual's CPFS structure.

Through one year experiment, we raised *Growth Teaching Strategy* (GTS) to improve students' CPFS structure. GTS forms a whole teaching process during which learners, teaching materials, teaching methods and teaching aids were reasonably organized together based on the internal growing law of mathematics knowledge, the cognitive features of learners and the growing law of mathematics knowledge in the learners' minds. It consists of 4 kinds of strategies.

① The growth strategy. The growth strategy includes the growth of nodes and the growth of structure. The growth of nodes means that the teaching about single mathematics knowledge should combine students' cognitive development feature and the function of mathematics knowledge itself so that students obtain new knowledge step by step. The growth of structure means that the teaching about knowledge structure should start from typical concept or proposition and help students construct concept field, concept system, proposition field, proposition system so that students' CPFS structure could be improved based on former CPFS structure.

② The variation strategy. The variation strategy includes the concept variation strategy and the process variation strategy. The concept variation strategy means that the teaching should give prominence to the essential nature of concept through vary the nonessential nature of concept and clear the extension of concept through vary the form of concept. The process variation strategy means that the teaching should construct appropriate scaffold so that students have rational relations between former knowledge and current knowledge.

③ The reflective strategy. The reflective strategy means that teachers should lead students to reflect mathematics learning content and learning process so that the relations between nodes in CPFS structure could be smooth.

④ The structure strategy. The structure strategy means that teachers should pay attention to whole teaching and lead students to construct mathematics knowledge so that CPFS structure could be formed.

In the teaching practice, these strategies were interdependent, and there were not changeless sequence. Furthermore, it is to say that these four strategies need not be appeared altogether in a lesson. There were difference emphases in difference lessons. For example, the growth strategy and the variation strategy were usually used in lessons that new concept was introduced into, and the reflective strategy and the structure strategy were usually used in review lessons.

We selected students in two classes to take the experiment. Students in the two classes had similar CPFS structure and mathematics achievement through pretest. GTS was used in one class, and traditional teaching was used in the other class. After one year experiment, we tested students in the two classes again. The test materials were as follows:

Question 1. Please explain and associate "function" and correlative mathematics knowledge, give conclusions as many as possible.

Question 2. What is the value field of function

$$y = 2 + 4\cos x - 4\sin^2 x \, (\pi/2 \le x \le \pi).$$

① Please answered following questions according to your solving process: What was your basic method for solving problems when you saw the problem at first? Was there a graph of "function" in your brain when you solved problems? That is to say, did you use the image of grape for solving problems? What roles did you think the grape plays in solving problems?

② Can you construct any problems if there is not given any conditions and only given the expression of function $y = 2 + 4\cos x - 4\sin^2 x$. Please give conclusions as more as possible and proved them.

③ What problems did you think were similar to our questions? Could you use the same methods to solve them? Please write them.

6. CONCLUSION

The CPFS structure, which is a special cognitive structure only existing in mathematics study, is composed of Concept Field, Concept System, Proposition Field and Proposition System. Based on the internal development of mathematics knowledge, cognitive features of learners and the growth of mathematics knowledge in the learners' minds, the Growth Teaching Strategy (GTS) describes the teaching process during which learners, teaching materials, teaching methods and teaching aids are reasonably organized together. Learners participated enthusiastically in activities such as research, exploration, invention and discovery in mathematics, which let the acquisition process of mathematics knowledge of learners become a naturally growing process.

The results showed that using GTS during the teaching was helpful for learners to improve their CPFS structure and their mathematics achievement.

There are still large numbers of positive researches about CPFS structure to do and these are our future works.

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