

Joint Decision of Optimal Procurement Policy and Optimal Order Size for a Product Recovery System

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회수제품 재가공이 이루어지는 시스템에서 최적 신제품 구매 정책과 최적 주문량에 대한 연구

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We consider a product recovery system that a single product is stocked in order to meet a demand from customers who may return products after usage. This paper addresses a problem of when to release a procurement process to replenish serviceable inventory and how many new products to procure. The structure of the optimal procurement policy is examined and numerically identified as a monotonic threshold curve. A numerical procedure is presented to jointly find the optimal procurement order size, optimal procurement policy, and optimal discounted profit. Sensitivity analysis also indicates that these optimal performance measurements have monotonic properties with respect to system parameters.

Keywords: Reverse Logistics, Remanufacturing, Product Recovery, Procurement, Inventory Management

1. Introduction

Environmental regulations and economic incentives motivate many companies to engage in product recovery activities such as remanufacturing and recycling of materials. European nations have mandated laws for product take back after the product's life ends, forcing companies to respond with creative solutions to the problem of product recovery (Mahadevan *et al.*, 2003). Inventory management with product recovery differs from traditional inventory management situations in essentially two aspects. First, product returns represent an exogenous inbound material flow causing a loss of

monotonicity of stock levels of serviceable products which serve customer demands. Second, two alternative supply sources are available for replenishing serviceable inventory. One source is to procure externally or to produce internally while the other comes from remanufacturing activity.

This paper addresses a problem of when to release a procurement process to replenish serviceable inventory and how many new products to procure. As a starting point for the analysis, we restrict our attention to the case that a single product is stocked in order to meet a demand from customers who may return products after usage. The primary goal of this paper is to examine an optimal procurement policy that maxi-

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zes the company's profit subject the system costs and, to present a numerical procedure which jointly finds the optimal procurement order size, optimal procurement policy, and optimal discounted profit.

We present the literature survey of product recovery models with procurement of new product according to the way of inventory review. Detailed literature surveys for product recovery models are found in Fleischmann *et al.* (1997), Mahadevan *et al.* (2003), and Savaskan *et al.* (2004). In product recovery models with periodic inventory review, research has been focused on deriving optimal control policies under various assumptions. Simpson (1978) considers the tradeoff between material savings due to reuse of old products versus additional inventory carrying costs and shows optimality of a three parameter policy to control replenishment order, recovery, and disposal when neither fixed costs nor lead times are considered. Inderfurth (1997) considers the effects of fixed and deterministic lead times for replenishment order and recovery without fixed costs. For identical lead times, he shows that the structure of the optimal policy is the same as Simpson (1978). Inderfurth (1997) also considers the case of recovery not allowing storage of recoverable products. For identical lead times, a two parameter order-up-to and dispose-down-to policy is shown to be optimal. Mahadevan *et al.* (2003) study the issue of when to release returned products to the remanufacturing line and how many new products to manufacture, and develop heuristic policies for approximating optimal order-up-to level. Teunter and Vlachos (2002) deal with the issue of what the cost reduction associated with having a disposal option for returned item is. Fleischmann and Kuik (2003) study a product recovery model without disposal which considers fixed cost as well as fixed lead time of replenishment order and recovery but does not distinguish between new and returned products.

A parallel stream of research has evolved for continuous inventory review models. Heyman (1977) studies disposal policies for a model where demands and returns are independent stochastic processes, remanufacturing and procurement are instantaneous, and no fixed costs of remanufacturing and procurement are taken into account. When demands and returns follow Poisson process, he shows the optimality of single parameter disposal policy and derives an explicit expression for the optimal disposal level. Muckstadt and Isaac (1981) consider a similar model with explicit

modeling of non-zero remanufacturing process. In contrast with Heyman (1977), disposal decisions are not treated, and demand and return processes are assumed to be unit quantity following a Poisson distribution. Van der Lann *et al.* (1996) present an alternative approximation for Muckstadt and Isaac (1981) and extend it with a disposal option under which the number of remanufacturable products is limited to a certain maximum level. Van der Lann *et al.* (1996) present a numerical comparison of several disposal policies and show that it is advantageous to base disposal decisions on both the inventory level of remanufacturable products and an adequately defined total inventory.

Our model contributes to the related research in the following aspect. Unlikely the periodic inventory review literature, continuous inventory review literature has been focused on optimizing system parameters given policies rather than exploited the issue of identifying the optimal replenishment policy for serviceable inventory. In this paper, we investigate the optimal policy based upon future demand and product return processes and serviceable and remanufacturable inventory processes. In particular, we address the issue of simultaneously determining the optimal replenishment policy and optimal order quantity.

This paper is organized as follows. Next section presents a formulation of the model. Then, section 3 examines the structure of the optimal procurement policy. In section 4, we provide a numerical example which graphically illustrates the optimal procurement policy. Section 5 describes a numerical procedure which jointly finds the optimal profit, optimal procurement order size, and optimal procurement policy. In section 6, we give various numerical test results about the optimal performance. Last section states the conclusion.

2. Problem Formulation

Demands for product occur randomly with rate λ_1 and are satisfied immediately from on-hand inventory of serviceable products. If they are not available, each arriving demand is lost. The sales price of each serviceable product is R . Product returns occur randomly with rate λ_2 and are accepted for remanufacturing. Remanufacturing time for transforming a returned pro-

duct into a serviceable one is a random variable with mean μ^{-1} and the unit cost of remanufacturing is c_R . It is further assumed that the unit cost of remanufacturing is the same for all remanufactured products and the quality of originally new product and remanufactured one are the same. This assumption is usually made in most of product recovery models. Even though serviceable products are recovered from returned products, the company replenishes serviceable inventory by purchasing new products of size Q . It is assumed that procurement lead time takes a randomly distributed amount of time with mean l^{-1} . Each procurement order incurs a lump-sum cost c_P . Holding costs are assessed at rate h_1 and h_2 for each product in serviceable inventory and each returned product in remanufacturable inventory, respectively.

Respectively denote $x_1(t)$ and $x_2(t)$ the serviceable inventory level and the remanufacturable inventory level at time t . We represent a state using the vector $(x_1(t), x_2(t), n)$ where n is an indicator such that $n = 1$ implies a procurement order in process while $n = 0$ means no procurement order in process. The state space is denoted by $S = Z^+ \times Z^+ \times 2$. At each epoch of a demand arrival, the company must decide whether or not it will purchase Q new products to replenish serviceable inventory. A control policy, π , specifies the action taken at any epoch of a demand arrival given the current state of the system. Denote the initial state by (x_1, x_2, n) and the interest rate by α . Then, the expected discounted profit given (x_1, x_2, n) over an infinite horizon under π can be written as

$$\begin{aligned} J^\pi(x_1, x_2, n) = & E_{(x_1, x_2, n)}^\pi \left[\int_0^\infty e^{-\alpha t} \left(- \sum_{i=1}^2 h_i x_i(t) dt \right. \right. \\ & \left. \left. + \sum_{t \in B_1^\pi(T)} R - \sum_{t \in B_2^\pi(T)} c_P - \sum_{t \in B_3^\pi(T)} c_R \right) \right] \end{aligned} \quad (1)$$

where $B_1^\pi(T)$, $B_2^\pi(T)$, and $B_3^\pi(T)$ respectively denote the set of random instances on $[0, T]$ of demand satisfaction, procurement order release, and remanufacturing completion under policy π .

Then, the goal of this paper is to find an optimal control policy π^* which maximizes the following expected discounted profit over an infinite horizon :

$$\begin{aligned} J(x_1, x_2, n) = & J^{\pi^*}(x_1, x_2, n) \\ = & \max_{\pi} J^\pi(x_1, x_2, n). \end{aligned} \quad (2)$$

Table 1. Summary of key notations

x_1	Serviceable inventory level
x_2	Remanufacturable inventory level
λ_1	Demand rate for serviceable products
μ	Remanufacturing rate
λ_2	Product return rate
l	Procurement lead time rate
h_1	Unit holding cost rate for serviceable inventory
h_2	Unit holding cost rate for remanufacturable inventory
R	Unit sales price of serviceable product
c_P	Procurement order setup cost
c_R	Unit remanufacturing cost

3. The Optimal Procurement Policy

This section is devoted to investigating the structure of an optimal procurement policy π^* . Since it is not tractable to identify the optimal policy under general probability distribution, we assume that demand and product return processes follow a Poisson process, and procurement and remanufacturing processes are exponentially distributed. This assumption allows us to formulate our model as a discrete time Markov decision problem. Even though the Markov model may be restricted for modeling the real world problem, it can provide us with insights into the effective procurement management (Carr and Duenyas(2000)).

Denote $D(x_1, x_2) = (x_1 - 1, x_2)$ if $x_1 > 0$; (x_1, x_2) otherwise, and $I(x_1, x_2) = (x_1 + 1, x_2 - 1)$ if $x_2 > 0$; (x_1, x_2) otherwise. Operator D and I respectively correspond to a demand arrival and a remanufacturing process completion. Let $\gamma = \lambda_1 + \lambda_2 + \mu + l$, that is, γ is the sum of all transition rates. Without loss of generality, assume that $\alpha + \gamma = 1$. Then, from the theory of Markov decision processes(see chapter 6 in Puterman (2005)), we know that the optimal profit function $J(x_1, x_2, n)$ in (2) satisfies the following optimality equation :

$$\begin{aligned} J(x_1, x_2, n) = & - \sum_{i=1}^2 x_i h_i + \lambda_1 R 1(x_1 > 0) \\ & + \lambda_1 [(1 - n) \max(J(D(x_1, x_2), n), \\ & J(D(x_1, x_2), 1 - n) - c_P) + n J(D(x_1, x_2), n)] \\ & + \mu [J(I(x_1, x_2), n) - c_R 1(x_2 > 0)] \\ & + \lambda_2 J(x_1, x_2 + 1, n) \\ & + l [(1 - n) J(x_1, x_2, n) + n J(x_1 + Q, x_2, 1 - n)] \end{aligned} \quad (3)$$

In (3), $\lambda_1 R1(x_1 > 0)$ means the expected revenue rate from sales when the serviceable inventory is not empty and $\mu c_R 1(x_2 > 0)$ implies that the expected remanufacturing cost rate when the remanufacturable inventory is not empty. The terms multiplied by λ_1 represent sales revenues and transitions generated with the arrival of demand, the terms multiplied by μ imply penalties and transitions associated with a remanufacturing completion, the term multiplied by λ_2 represents transitions generated with the arrival of a product return, and the terms multiplied by l imply transitions associated with the arrival of Q new products.

Define the value iteration operator T by

$$\begin{aligned}
 TJ(x_1, x_2, n) = & - \sum_{i=1}^2 x_i h_i + \lambda_1 R1(x_1 > 0) \\
 & + \lambda_1 [(1-n)\max(J(D(x_1, x_2), n), \\
 & J(D(x_1, x_2), 1-n) - c_P) + nJ(D(x_1, x_2), n)] \quad (4) \\
 & + \mu [J(I(x_1, x_2), n) - c_R 1(x_2 > 0)] \\
 & + \lambda_2 J(x_1, x_2 + 1, n) \\
 & + l [(1-n)J(x_1, x_2, n) + nJ(x_1 + Q, x_2, 1-n)]
 \end{aligned}$$

Then, the optimality equation can be rewritten as (see Equation (6.2.8) in Puterman (2005))

$$J(x_1, x_2, n) = TJ(x_1, x_2, n) \quad (5)$$

Next, consider the following value iteration (VI) algorithm (see Bertsekas (1987) for the detail) to solve for (5) :

$$J^{k+1}(x_1, x_2, n) = TJ^k(x_1, x_2, n), \quad k = 0, 1, \dots, \quad (6)$$

where $J^0(x_1, x_2, n) = 0$ for every state (x_1, x_2, n) . Here $J^k(x_1, x_2, n)$ can be viewed as the optimal profit in state (x_1, x_2, n) when the problem is terminated after k iterations. Since T is a contraction mapping, it is well known that $J^k(x_1, x_2, n)$ converges to $J(x_1, x_2, n)$ as k goes to the infinite.

To implement a marginal value analysis on the optimal profit function J , we define the following operators D_1 and D_2 on J :

$$\begin{aligned}
 D_1 J(x_1, x_2, n) &= J(x_1 + 1, x_2, n) - J(x_1, x_2, n) \\
 D_2 J(x_1, x_2, n) &= J(x_1, x_2 + 1, n) - J(x_1, x_2, n) \\
 D_3 J(x_1, x_2, 0) &= J(x_1, x_2, 1) - J(x_1, x_2, 0)
 \end{aligned}$$

$D_1 J(x_1, x_2, n)$ and $D_2 J(x_1, x_2, n)$ respectively imply the marginal profit attained when there is one more unit in serviceable inventory and one more unit in re-

manufacturable inventory. $D_3 J(x_1, x_2, 0)$ means the marginal profit expected when there is a procurement order in process.

Numerical investigation suggests that the optimal profit function J has the following set of the functional properties :

- (i) $D_1 J(x_1, x_2, 0) \geq D_1 J(x_1, x_2, 1)$
- (ii) $D_2 J(x_1, x_2, 0) \geq D_2 J(x_1, x_2, 1)$
- (iii) $D_3 J(x_1, x_2 + 1, 0) \geq D_3 J(x_1 + 1, x_2, 0)$

Property (i) ((ii)) says that the incremental profit of holding one more serviceable (remanufacturable) is larger when a procurement order is not in process than when it is. Property (iii) states that relative to a certain state, the marginal profit expected from having a procurement order in process is larger when there is one more unit of remanufacturable inventory than when there is one more unit of serviceable inventory.

From the above observations, it is conjectured that the optimal procurement policy may be established by the following monotonic threshold structure under the general probability distributions :

- (S1) *If the company purchases Q units of new product in state $(x_1, x_2 + 1, n)$, then it also does in state (x_1, x_2, n) (from observation (i)).*
- (S2) *If the company purchases Q units of new product in state $(x_1 + 1, x_2, n)$, then it also does in state (x_1, x_2, n) (from observation (ii)).*
- (S3) *If the company purchases Q units of new product in state $(x_1 + 1, x_2, n)$, then it also does in state $(x_1, x_2 + 1, n)$ (from observation (iii)).*

The following theorem states that when serviceable inventory becomes sufficiently large, then the company should not purchase new products, which is an intuitive result.

Theorem 1

$$\lim_{x_1 \rightarrow \infty} J(x_1, x_2, 0) > \lim_{x_1 \rightarrow \infty} J(x_1, x_2, 1) - c_P \quad (7)$$

Proof : See the Appendix.

The following theorem shows that the profit function under the optimal policy is asymptotically linear. These results are used in truncating the state space when the value iteration method finds the optimal policy.

Theorem 2

$$(i) \lim_{x_1 \rightarrow \infty} D_1 J(x_1, x_2, n) = \frac{-h_1}{\alpha} \tag{8}$$

$$(ii) \lim_{x_2 \rightarrow \infty} D_2 J(x_1, x_2, n) = \frac{-h_2}{\alpha} \tag{9}$$

Proof : See the Appendix.

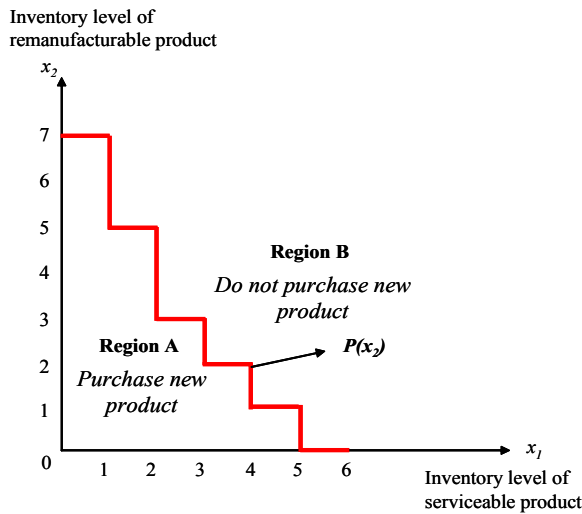


Figure 1. Graphical illustration of the optimal policy

The intuition for Theorem 2 is as follows. Suppose that there is a very large inventory of serviceable product. Having one more unit of serviceable product incurs an additional cost h_1 over the time until excessive inventory has been used. In the limit, the net present value of this additional cost becomes $-h_1 \int_0^\infty e^{-\alpha t} dt = -h_1/\alpha$. The same argument can be applied to (9).

4. A Numerical Example for an Optimal Procurement Policy

We present an example to illustrate the results obtained in the previous section. Arrivals for demand and product return are Poisson distribution with rates $\lambda_1 = 1$ and $\lambda_2 = 0.2$, respectively. Remanufacturing and procurement lead times are exponential with rate $\mu = 1$ and $l = 0.1$, respectively. Holding cost rates are $h_1 = 1$ and $h_2 = 0.2$ for each unit of serviceable and remanufacturable inventory, respectively. Sales price

of a serviceable product is $R = 100$. Procurement cost of Q new products is $c_p = 400$. Procurement order size is $Q = 15$. The unit cost of remanufacturing is $c_R = 5$. The discount factor is set to 0.99.

The optimal procurement policy in state $(x_1, x_2, 0)$ is graphically represents in <Figure 1>. It is characterized by a monotonic threshold curve, $P(x_2)$, which is decreasing in x_2 . $P(x_2)$ separates the state space $(x_1, x_2, 0)$ into two regions : 1) Purchase Q new products and 2) Do not purchase. In this example, if a demand occurs in state $(1, 3, 0)$, the company should purchase Q new products. If a demand occurs in state $(10, 0, 0)$, a procurement order should not triggered.

If the system starts within Region B , we note that the state $(x_1, x_2, 0)$ cannot move down across the boundary of $P(x_2)$. To see this, suppose that the system starts in state $(4, 3, 0)$ where there are following three possible transitions : *i*) x_2 decreases by one and x_1 increases by one corresponding to a remanufacturing completion, *ii*) x_2 increases by one corresponding to a remanufacturing completion, and *iii*) x_1 decreases by one corresponding to a demand and places a procurement order which transits the current state $(4, 3, 0)$ into $(3, 3, 1)$. All the transition makes the system not move down across the boundary of $P(x_2)$.

5. Numerical Procedure for Jointly Finding Optimal Profit, Optimal Procurement Order Size, and Optimal Procurement Policy

In the following, we present a numerical procedure which jointly finds the optimal procurement order size, Q^* and the optimal profit function, J , and the optimal procurement curve, $P(x_2)$, given Q^* . Even though the procedure relies on one dimensional search to find Q^* , our numerical investigation implemented with a variety of test examples suggests that J is a concave function of Q . <Figure 2> exhibits $J(0, 0, 0)$ as a function of Q for the example used in the previous section. It is observed that $J(0, 0, 0)$ is concave in Q and $Q^* = 20$. Based on our numerical study, it is conjectured that concavity (convexity) of the order size on the profit (cost) function, well established in the literature of inventory management without product returns, may also hold for product recovery system.

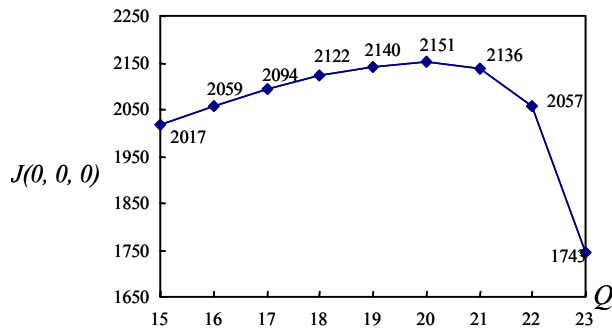


Figure 2. Optimal profit $J(0, 0, 0)$ as a function of Q

To guarantee that one dimensional search finds Q^* , we show that there exists the following upper bound on Q :

Theorem 3 : To maximize (3) which is a function of Q , the purchasing order size Q must be less than $1 + c_P \lambda / h_1$, regardless of the serviceable inventory level.

Proof : It can be shown using an argument similar to the one used in the proof of Lemma 3.1 in He *et al.* (2002).

The following procedure, combining value iteration and one dimensional search, jointly finds the optimal profit, optimal purchasing order size and optimal purchasing policy.

Optimal solution procedure :

Denote by $J_Q^k(x_1, x_2, n)$ be the optimal profit in state (x_1, x_2, n) when the problem is terminated after k iterations, given Q . Let ϵ be the termination criterion of VI.

1. Start with $Q > 0$
2. Implementation of Value Iteration subroutine
 - (a) **Initialization** : Set $k = 0$, and for each state (x_1, x_2, n) , pick the value function $J_Q^0(x_1, x_2, n) = 0$.
 - (b) **Value iteration step** : Implement a VI on the current value function estimate J^k : For the detail of implementing the value iteration algorithm, see Section 5.2 of Bertsekas(1987).

$$\begin{aligned}
 TJ_Q^k(x_1, x_2, n) = & - \sum_{i=1}^2 x_i h_i + \lambda_1 R 1(x_1 > 0) \\
 & + \lambda_1 [(1-n)\max(J_Q^k(D(x_1, x_2), n), \\
 & J_Q^k(D(x_1, x_2), 1-n) - c_P) + nJ_Q^k(D(x_1, x_2), n) \\
 & + \mu [J_Q^k(I(x_1, x_2), n) - c_R 1(x_2 > 0)] \quad (10) \\
 & + \lambda_2 J_Q^k(x_1, x_2 + 1, n) + l [(1-n)J_Q^k(x_1, x_2, n) \\
 & + nJ_Q^k(x_1 + Q, x_2, 1-n)]
 \end{aligned}$$

(c) **Termination test** : Perform the following convergence test :

$$b_k = \min_{(x_1, x_2, n) \in S} \{ TJ_Q^k(x_1, x_2, n) - J_Q^k(x_1, x_2, n) \}, \quad (11)$$

$$b_k = \max_{(x_1, x_2, n) \in S} \{ TJ_Q^k(x_1, x_2, n) - J_Q^k(x_1, x_2, n) \}. \quad (12)$$

If $(b_k - b_{k-1}) \geq \epsilon$, for every state (x_1, x_2, n) , let $J_Q^{k+1}(x_1, x_2, n) = TJ_Q^k(x_1, x_2, n)$, (13)

increase k by one, and go to *Value iteration step*. otherwise, go to *Evaluation step*.

(d) **Optimal order quantity and Optimal policy evaluation step** : Let the optimal discounted profit given Q be

$$J_Q(x_1, x_2, n) = TJ_Q^k(x_1, x_2, n).$$

Using one dimensional search, find Q^* that maximizes $J_Q(x_1, x_2, n)$ for $1 \leq Q \leq \bar{Q}$ where $\bar{Q} = 1 + c_P \lambda / h_1$. Let $P(x_2) = \max_{x_1} \{ J_Q^k(x_1, x_2, 0) : J_Q^k(x_1, x_2, 0) < J_Q^k(x_1, x_2, 1) - c_P \}$ and stop the procedure.

6. A Sensitivity Analysis

In this section, we investigate (1) what is the effects of cost parameters on the optimal profit and procurement order size? and (2) What is the effects of mean time parameters on the optimal profit and procurement order size? First, in the following theorems, we show how the optimal discounted profit changes as a function of system parameters, which is intuitive.

Theorem 4 : *The optimal discounted profit function J decreases in c_P, c_R, h_1 , and h_2 and increases in R .*

Proof : If a new system has $c'_P < c_P$, one can couple the new system to the original by applying the policy π^* that was optimal for the original system to the new one as well. Along every sample path, the cost of the new system is not increased under π^* , therefore an optimal policy for the new system will perform at least as well. The same argument is applied to the costs other than c_P .

Theorem 5 : *The increase in demand rate result in an equal or larger expected profit.*

Proof : We apply an argument similar to the one used in the proof of Theorem 2 in Carr and Duenyas (2000). Suppose that in the new system, demand rate is increased from λ to λ' , ceteris paribus. Consider a policy π' for the new system which works as follows. Whenever a demand arrives at the system, it is rejected with probability $(\lambda' - \lambda)/\lambda'$ and the sequence of procurement order is done using the optimal policy for the original system. Under π' , the new system has the same sample path as the original system and thus achieves the same expected discounted profit as the original system. Because π' may not necessary be optimal for the new system, the optimal policy in the new system will perform at least as well π' .

Next, we numerically implemented a sensitivity analysis. To this end, we have investigated the following scenarios using the standard parameter settings in <Table 2> :

- Scenario 1 : Vary procurement setup cost c_P , ceteris paribus
- Scenario 2 : Vary serviceable inventory holding cost h_1 , ceteris paribus
- Scenario 3 : Vary unit sales price R , ceteris paribus
- Scenario 4 : Vary demand rate λ_1 , ceteris paribus
- Scenario 5 : Vary return rate λ_2 , ceteris paribus
- Scenario 6 : Vary remanufacturing rate μ , ceteris paribus
- Scenario 7 : Vary lead time rate l , ceteris paribus

Besides the reference example in <Table 2>, we extensively tested more than 100 examples with varying the values of system parameters and observed that test results with those examples are very similar to the ones obtained using the reference example. For this reason, we only report the observations and insights that are derived by the reference example.

Table 2. Reference example used in the numerical test

λ_1	μ	λ_2	l	h_1	h_2	R	c_P	c_R	Discount factor
1	1	0.2	0.2	1	0.2	100	400	5	0.99

6.1 Effects of Cost Parameters on the Optimal Profit And Procurement Order Size

Based on the computational experiment, we observe the following monotonicity of the optimal procurement order size Q^* with respect to cost parameters :

- Q^* is increasing as the fixed cost of procurement, c_P , increases (see <Figure 3>).
- Q^* is decreasing as the serviceable inventory cost, h_1 , increases (see <Figure 4>).
- Q^* is decreasing as the sales price, R , increases (see <Figure 5>).

The first and second phenomenon can be explained using the reasoning of the economic order quantity (EOQ) model. In EOQ model, it can be easily seen that the optimal order size decreases in inventory holding cost and increases in order setup cost. The third observation indicates that a thorough study of the optimal selection of Q^* is needed for our model, which focuses on how best to use procurement order and remanufacturing. We believe that these monotonicity properties will be very useful in developing a heuristic formula which approximates Q^* .

We would like to note that the unit remanufacturing cost, c_R , and remanufacturable inventory holding cost, h_2 , do not affect Q^* since the remanufacturing process in our model is not a controllable variable. In contrast, if it is controlled, it is obvious that Q^* also depends on these cost parameters.

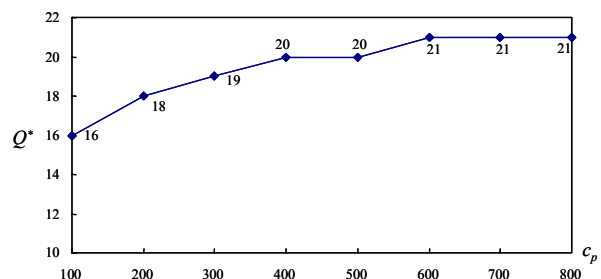


Figure 3. Optimal order size as a function of c_P

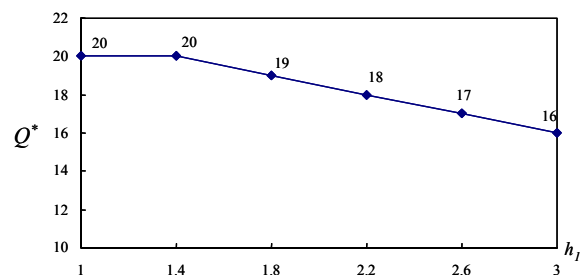


Figure 4. Optimal order size as a functional of h_1

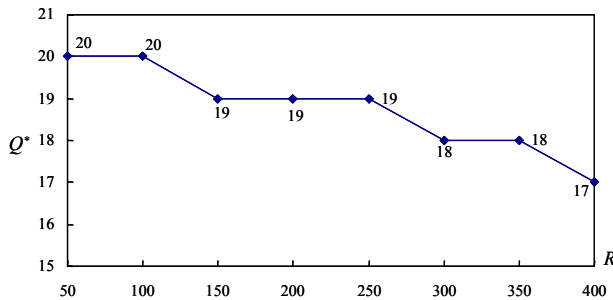


Figure 5. Optimal order size as a function of R

6.2 Effects of Mean Time Parameters on the Optimal Profit and Procurement Order Size

From the computational experiment, we also observe the following monotonic properties of Q^* and J with respect to time distribution parameters :

- Q^* is increasing as the demand rate λ_1 increases (see <Figure 6>).
- Q^* is decreasing as the return rate λ_2 increases (see <Figure 7>).
- J is concave with respect to the remanufacturing rate μ (see <Figure 8>).
- J is concave with respect to the procurement lead time rate l (see <Figure 9>).

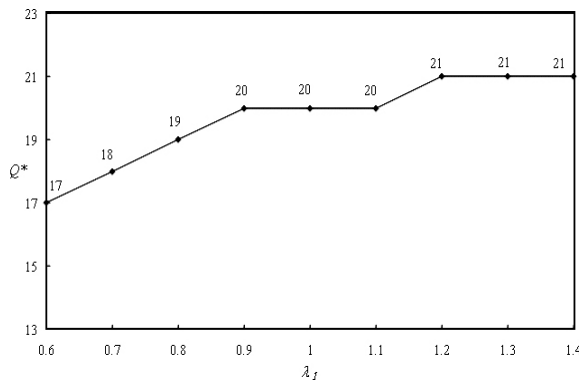


Figure 6. Optimal order size as a function of λ_1

If demand rate increases while other conditions remain the same, the chance of enhancing sales revenue also increases. Hence, the firm will increase the procurement order size to raise the inventory level of serviceable product. To explain the second observation, suppose that product return rate increases, ceteris paribus. Increased returned products results in increased inventory level of remanufacturable product which will be converted into serviceable inventory through remanufacturing. Then, it is reasonable to expect that the firm will try to balance the inventory level between

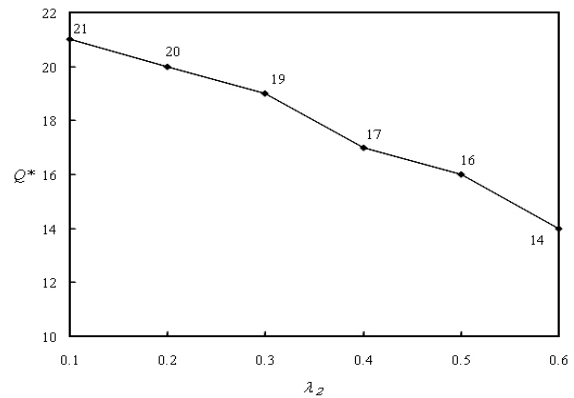


Figure 7. Optimal order size as a function of λ_2

serviceable and remanufacturable products and thus to reduce the procurement order size.

The third observation indicates that there exists an optimal remanufacturing rate that maximizes the company's profit. The intuition behind this is as follows. The increase in the remanufacturing rate will more rapidly convert remanufacturable inventory into serviceable inventory. This causes the company to pay more serviceable inventory costs than before, since other conditions remain the same. In contrast, if the remanufacturing rate becomes decreasing, the delay in the remanufacturing process results in keeping remanufacturable inventory for a longer time than before. Hence, larger remanufacturable holding costs than before will shrink the company's profit.

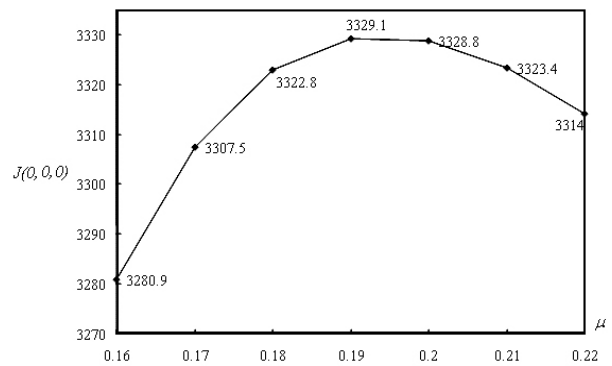


Figure 8. Optimal profit $J(0, 0, 0)$ as a functional μ

Similarly, the fourth observation indicates that there exists an optimal procurement lead time rate that maximizes the company's profit. The intuition behind this is as follows. Numerical test demonstrates that both optimal procurement policy and optimal order size are little affected by the change in the procurement lead time rate. It comes from the fact that serviceable inventory can be replenished through remanufacturing

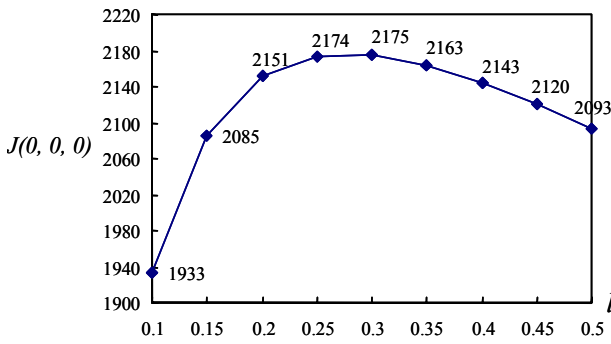


Figure 9. Optimal profit $J(0, 0, 0)$ as a function of l

process as well as procurement process. Hence, when procurement lead time is too short while other conditions remain the same, a procurement action will pile up serviceable inventory immediately, which causes the company to pay more inventory holding costs. In contrast, if procurement lead time is too long while other conditions remain the same, the possibility of stockout will be increased and the lost of sales opportunities will be also increased.

7. Conclusion

In this paper we considered a product recovery system with a continuous inventory review and studied the issue of when to release a procurement order to replenish serviceable inventory and how many new products to procure. The aim of this paper was to help us to gain insights into the nature of problems with product recovery. We examined the structure of the optimal procurement policy, which has not been treated in the literature. In particular, we examined the policy that is derived based on the inventory level of both serviceable and remanufacturable products.

By modeling our problem as the Markov decision process and using value iteration, we numerically characterized the structure of an optimal procurement policy as a monotonic threshold function. It was shown that a procurement decision is allowed only when serviceable inventory level drops down below a procurement curve. This curve is the function of the remanufacturable inventory level and is decreasing as it increases. We also implemented a sensitivity analysis and observed many meaningful monotonic properties of the optimal order size and the optimal discounted profit with respect to system parameters.

The insights obtained in this paper will be very useful for studying more realistic models with arbitrary probability distributions other than exponential ones. One of the major extensions to the current model is to include a disposal option of returned product, since it becomes a more effective strategy in handling both remanufacturable and serviceable inventories. The other direction of the future research is to control the remanufacturing process and decide when to start and when to end remanufacturing of returned products.

Appendix

Proof of Theorem 1 : From the definition of value iteration operator T ,

$$\begin{aligned}
 & \lim_{x_1 \rightarrow \infty} TJ(x_1, x_2, 0) - \lim_{x_1 \rightarrow \infty} TJ(x_1, x_2, 1) \\
 &= \lambda_1 \left[\max \left(\lim_{x_1 \rightarrow \infty} J(x_1 - 1, x_2, 0), \right. \right. \\
 & \quad \left. \left. - \lim_{x_1 \rightarrow \infty} J(x_1 - 1, x_2, 1) \right) \right] \\
 &+ \mu \left[\lim_{x_1 \rightarrow \infty} J(x_1 + 1, x_2 - 1, 0) \right. \\
 & \quad \left. - \lim_{x_1 \rightarrow \infty} J(x_1 + 1, x_2 - 1, 1) \right] \\
 &+ \lambda_2 \left[\lim_{x_1 \rightarrow \infty} J(x_1, x_2 + 1, 0) - \lim_{x_1 \rightarrow \infty} J(x_1, x_2 + 1, 1) \right] \\
 &+ l \left[\lim_{x_1 \rightarrow \infty} J(x_1, x_2, 0) - \lim_{x_1 \rightarrow \infty} J(x_1 + Q, x_2, 1) \right] \\
 &> \gamma(-c_p)(by(7)) \\
 &> -c_p \text{ (since } \alpha + \gamma = 1 \text{ by assumption and } \alpha > 0)
 \end{aligned}$$

Proof of Theorem 2 :

(i) by (7), the optimal action in state $(x_1, x_2, 0)$ when x_1 goes to the infinite is *do not purchase*. In addition, by (8), $\lim_{x_1 \rightarrow \infty} D_1 J(D(x_1, x_2), n)$, $\lim_{x_1 \rightarrow \infty} D_1 J(I(x_1, x_2), n)$, $\lim_{x_1 \rightarrow \infty} D_1 J(x_1, x_2 + 1, n)$, and $\lim_{x_1 \rightarrow \infty} D_1 J(x_1, x_2, n)$ exist and finite. Hence,

$$\begin{aligned}
 & \lim_{x_1 \rightarrow \infty} D_1 TJ(x_1, x_2, n) \\
 &= -h_1 + \lambda_1 \lim_{x_1 \rightarrow \infty} D_1 J(x_1 - 1, x_2, n) \\
 &+ \mu \lim_{x_1 \rightarrow \infty} D_1 J(x_1 + 1, x_2 - 1, n) \\
 &+ \lambda_2 \lim_{x_1 \rightarrow \infty} D_1 J(x_1, x_2 + 1, n) + l \lim_{x_1 \rightarrow \infty} D_1 J(x_1, x_2, n) \\
 &= -h_1 + \lambda_1 \frac{-h_1}{\alpha} + \mu \frac{-h_1}{\alpha} + \lambda_2 \frac{-h_1}{\alpha} + l \frac{-h_1}{\alpha} by(8)
 \end{aligned}$$

$$\begin{aligned}
&= -h_1 + \gamma \frac{-h_1}{\alpha} (\text{by } \lambda_1 + \mu + \lambda_2 + l = \gamma) \\
&= \frac{\alpha + \gamma}{\alpha} (-h_1) \\
&= \frac{-h_1}{\alpha} (\text{since } \alpha + \gamma = 1 \text{ by assumption}).
\end{aligned}$$

(ii) The arguments similar to providing (i) can be applied here and we omit the detailed proof.

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