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Cap Pricings under the Fractional Brownian Motion†

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Abstract

We present formulas for two types of cap pricing under fBm-HJM model reflecting the empirical long range dependence in the interest rate model. In particular, we propose a new approach to pricing the cap with the default risk.

Keywords: Fractional Brownian motion; HJM; Wick integral; defaultable bond; cap.

1. Introduction

This paper studies the cap pricing reflecting the long range dependence in the interest rate model. Recently, empirical researches on interest rate dynamics such as Cajueiro and Tabak (2007) imply that short rates show some long memories and non-Markovian. It is well-known that fractional Brownian motion(fBm) is a proper candidate for modelling these empirical phenomena. Since fBm, however, is not a semimartingale process, it is difficult to apply such a process to asset price modelling. Oksendal (2004), however, overcomes those difficulties by using Wick integral introduced by Bender (2003) and Duncan *et al.* (2000). We give a brief review fBm -HJM interest rate theory, and obtain two types of closed form solutions of the cap prices by using the Wick integral.

2. No Arbitrage in fBm Model

We denote by (B_t^H) the one parameter fractional Brownian motion(fBm) with Hurst parameter $H \in (0,1)$, *i.e.,* fBm is the Gaussian process $B_t^H = B_H(t,w)$, $t \in \mathrm{R}$, $w \in \Omega$, satisfying

$$B_0^H = E[B_t^H] = 0$$

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for all t and
$$E[B_s^H B_t^H] = \frac{1}{2}\left[|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right],$$
where the expectation is taken under the probability measure P and $(\Omega, \mathcal{F})$ is a measurable space.

In order to apply fBm to interest rate model, we need a new type of stochastic calculus. Note that fBm is not a semimartingale except that H = 1/2. So we cannot use the theory of stochastic calculus for semimartingale on B_t^H. In view of the theory of Wick-Ito integral studied by Oksendal (2004), this integral is defined by

$$\int_0^T \phi(t,w)dB_t^H \triangleq \int_0^T \phi(t,w)\delta B_t^H = \lim_{\triangle t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \diamond (B_{t_{k+1}} - B_{t_k}),$$

where $\diamond$ denotes the Wick product. We will refer to it as the Skorohod stochastic integral or Wick-Ito integral of the process ϕ. Note that this integral compared with the ordinary integral (pathwise or forward integral) uses the Wick product instead of usual product. Then, we have
$$E\left[\int_0^T \phi(t,w)\delta B_t^H\right] = 0,$$
if the integral belongs to $L^2(P)$. Following the method of Oksendal (2004), we re-define no arbitrage in fBm setting.

Definition 2.1 *(a) The total wealth process $V^\theta(t)$ corresponding to a portfolio $\theta(t)$ in the Wick-Skorohod model is defined by*

$$V^\theta(t) = \theta(t) \diamond S(t),$$

where $S(t)$ can be any asset. (b) A portfolio $\theta(t)$ is called Wick-Skorohod self-financing if
$$\delta V^\theta(t) = \theta(t)\delta S(t).$$

Definition 2.2 *For $t \in [0,T]$, a Wick-Skorohod admissible portfolio $\theta(t)$ is called a strong arbitrage if the corresponding total wealth process $V^\theta(t)$ satisfies*

$$\begin{aligned} V^\theta(0) &= 0, \\ V^\theta(T) &\geq 0, \quad a.s. \\ P[V^\theta(T) > 0] &> 0. \end{aligned}$$

Then we can derive the bond pricing under the strong arbitrage framework.

3. HJM Representation by fBm

In this section, we consider the single factor HJM under the Wick derivative. Define a pure discount bond price as

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right).$$

The forward rate, then, is given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dB_s^H.$$

By using the definition of the pure discount bond and Fübini thoerem, we get

$$P(t,T) = \exp\left(-\int_t^T f(0,s)ds - \int_0^t\int_t^T \alpha(s,u)duds - \int_0^t\int_t^T \sigma(s,u)dudB_s^H\right).$$

Let us set $Z(t,T) = B^{-1}(t)P(t,T)$, where $B(t)$ is the money market account. By applying Itô lemma for fBm, we have

$$\frac{dZ(t,T)}{Z(t,T)} = -\left(\alpha'(t,T)dt + \sigma'(t,T)\phi'(t,T)dt + \sigma'(t,T)dB_t^H\right),$$

where

$$\begin{aligned} \phi(s,u) &= H(2H-1)\,|s-u|^{2H-1}, \\ \alpha'(t,T) &= \int_t^T \alpha(t,u)du,\ \sigma'(t,T) = \int_t^T \sigma(t,u)du \end{aligned}$$

and

$$\phi'(t,T) = \int_0^t \phi(t,u)\sigma'(u,T)du.$$

To make the discounted bond price driftless, we set

$$dB_t^H = \gamma_t dt + dB_t^H(Q),$$

where

$$\gamma_t = -\sigma'(t,T)^{-1}\alpha'(t,T) - \phi'(t,T),$$

and $(B_t^H(Q), t \in \mathrm{R})$ is the Q-fBm defining an equivalent martingale measure such that

$$\varepsilon' = \exp\left(\int_0^t \gamma_s dB_s^H - \frac{1}{2}\,|\gamma|_\phi^2\right). \tag{3.1}$$

The following theorem is the part of Theorem 3 of Rhee (2007), which shows the drift condition for fBm HJM.

Theorem 3.1 *The drift condition of HJM in fBm setup is given by*

$$\alpha(t,T) = -\left(\sigma(t,T)\phi'(t,T) + \sigma'(t,T)\frac{\partial\phi'(t,T)}{\partial T}\right) - \sigma(t,T)\gamma_t.$$

To price interest rate derivatives, we need the measure change from the risk neutral measure to so called the forward measure. In this case, Rhee (2007) also shows that the Radon-Nikodym derivative is given by

$$\varepsilon = \exp\left(-\int_0^t \sigma'(s,T)dB_s^H(Q) - \frac{1}{2}|\sigma'|_\phi^2\right), \tag{3.2}$$

where

$$|\gamma|_\phi^2 = \int_0^T\int_0^T \gamma_s\gamma_t\phi(s,t)dsdt.$$

We will price the forward cap using the HJM model in the next section.

3.1. Cap Pricing without Default Risk

We begin with the interest rate model under the measure Q. In the HJM setting, the price of a pure discount bond is given by

$$\begin{aligned} P(t,T) &= P(0,T) \\ &\times \exp\left(\int_0^t r_s ds + \int_0^t \sigma'(s,T)\phi'(s,T)ds - \int_0^t \sigma'(s,T)dB_s^H(Q)\right), \end{aligned} \tag{3.3}$$

where $B^H(Q) = (B_s^H(Q))_{s\geq 0}$ a Q-fBm and r_t is the spot rate. The price dynamics of all securities considered are assumed to be described by a cadlag adapted process. Denote the spot LIBOR rate by

$$L(T_{j-1}) \doteq L(T_{j-1},T_{j-1}) = \frac{1}{\delta}\left(\frac{1}{P(T_{j-1},T_j)} - 1\right).$$

The caplet price at time t is given by

$$caplet_t = P(t,T_j)E^{T_j}[(L(T_{j-1}) - k)^+\delta|\mathcal{F}_t].$$

Then the price at time t of a forward cap, denoted by FC_t, becomes

$$FC_t = \sum_{j=1}^n P(t,T_{j-1})E^{T_{j-1}}[(L(T_{j-1}) - k)^+\delta|\mathcal{F}_t].$$

Now we introduce the definition and lemma in order to derive the formula for the fBm cap pricing.

Definition 3.1 *Let* $f : \mathrm{R} \to \mathrm{R}$ *measurable. Then* $f \in L^2_\phi$ *if*

$$|f|^2_\phi = \int_{\mathrm{R}} \int_{\mathrm{R}} f(s)f(t)\phi(s,t)dsdt < \infty.$$

The following lemma is given in Oksendal (2004).

Lemma 3.1 *Let* $f \in L^2_\phi$ *and* $M(t) = \int_0^t f(s,\omega)dB^H_s$. *Then* $M(t)$ *is a quasi-martingale.*

Finally we prove the main result in this section.

Theorem 3.2 *Assume that term structure of interest rate follows fBm type HJM as (3.3). Suppose that the forward cap starting at time* T_0 *is settled in arrears at dates* T_j, $j = 1, 2, \ldots, n$. *Then the price at time* $t \le T_0$ *of forward cap is given by*

$$\begin{aligned} FC_t &= \sum_{j=1}^{n} P(t, T_{j-1}) \left[\Phi\left(\frac{-\ln a'_j - \mu_j}{\sigma_j} \right) - a'_j \exp\left(\mu_j + \frac{\sigma_j^2}{2} \right) \right. \\ &\quad \left. \times \Phi\left(\frac{-\ln a'_j - \mu_j - \sigma_j^2}{\sigma_j} \right) \right], \end{aligned}$$

where

$$\begin{aligned} \mu_j &\triangleq E^{T_{j-1}}[X_{j-1}|\mathcal{F}_t] = \int_0^t (\sigma'(s, T_{j-1}) - \sigma'(s, T_j)) dB^H_s(Q_{T_{j-1}}), \\ \sigma_j^2 &\triangleq \mathrm{Var}^{T_{j-1}}[X_{j-1}|\mathcal{F}_t] = \left| f\chi_{[0,T_{j-1}]} \right|^2_\phi - \left| f\chi_{[0.t]} \right|^2_\phi, \\ a'_j &= \delta' P(0, T_j) P(0, T_{j-1})^{-1} \\ &\quad \times \exp\left\{ \int_0^{T_{j-1}} (\sigma'(s, T_j)\phi'(s, T_j) - \sigma'(s, T_{j-1})\phi'(s, T_{j-1})) ds \right\} \\ &\quad \times \exp\left\{ -\int_0^{T_{j-1}} (\sigma'(s, T_{j-1}) - \sigma'(s, T_j))\sigma'(s, T_{j-1}) ds \right\}, \end{aligned}$$

and $\Phi(\cdot)$ *is the cumulative standard Normal.*

Proof: First we rearrange the caplet price as follows :

$$\begin{aligned} caplet_t &= P(t, T_j) E^{T_j}[(L(T_{j-1}) - k)^+ \delta | \mathcal{F}_t] \\ &= P(t, T_{j-1}) E^{T_{j-1}}[(1 - \delta' P(T_{j-1}, T_j))^+ | \mathcal{F}_t] \\ &= P(t, T_{j-1}) E^{T_{j-1}} \left[\left(1 - \delta' P(0, T_j) \exp\left\{ \int_0^{T_{j-1}} r_s ds \right. \right. \right. \\ &\quad \left. \left. \left. + \int_0^{T_{j-1}} \sigma'(s, T_j)\phi'(s, T_j) ds - \int_0^{T_{j-1}} \sigma'(s, T_j) dB^H_s(Q) \right\} \right)^+ | \mathcal{F}_t \right] \\ &= P(t, T_{j-1}) E^{T_{j-1}}[(1 - a_j \exp(X_{j-1}))^+ | \mathcal{F}_t], \end{aligned}$$

where

$$\delta' = 1 + k\delta,$$

$$\begin{aligned} a_j &= \delta' P(0,T_j)P(0,T_{j-1})^{-1} \\ &\quad \times \exp\left\{ \int_0^{T_{j-1}} (\sigma'(s,T_j)\phi'(s,T_j) - \sigma'(s,T_{j-1})\phi'(s,T_{j-1}))ds \right\} \end{aligned}$$

and

$$X_{j-1} = \int_0^{T_{j-1}} (\sigma'(s,T_{j-1}) - \sigma'(s,T_j))dB_s^H(Q).$$

To evaluate the expectation under the measure T_{j-1}, we change $B_s^H(Q)$ into $B_s^H(Q_{T_{j-1}})$ by using the equation (3.2). Then $caplet_t$ can be represented as

$$caplet_t = P(t,T_{j-1})E^{T_{j-1}}[(1 - a_j' \exp(X_{j-1}))^+ | \mathcal{F}_t],$$

where

$$X_{j-1} = \int_0^{T_{j-1}} (\sigma'(s,T_{j-1}) - \sigma'(s,T_j))dB_s^H(Q_{T_{j-1}})$$

and

$$\begin{aligned} a_j' &= \delta' P(0,T_j)P(0,T_{j-1})^{-1} \\ &\quad \times \exp\left\{ \int_0^{T_{j-1}} (\sigma'(s,T_j)\phi'(s,T_j) - \sigma'(s,T_{j-1})\phi'(s,T_{j-1}))ds \right\} \\ &\quad \times \exp\left\{ - \int_0^{T_{j-1}} (\sigma'(s,T_{j-1}) - \sigma'(s,T_j))\sigma'(s,T_{j-1})ds \right\}. \end{aligned}$$

The price at time t of a forward cap, denoted by FC_t, is given by

$$FC_t = \sum_{j=1}^{n} P(t,T_{j-1})E^{T_{j-1}}[(1 - a_j' \exp(X_{j-1}))^+ | \mathcal{F}_t].$$

Lemma 3.1 and change of variables give the result. □

4. Cap Pricing with Default Risk

In real world, interest rate derivatives such as caps and swaps have the default risk. In this section, we focus on the cap pricing with counterpart default risk. Formally, we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The default time τ is assumed to be an $\mathcal{F}_t$-stopping time. Hence define the default process by

$$H(t) = 1_{\{\tau \le t\}},$$

which is $\mathcal{F}_t$-adapted. Obviously, H is a uniformly integrable submartingale. By the Doob-Meyer decomposition, there exists a unique $\mathcal{F}_t$-predictable increasing process $A(t)$ such that $M(t) \doteq H(t) - A(t)$ is a martingale. We give more restrictive assumptions.

(A1) There exist a strict sub-filtration $(\mathcal{G}_t) \subset (\mathcal{F}_t)$ and a $\mathcal{G}_t$-adapted process Λ such that

$$A(t) = \Lambda(t \wedge \tau) \text{ and } \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t,$$

where $\mathcal{H}_t \doteq \sigma(H(s)|s \leq t)$ and $\mathcal{G}_t \vee \mathcal{H}_t$ represents for the smallest $\sigma-$algebra containing $\mathcal{G}_t$ and $\mathcal{H}_t$.

(A2) We define the positive $\mathcal{G}_t$-adapted hazard process Γ by

$$\exp(-\Gamma(t)) \doteq \mathcal{P}(\tau > t|\mathcal{G}_t).$$

(A3) B^H is a $(Q, \mathcal{G}_t)$ fBm.

We give the following lemma which is very useful for asset pricing with default risk (see Duffie and Lando, 2001).

Lemma 4.1 *Let $t \in R_+$ and Y a random variable. Then*

$$E\left[1_{\{\tau>t\}}Y|\mathcal{F}_t\right] = 1_{\{\tau>t\}} \exp(\Gamma(t))\, E\left[1_{\{\tau>t\}}Y|\mathcal{G}_t\right].$$

With this background, we price the cap with the default risk. The caplet price at time t with the default risk is given by

$$caplet_t = P(t, T_j)E^{T_j}[(L(T_{j-1}) - k)^+ \delta 1_{\{\tau>T_j\}}|\mathcal{F}_t].$$

Note that since the default indicator function is inside the expectation, the ordinary forward measure does not work. We give a fundamental theorem for pricing the interest rate derivatives with default risk, defining a new type of forward measure.

Theorem 4.1 *Denote the money market account and the pure discount bond with the default risk as*

$$\widetilde{B}(t) = \exp\left(\int_0^t r_d(s)ds\right) \quad \text{and} \quad \widetilde{P}(t,T) = E^Q\left[\exp\left(-\int_t^T r_d(s)ds\right)|\mathcal{F}_t\right]$$

respectively. Here

$$r_d(s) = r(s) + \lambda(s),$$

λ is the default risk premium, and the hazard rate process is represented by

$$\Gamma(t) = \int_0^t \lambda(s)ds,$$

i.e., the default stopping time is totally inaccessible. Then

$$caplet_t = P(t,T_j)E^{T_j}[(L(T_{j-1})-k)^+\delta 1_{\{\tau>T_j\}}|\mathcal{F}_t]$$
$$= 1_{\{\tau>t\}}\widetilde{P}(t,T_j)E^{\widetilde{T_j}}[(L(T_{j-1})-k)^+\delta|\mathcal{G}_t],$$

and the density process is given by

$$\frac{d\widetilde{Q}_{T_j}}{dQ_{T_j}} = \exp\left(-\int_0^t \widetilde{\sigma}'(s,T_j)dB_s^H(Q) - \frac{1}{2}|\widetilde{\sigma}'|_\phi^2\right)$$
$$\times \exp\left(-\int_0^t \sigma'(s,T_j)dB_s^H(Q) - \frac{1}{2}|\sigma'|_\phi^2\right),$$

where $\widetilde{\sigma}'(t,T_j)$ is the defaultable bond price volatility with maturity T_j.

In this case, the price at time t of a forward cap with the default risk, denoted by $\widetilde{FC_t}$, is represented by

$$\widetilde{FC_t} = \sum_{j=1}^{n}\widetilde{P}(t,T_{j-1})E^{\widetilde{Q}_{T_{j-1}}}[(L(T_{j-1})-k)^+\delta|\mathcal{G}_t].$$

Proof: By the lemma 4.1, we obtain

$$caplet_t = P(t,T_j)E^{T_j}\left[(L(T_{j-1})-k)^+\delta 1_{\{\tau>T_j\}}|\mathcal{F}_t\right]$$
$$= E^Q\left[\exp\left(-\int_t^{T_j} r(s)ds\right)(L(T_{j-1})-k)^+\delta 1_{\{\tau>T_j\}}|\mathcal{F}_t\right]$$
$$= 1_{\{\tau>t\}}E^Q\left[\exp\left(-\int_t^{T_j}(r(s)+\lambda(s))\,ds\right)(L(T_{j-1})-k)^+\delta|\mathcal{G}_t\right].$$

We define a new measure, being called "Default Risk Forward Measure", as

$$\frac{d\widetilde{Q}_{T_j}}{dQ_{T_j}} = \frac{\widetilde{P}(T_j,T_j)\widetilde{P}(0,T_j)^{-1}}{\widetilde{B}(T_j)} = \frac{\exp\left(-\int_0^{T_j} r_d(s)ds\right)}{\widetilde{P}(0,T_j)}.$$

Then we have

$$1_{\{\tau>t\}}E^Q\left[\exp\left(-\int_t^{T_j}(r(s)+\lambda(s))\,ds\right)(L(T_{j-1})-k)^+\delta|\mathcal{G}_t\right]$$
$$= 1_{\{\tau>t\}}\widetilde{P}(t,T_j)E^{\widetilde{Q}_{T_j}}[(L(T_{j-1})-k)^+\delta|\mathcal{G}_t],$$

which implies

$$\frac{d\widetilde{Q}_{T_j}}{dQ_{T_j}} = \exp\left(-\int_0^t \widetilde{\sigma}'(s,T_j)dB_s^H(Q) - \frac{1}{2}|\widetilde{\sigma}'|_\phi^2\right)$$
$$\times \exp\left(-\int_0^t \sigma'(s,T_j)dB_s^H(Q) - \frac{1}{2}|\sigma'|_\phi^2\right).$$

This ends the proof. □

5. Conclusion

This short paper demonstrates the fBm cap price with the default risk under the HJM framework, and also without the default risk. In particular, we propose a new fundamental approach to pricing the cap with the default risk. The Itô integral is replaced by Wick Integral for maintaining no arbitrage, which is a crucial tool for asset pricing. Our result shows that the adoption of the Wick Integral for fBm-type cap pricings provides a similar framework as the Itô integral with respect to the Brownian motion or Levy process.

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