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NEW COMPLEXITY ANALYSIS OF IPM FOR $P_*(\kappa)$ LCP BASED ON KERNEL FUNCTIONS

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ABSTRACT. In this paper we extend primal-dual interior point algorithm for linear optimization (LO) problems to $P_*(\kappa)$ linear complementarity problems(LCPs) ([1]). We define proximity functions and search directions based on kernel functions, $\psi(t) = \frac{t^{p+1}-1}{p+1} - \log t, p \in [0,1]$, which is a generalized form of the one in [16]. It is the first to use this class of kernel functions in the complexity analysis of interior point method(IPM) for $P_*(\kappa)$ LCPs. We show that if a strictly feasible starting point is available, then new large-update primal-dual interior point algorithms for $P_*(\kappa)$ LCPs have $O((1+2\kappa)n \log \frac{n}{\varepsilon})$ complexity which is similar to the one in [16]. For small-update methods, we have $O((1+2\kappa)\sqrt{n} \log \frac{n}{\varepsilon})$ which is the best known complexity so far.

1. INTRODUCTION

In this paper we consider the following linear complementarity problem (LCP) as follows :

$$s = Mx + q, \ xs = 0, \ x \geq 0, \ \ s \geq 0, \tag{1}$$

where $M \in R^{n \times n}$ is a $P_*(\kappa)$ matrix and $q \in R^n$.

LCPs have many applications in mathematical programming and equilibrium problems. Indeed, it is known that by exploiting the first-order optimality conditions of the optimization problem, any differentiable convex quadratic program can be formulated into a monotone linear complementarity problem, i.e. $P_*(0)$ LCP, and vice versa([17]). And variational inequality problems are widely used in the study of equilibrium in economics, transportation planning and game theory. And variational inequality problems have a close connection to the LCPs. The reader can refer to [4] for the basic theory, algorithms and applications.

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The primal-dual IPM for LO problem was first introduced in [7, 11]. Kojima et al.([7]) proved the polynomial computational complexity of the algorithm for LO problem, and since then many other algorithms have been developed based on the primal-dual strategy. Kojima et al.([9]) proposed a polynomial time algorithm for monotone linear complementarity problems, i.e. $P_*(0)$ LCPs. Kojima et al.([8]) proved the existence of the central path for any $P_*(\kappa)$ LCP and generalized previously known results to the wider class of so called $P_*(\kappa)$ LCPs and unified interior point methods(IPMs) for LCPs. Since then an interior point algorithm's quality is measured by the fact whether it can be generalized to $P_*(\kappa)$ LCPs or not([6]). Miao ([12]) extended the Mizuno-Todd-Ye predictor-corrector method to $P_*(\kappa)$ LCPs and his algorithm has $O((1+\kappa)\sqrt{n}L)$ iteration complexity. Recently, Illés and Nagy([6]) gives a version of Mizuno-Todd-Ye predictor-corrector interior point algorithm for the $P_*(\kappa)$ LCP and show the iteration complexity $O((1+\kappa)^{\frac{3}{2}}\sqrt{n}L)$.

In this paper we propose new large-update primal-dual interior point algorithms for $P_*(\kappa)$ LCP and show that the algorithm has $O((1+2\kappa)n\log\frac{n}{\varepsilon})$ iteration complexity which is similar to the one in [16]. For $p=1$, the kernel function is a classical logarithmic barrier function which is studied in [16] for LO. Since $P_*(\kappa)$ LCP is a generalization of LO problem, we loose the orthogonality of the vectors dx and d_s. So our analysis is different from the one in [13] and [15]. We define a neighborhood and use a search direction based on a specific class of kernel functions which are eligible. However we don't use the condition (iii) of eligible function (See Definition 2.4). Thus the analysis is different from others in [6], [8], [9], [10], and [12].

This paper is organized as follows. In Section 2 we recall some basic concepts and properties of the kernel functions. In Section 3 we show the complexity result.

We use the following notations throughout the paper : R^n_+ denotes the set of n dimensional nonnegative vectors and R^n_{++}, the set of n dimensional positive vectors. For $x = (x_1, x_2, \cdots, x_n)^T \in R^n$, $x_{min} = \min\{x_1, x_2, \cdots, x_n\}$, i.e. the minimal component of x, $\|x\|$ is the 2-norm of x, and X is the diagonal matrix from vector x , i.e. $X = diag(x)$. xs denotes the componentwise product (Hadamard product) of vectors x and s. x^Ts is the scalar product of the vectors x and s. e is the n-dimensional vector of ones and I is the n-dimensional identity matrix. J is the index set, i.e. $J = \{1, 2, \cdots, n\}$. We write $f(x) = O(g(x))$ if $\mid f(x) \mid \leq k \mid g(x) \mid$ for some positive constant k and $f(x) = \Theta(g(x))$ if $k_1 \mid g(x) \mid \leq \mid f(x) \mid \leq k_2 \mid g(x) \mid$ for some positive constants k_1 and k_2.

2. Preliminaries

In this section we give some basic definitions and properties.

Definition 2.1. *Let $\kappa \geq 0$. A matrix $M \in R^{n\times n}$ is called a $P_*(\kappa)$ matrix if*

$$(1+4\kappa)\sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_-(x)} x_i(Mx)_i \geq 0\,,$$

for all $x \in R^n$, where $J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\}$ and $J_-(x) = \{i \in J : x_i(Mx)_i < 0\}$.

Remark 2.2. *The class $P_*(\kappa)$ contains the class PSD of positive semi-definite matrices, i.e. matrices M satisfying $x^TMx \geq 0$ for all $x \in R^n$, and the class P of matrices with all the principal minors positive.*

Definition 2.3. *A function $\psi : R_+ \rightarrow R_+$ is called a kernel function if ψ is twice differentiable and satisfies the following conditions:*
(i) $\psi^{'}(1) = \psi(1) = 0,$
(ii) $\psi^{''}(t) > 0,$ *for all* $t > 0,$
(iii) $\lim_{t\rightarrow 0^+} \psi(t) = \lim_{t\rightarrow\infty} \psi(t) = \infty.$

Definition 2.4. *A function $\psi(\in \mathcal{C}^3) : (0, \infty) \rightarrow R$ is eligible if it satisfies the following conditions:*
(i) $t\psi^{''}(t) + \psi^{'}(t) > 0,\ t > 0.$
(ii) $\psi^{'''}(t) < 0,\ t > 0,$
(iii) $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) > 0,\ 0 < t \leq 1.$
(iv) $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1.$

Definition 2.5. *A function $f : D(\subset R) \rightarrow R$ is exponentially convex if and only if $f(\sqrt{x_1x_2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$ for all $x_1, x_2 \in D$.*

We use the following lemma to prove that the modified Newton-system has a unique solution.

Lemma 2.6. (Lemma 4.1 in [8]) *Let $M \in R^{n\times n}$ be a $P_*(\kappa)$ matrix and $x,\ s \in R^n_{++}$. Then for all $a \in R^n$ the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

We denote the strictly feasible set of LCP (1) by $\mathcal{F}^o$, i.e.,

$$\mathcal{F}^o := \{(x, s) \in R^{2n}_{++} : s = Mx + q)\}.$$

Definition 2.7. *A $(x, s) \in \mathcal{F}^o$ is an ε-approximate solution if and only if $x^Ts \leq \varepsilon$ for $\varepsilon > 0$.*

To find an ε-approximate solution for (1) we relax the complementarity condition, i.e. the second equation in (1) and introduce the following parameterized system :

$$s = Mx + q,\ xs = \mu e,\ x > 0,\ s > 0, \tag{2}$$

where $\mu > 0$. Without loss of generality, we assume that (1) is strictly feasible, i.e. there exists (x^0, s^0) such that $s^0 = Mx^0 + q,\ x^0 > 0,\ s^0 > 0$ ([8]). For given strictly feasible point (x^0, s^0) we can always find $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$. Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, (2) has a unique solution for any $\mu > 0$. We denote the solution of (2) as $(x(\mu), s(\mu))$ for given $\mu > 0$ and call it $\mu-$*center*. We define the solution set

$\{(x(\mu), s(\mu)) \mid \mu > 0\}$ as the *central path* of system (2). As $\mu \to 0$, the sequence $(x(\mu), s(\mu))$ approaches to the solution (x, s) of the system (1) ([8]). By defining the following notations:

$$d = \sqrt{\frac{x}{s}},\ v = \sqrt{\frac{xs}{\mu}},\ d_x = \frac{v\Delta x}{x},\ d_s = \frac{v\Delta s}{s}, \tag{3}$$

we can write the scaled Newton-system as follows :

$$\begin{cases} -\bar{M}d_x + d_s & = \ 0, \\ d_x + d_s & = \ v^{-1} - v, \end{cases} \tag{4}$$

where $\bar{M} = DMD$ and $D = diag(d)$. Note that $v^{-1} - v$ in (4) is exactly the negative gradient of the logarithmic barrier function $\Psi_l(v) = \sum_{i=1}^n ((v_i^2 - 1)/2 - \log\ v_i)$, i.e.

$$d_x + d_s = -\nabla\Psi_l(v). \tag{5}$$

In this paper we replace logarithmic barrier function with the generalized log-barrier function,

$$\Psi(v) = \sum_{i=1}^n \psi(v_i),\ \psi(t) = \frac{t^{p+1} - 1}{p+1} - \log t,\ p \in [0, 1]. \tag{6}$$

ψ is called the kernel function of $\Psi(v)$ and corresponding Newton system is given as follows:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = \mu e - \mu v^{p+1}. \end{cases} \tag{7}$$

This system uniquely defines a search direction $(\Delta x, \Delta s)$ by Lemma 2.6, since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible. Throughout the paper, we assume that a proximity parameter τ and a barrier update parameter θ are given and $\tau = O(n)$ and $0 < \theta < 1$, fixed. The algorithm works as follows. We assume that strictly feasible point (x, s) with $\psi(x, s, \tau) \leq \tau$ is given. Then after decreasing μ to $\mu_+ = (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$, we solve Newton system (7) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) which is in a $\tau-$neighborhood of the μ_+-center and let $\mu := \mu_+$, $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1 - \theta$ and we solve Newton system (7) targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, e.g. $n\mu \leq \varepsilon$. One distinguishes IPMs as large-update methods when $\theta = \Theta(1)$ and small-update methods when $\theta = \Theta(\frac{1}{\sqrt{n}})$. The small-update methods have the best known iteration complexity, but in practice large-update methods are more efficient than small-update. In this paper we define a large-update IPM and the algorithm is defined as follows :

Algorithm

Input:
A threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
starting point (x^0, s^0) and $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$;
begin
$x := x^0;\ s := s^0;\ \mu := \mu^0;$
while $n\mu \geq \varepsilon$ do
begin
$\mu := (1-\theta)\mu;$
while $\Psi(v) > \tau$ do
begin
solve (7) for Δx and Δs;
determine a step size α from (18);
$x := x + \alpha\Delta x;$
$s := s + \alpha\Delta s;$
end
end
end

For ψ we have

$$\psi^{'}(t) = t^p - \frac{1}{t},\ \psi^{''}(t) = pt^{p-1} + \frac{1}{t^2},\ \psi^{'''}(t) = p(p-1)t^{p-2} - \frac{2}{t^3}. \tag{8}$$

Since $\psi^{''}(t) > 0$, ψ is strictly convex. Note that for $p \in [0,1]$, $\psi(1) = \psi^{'}(1) = 0$. From this fact ψ is determined by the second derivative, i.e., $\psi(t) = \int_1^t \int_1^\xi \psi^{''}(\varsigma)d\varsigma d\xi$. We also define the norm-based proximity measure $\delta(v)$ as follows :

$$\delta(v) = \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2} \parallel d_x + d_s \parallel . \tag{9}$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have $\Psi(v) = 0$ which is equivalent to $\delta(v) = 0$ and to $v = e$. For the notational convenience we denote $\delta(v)$ by δ.

Lemma 2.8. *Let δ be the value defined in (9). Then we have*

$$v_{min} \geq \frac{1}{1+2\delta}.$$

Proof: First, if $v_{min} \leq 1$, then we have

$$\delta = \frac{1}{2}\| - \nabla\Psi(v)\| = \frac{1}{2}\|v^{-1} - v^p\| \geq \frac{1}{2} \mid v_{min}^{-1} - v_{min}^p \mid \geq \frac{1}{2}(v_{min}^{-1} - 1).$$

Thus we have $v_{min} \geq (1+2\delta)^{-1}$. Secondly, if $v_{min} > 1$, then we have $v_{min} > 1 \geq \frac{1}{1+2\delta}$. $\square$

In the following lemma we give key properties which are crucial in the analysis of the algorithm.

Lemma 2.9. *Kernel function ψ in (6) satisfies the following properties.*

(i) $t\psi''(t) + \psi'(t) > 0,\ t > 0.$

(ii) $\psi'''(t) < 0,\ t > 0,$

(iii) $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ t > 1,\ \beta > 1.$

Proof: (i): From (8), $t\psi''(t)+\psi'(t) = t(pt^{p-1} + \frac{1}{t^2}) + (t^p - \frac{1}{t}) = (p+1)t^p > 0$, for $t > 0$.
(ii): By (8), it is obvious.
(iii): By (8), $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) = \frac{t^p(1+p)(\beta^{p+1}-1)}{\beta t^2} > 0$, for $p \in [0,1]$, $t > 1$, $\beta > 1$. □

By Lemma 2.9 (i) and Lemma 1 in [14], ψ is exponentially convex. Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of ψ for $t \geq 1$, $\rho : [0,\infty) \to (0,1]$ the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \in (0,1]$. Then we have the following lemma.

Lemma 2.10. (Example 9 in [5])

(i) $(1 + (p+1)s)^{\frac{1}{p+1}} \leq \varrho(s) \leq 1 + s + \sqrt{s^2 + 2s},\ s \geq 0.$

(ii) $\rho(s) \geq \frac{1}{(2s+1)},\ s > 0.$ □

Now we obtain a lower bound for δ in terms of the proximity function $\Psi(v)$.

Theorem 2.11. *Let δ be the norm-based proximity measure as defined in (9). If $\Psi(v) \geq \tau$ for $\tau \geq 1$, then we have*

$$\delta \geq \frac{1}{6}(\Psi(v))^{\frac{p}{p+1}}.$$

Proof: By Theorem 4.9 in [1], Lemma 2.10 (i) and $\Psi(v) \geq 1$, we get

$$\begin{aligned}\delta &\geq \frac{1}{2}\left(\varrho(\Psi(v))^p - \frac{1}{\varrho(\Psi(v))}\right) \geq \frac{1}{2}\left(((p+1)\Psi(v)+1)^{\frac{p}{p+1}} - \frac{1}{((p+1)\Psi(v)+1)^{\frac{1}{p+1}}}\right)\\ &= \frac{1}{2}\frac{(p+1)\Psi(v)}{((p+1)\Psi(v)+1)^{\frac{1}{p+1}}} \geq \frac{(p+1)\Psi(v)}{2(1+2\Psi(v))^{\frac{1}{p+1}}} \geq \frac{(p+1)\Psi(v)}{6\Psi(v)^{\frac{1}{p+1}}} \geq \frac{1}{6}\Psi(v)^{\frac{p}{p+1}},\end{aligned}$$

where $p \in [0,1]$. □

Note that at the start of outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. By using the following lemma, we obtain an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Lemma 2.12. *If $t > 1$, then $\psi(t) < \frac{1}{2}\psi''(1)(t-1)^2$.*

Proof: By using Taylor's Theorem and $\psi(1) = \psi^{'}(1) = 0$,

$$\begin{aligned}\psi(t) &= \psi(1) + \psi^{'}(1)(t-1) + \frac{1}{2}\psi^{''}(t-1)^2 + \frac{1}{3}\psi^{'''}(\xi)(\xi-1)^3 \\ &= \frac{1}{2}\psi^{''}(t-1)^2 + \frac{1}{3}\psi^{'''}(\xi)(\xi-1)^3,\end{aligned}$$

where $1 \leq \xi \leq t$ if $t > 1$. Since $\psi^{'''} < 0$, we obtain the desired result. □

Lemma 2.13. *If* $\Psi(v) \leq \tau$*, then we have for* $0 < \theta < 1$

$$\Psi(\frac{v}{\sqrt{1-\theta}}) \leq \frac{(1+p)n}{2(1-\theta)}\left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2.$$

Proof: By the definition of ϱ and $\frac{1}{\sqrt{1-\theta}} \geq 1$, $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$. By Theorem 3.2 in [1], Lemma 2.12 (i) with $\psi^{''}(1) = 1+p$, and Lemma 2.10, we have

$$\begin{aligned}\Psi(\frac{v}{\sqrt{1-\theta}}) &\leq n\psi\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}}\right) \leq \frac{(1+p)n}{2}\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2 \\ &\leq \frac{(1+p)n}{2}\left(\frac{1 + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}} - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\ &\leq \frac{(1+p)n}{2(1-\theta)}\left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2.\end{aligned}$$

By using $1 - \theta = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, the last inequality holds. □

Define Ψ_0 as the value of $\Psi(v)$ after the μ-update. Then we have

$$\Psi_0 \leq \frac{(1+p)n}{2(1-\theta)}\left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2. \tag{10}$$

For large-update methods with $\tau = O(n)$ and $\theta = O(1)$, we have $\Psi_0 = O(n)$.

3. Complexity Analysis

In this section we analyze the complexity of the algorithm. Since $P_*(\kappa)$ LCPs are generalization of LO problems, we loose the orthogonality of vectors d_x and d_s. So the analysis is different from LO case([1], [5], [7], [13], [16]). After a damped step for fixed μ we have $x_+ = x + \alpha\Delta x$, $s_+ = s + \alpha\Delta s$. From (3), we have $x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x)$, $s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s)$. Thus we have $v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s)$. Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (7), for $\Delta x \in R^n$ we have

$$(1+4\kappa)\sum_{i\in J_+} \Delta x_i \Delta s_i + \sum_{i\in J_-} \Delta x_i \Delta s_i \geq 0,$$

where $J_+ = \{\ i \in J\ :\ \Delta x_i \Delta s_i \geq 0\ \}$, $J_- = J - J_+$ and $\Delta x_i,\ \Delta s_i$ denote the $i-$th component of the vector Δx and Δs, respectively. Since $d_x d_s = \frac{v^2 \Delta x \Delta s}{xs} = \frac{\Delta x \Delta s}{\mu}$ and $\mu > 0$,

$$(1+4\kappa)\sum_{i\in J_+} [d_x]_i [d_s]_i + \sum_{i\in J_-} [d_x]_i [d_s]_i \geq 0, \tag{11}$$

where $[d_x]_i$ and $[d_s]_i$ denote the $i-$th component of the vector d_x and d_s, respectively. In the following lemma we obtain the bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 3.1. (Lemma 4.2 in [3]) $\sum_{i=1}^n ([d_x]_i^2 + [d_s]_i^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\sqrt{1+2\kappa}\ \delta$ *and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\ \delta$.

Lemma 3.2. *Let δ be the value defined in (9). Then we have*

$$\|(x^{-1}\Delta x, s^{-1}\Delta s)\| \leq 2\sqrt{1+2\kappa}\ \delta(1+2\delta). \tag{12}$$

Proof: By using Lemma 3.1 and Lemma 2.8, we have

$$\begin{aligned}\|(x^{-1}\Delta x, s^{-1}\Delta s)\| = \|(v^{-1}d_x, v^{-1}d_s)\| &\leq \frac{1}{v_{min}}\sqrt{\|d_x\|^2 + \|d_s\|^2} \\ &\leq \frac{1}{v_{min}} 2\sqrt{1+2\kappa}\delta = 2\sqrt{1+2\kappa}\ \delta(1+2\delta).\end{aligned}$$

□

Define

$$\hat{\alpha} = \frac{1}{2\sqrt{1+2\kappa}\ \delta(1+2\delta)}. \tag{13}$$

Then for $\alpha \in [0, \hat{\alpha}]$, we get $x(\alpha) = x + \alpha\Delta x > 0$ and $s(\alpha) > 0$. Indeed, if $\Delta x > 0$, it is clear. Otherwise, there exists an index set $\bar{J}$ such that $\bar{J} = \{i \in J : \Delta x_i < 0\}$. From (14),

$$\max_{i\in\bar{J}}(-x^{-1}\Delta x)_i \leq \|-x^{-1}\Delta x\| \leq 2\sqrt{1+2\kappa}\ \delta(1+2\delta) = \hat{\alpha}^{-1}.$$

Thus $\min_{i\in\bar{J}}(-x(\Delta x)^{-1})_i \geq \hat{\alpha} \geq \alpha$ and $x_i + \alpha\Delta x_i > 0$ for $i \in \bar{J}$ and $\alpha \in [0, \hat{\alpha}]$. Hence $x + \alpha\Delta x > 0$ for $\alpha \in [0, \hat{\alpha}]$. By the same way, we can get the case $s(\alpha) = s + \alpha\Delta s(\alpha) > 0$ for $\alpha \in [0, \hat{\alpha}]$. Since $\psi(v)$ is exponentially convex, we have

$$\Psi(v_+) = \Psi(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\ )\ \leq\ \frac{1}{2}\,(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\ ).$$

For given $\mu > 0$ by letting $f(\alpha)$ the difference of the new and old proximity measures, i.e.

$$f(\alpha) = \Psi(v_+) - \Psi(v),$$

we have $f(\alpha) \leq f_1(\alpha)$, where $f_1(\alpha) := \frac{1}{2}(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)) - \Psi(v)$. Note that $f(0) = f_1(0) = 0$. By taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{'}(v_i + \alpha[d_x]_i)[d_x]_i + \psi^{'}(v_i + \alpha[d_s]_i)[d_s]_i).$$

Using (5) and the definition of δ, we have

$$f_1^{'}(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta^2. \tag{14}$$

By taking the derivative of $f_1^{'}(\alpha)$ with respect to α, we have

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{''}(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi^{''}(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{15}$$

In the followings we cite some technical lemmas without proof. We obtain the feasible step size α such that the proximity measure is decreasing when we take a new iterate for fixed μ in Lemma 3.5.

Lemma 3.3. (Lemma 4.3 in [3]) $f_1^{''}(\alpha) \leq 2(1+2\kappa)\,\delta^2\psi^{''}(v_{min} - 2\alpha\sqrt{1+2\kappa}\,\delta)$.

Lemma 3.4. (Lemma 4.4 in [3]) $f_1^{'}(\alpha) \leq 0$ *if* α *is satisfying*

$$-\psi'(v_{min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi^{'}(v_{min}) \leq \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{16}$$

Lemma 3.5. (Lemma 4.5 in [3]) *Let* $\rho : [0,\infty) \to (0,1]$ *denote the inverse function of the restriction of* $-\frac{1}{2}\psi^{'}(t)$ *to the interval* $(0,1]$. *Then the largest step size* α *that satisfies* (16) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{17}$$

In the following lemma we obtain the lower bound for $\bar{\alpha}$ in Lemma 3.5.

Lemma 3.6. *Let* ρ *and* $\bar{\alpha}$ *be the values as defined in* $Lemma$ 3.5. *Then we have*

$$\bar{\alpha} \geq \frac{1}{256(1+2\kappa)\delta^2}.$$

Proof: By the definition of ρ, $-\psi^{'}(\rho(\delta)) = 2\delta$. By taking the derivative with respect to δ, we get $-\psi^{''}(\rho(\delta))\rho^{'}(\delta) = 2$. Since $\psi^{''} > 0$, we have $\rho^{'}(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0$. Hence ρ is monotonically decreasing. By (17) and the fundamental theorem of calculus, we have

$$\begin{aligned}\bar{\alpha} &= \frac{1}{2\delta\sqrt{1+2\kappa}}(\rho(\delta) - \rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta)) \\ &= \frac{1}{2\delta\sqrt{1+2\kappa}}\int_{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}^{\delta}\rho^{'}(\xi)d\xi = \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}\frac{d\xi}{\psi''(\rho(\xi))}.\end{aligned}$$

Since $\delta \le \xi \le (1+\frac{1}{\sqrt{1+2\kappa}})\delta$ and ρ is monotonically decreasing, $\rho(\xi) \ge \rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta)$. Since ψ'' is monotonically decreasing, $\psi''(\rho(\xi)) \le \psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))$. Hence $\frac{1}{\psi''(\rho(\xi))} \ge \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}$. Therefore we have

$$\bar{\alpha} \ge \frac{1}{\delta\sqrt{1+2\kappa}} \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))} \int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta} d\xi = \frac{1}{1+2\kappa} \frac{1}{\psi''(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}.$$

Let $a := 1+\frac{1}{\sqrt{1+2\kappa}}$ and $t = \rho(a\delta)$. Note that $a \le 2$. Then by Lemma 2.10 (ii) and the definition of ρ,

$$1 \ge t = \rho(a\delta) \ge \frac{1}{2a\delta+1}.$$

Since $p \in [0,1]$ and $t \le 1$, we have $pt^p \le 1$. Using Theorem 2.11, we have for $p \in [0,1]$,

$$\begin{aligned}
\bar{\alpha} &\ge \frac{1}{(1+2\kappa)} \frac{1}{\psi''(\rho(a\delta))} = \frac{1}{(1+2\kappa)} \frac{1}{\psi''(t)} = \frac{1}{(1+2\kappa)} \frac{1}{pt^{p-1}+t^{-2}} \\
&= \frac{1}{(1+2\kappa)} \frac{1}{t^{-1}(pt^p+t^{-1})} \ge \frac{1}{(1+2\kappa)} \frac{1}{t^{-1}(1+t^{-1})} \\
&\ge \frac{1}{(1+2\kappa)} \frac{1}{(2a\delta+1)(2a\delta+2)} = \frac{1}{2(1+2\kappa)} \frac{1}{(2a\delta+1)(a\delta+1)} \ge \frac{1}{4(1+2\kappa)} \frac{1}{(a\delta+1)^2} \\
&\ge \frac{1}{4(1+2\kappa)} \frac{1}{(2\delta+6\delta)^2} = \frac{1}{256(1+2\kappa)\delta^2}.
\end{aligned}$$

□

Define

$$\tilde{\alpha} = \frac{1}{256(1+2\kappa)\delta^2}. \tag{18}$$

Then using Lemma 3.6 and (13), we have

$$\tilde{\alpha} = \frac{1}{256(1+2\kappa)\delta^2} \le \frac{1}{2(1+2\kappa)} \frac{1}{(2a\delta+1)(a\delta+1)} \le \frac{1}{2\sqrt{1+2\kappa}\,\delta(2\delta+1)} = \hat{\alpha}.$$

By Lemma 3.2, $\tilde{\alpha}$ is strictly feasible step size. Thus we will use $\tilde{\alpha}$ as the default step size in the algorithm.

Lemma 3.7. (Lemma 3.12 in [14]) *Let h be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let h attain its (global) minimum at $t^* > 0$. If h'' is increasing for $t \in [0, t^*]$, then $h(t) \le \frac{th'(0)}{2}$ for $0 \le t \le t^*$.*

Lemma 3.8. (Lemma 4.8 in [3]) *If the step size α satisfies $\alpha \le \bar{\alpha}$, then $f(\alpha) \le -\alpha\delta^2$.*

In the following theorem we obtain the upper bound for the difference $f(\alpha)$.

Theorem 3.9. *Let $\tilde{\alpha}$ be a step size as defined in* (18). *Then we have*

$$f(\tilde{\alpha}) \leq -\frac{1}{256(1+2\kappa)}. \tag{19}$$

Proof: By Lemma 3.6 and (18), $\tilde{\alpha} \leq \bar{\alpha}$. Thus by Lemma 3.8,

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 = -\frac{\delta^2}{256(1+2\kappa)\delta^2} = -\frac{1}{256(1+2\kappa)}.$$

□

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as $\Psi_k,\ k = 1,\ 2, \cdots$. Let K denote the total number of inner iterations in the outer iteration. Then by the definition of K, we have

$$\Psi_{K-1} > \tau,\ 0 \leq \Psi_K \leq \tau.$$

In the following lemma, we obtain the upper bound for the total number of inner iterations which we needed to return to the τ-neighborhood again.

Lemma 3.10. *Let K be the total number of inner iterations in an outer iteration. Then we have*

$$K \leq 256(1+2\kappa)\Psi_0,$$

where Ψ_0 denotes the value of $\Psi(v)$ after the μ-update.

Proof: Using Theorem 3.9, we have $f(\tilde{\alpha}) \leq -\frac{1}{256(1+2\kappa)}$. This implies that $\Psi_K - \Psi_{K-1} \leq -\frac{1}{256(1+2\kappa)}$ and $0 \leq \Psi_K \leq \Psi_{K-1} - \frac{1}{256(1+2\kappa)} \leq \Psi_{K-2} - \frac{2}{256(1+2\kappa)} \leq \cdots \leq \Psi_0 - \frac{K}{256(1+2\kappa)}$. Therefore K is bounded above by

$$K \leq 256(1+2\kappa)\Psi_0.$$

This completes the proof. □

From (10), we have $\Psi_0 \leq \frac{(1+p)n}{2(1-\theta)} \left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2$ and from Lemma 3.10, we have $K \leq \frac{128(1+2\kappa)(p+1)n}{(1-\theta)} \left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2$. Thus the upper bound for the total number of iterations is obtained by multiplying the number K by the number of outer iterations. If the central path parameter μ has the initial value μ^0 and is updated by multiplying $1-\theta$, with $0 < \theta < 1$, then after at most $\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon} \right\rceil$ iterations we have $n\mu \leq \epsilon$ ([16]). Thus the total number of iterations is bounded above by $\frac{128(1+2\kappa)(p+1)n}{\theta(1-\theta)} \left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2 \log \frac{n\mu^0}{\epsilon}$. So we obtain the main result as follows.

Theorem 3.11. *Let a $P_*(\kappa)$ linear complementarity problem be given, where $\kappa \geq 0$. Assume that a strictly feasible starting point (x^0, s^0) is available with $\Psi(x^0, s^0, \mu^0) \leq \tau$ for some*

$\mu^0 > 0$. Then the total number of iterations to have solution (x, s) with $x^T s \le \varepsilon$ is bounded above by

$$\left\lceil \frac{128(1+2\kappa)(p+1)n}{(1-\theta)}\left(\theta + \frac{\tau}{n} + \sqrt{\left(\frac{\tau}{n}\right)^2 + \frac{2\tau}{n}}\right)^2 \right\rceil \left\lceil \frac{1}{\theta}\log\frac{n\mu^0}{\epsilon} \right\rceil .$$

Remark 3.12. *For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, the algorithm has $O((1+2\kappa)n\log\frac{n}{\varepsilon})$ iteration complexity. Note that for small-update methods with $\theta = n^{-\frac{1}{2}}$ and $\tau = 1$, we have $O((1+2\kappa)\sqrt{n}\log\frac{n}{\varepsilon})$ which is the best known complexity so far.*

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