J. KSIAM Vol.12, No.3, 161–169, 2008

MULTIPLICITY AND NONLINEARITY IN THE NONLINEAR ELLIPTIC SYSTEM

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ABSTRACT. We investigate the existence of solutions $u(x,t)$ for perturbations of the elliptic system with Dirichlet boundary condition

$$\begin{aligned} L\xi + \mu g(\xi + 2\eta) = f \quad \text{in} \quad \Omega, \\ L\eta + \nu g(\xi + 2\eta) = f \quad \text{in} \quad \Omega, \end{aligned} \tag{0.1}$$

where $g(u) = Bu^+ - Au^-$, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, μ, ν are nonzero constants and the nonlinearity $(\mu + 2\nu)g(u)$ crosses the eigenvalues of the elliptic operator L.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1 \le i,j \le n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}),$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$. In [2] the authors investigate multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$\begin{aligned} &Lu + g(u) = f(x) \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where the semilinear term $g(u) = bu^+ - au^-$ and L is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \le \cdots \le \lambda_i \le \cdots \to \infty.$$

2000 *Mathematics Subject Classification.* 35J05, 35J25, 35J55.

Key words and phrases. Elliptic system, eigenvalue problem, Dirichlet boundary condition, uniqueness, multiple solutions.

This work was supported by Inha University Research grant.

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Here the source term f is generated by the eigenfunctions of the second order elliptic operator with Dirichlet boundary condition.

In [5, 7, 8], the authors have investigated multiplicity of solutions of (1.1) when the forcing term f is supposed to be a multiple of the first eigenfunction and the nonlinearity $-(bu^+ - au^-)$ crosses eigenvalues. In [4], the authors investigated a relation between multiplicity of solutions and source terms of (1.1) when the forcing term f is supposed to be spanned two eigenfunction ϕ_1, ϕ_2 and the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues λ_1, λ_2.

In this paper we investigate the existence of solutions $u(x,t)$ for perturbations of the elliptic system with Dirichlet boundary condition

$$\begin{aligned} &L\xi + \mu(B(\xi + 2\eta)^+ - A(\xi + 2\eta)^-) = f \quad \text{in} \quad \Omega, \\ &L\eta + \nu(B(\xi + 2\eta)^+ - A(\xi + 2\eta)^-) = f \quad \text{in} \quad \Omega, \\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, μ, ν are nonzero constants and the nonlinearity $(\mu + 2\nu)(B(\xi + 2\eta)^+ - A(\xi + 2\eta)^-)$ crosses the eigenvalues of the elliptic operator L.

Equation (1.1) and the following type nonlinear equation with Dirichlet boundary condition was studied by many authors:

$$\begin{aligned} Lu &= b[(u + 2)^+ - 2] \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.3}$$

In [9] Lazer and McKenna point out that this kind of nonlinearity $b[(u + 2)^+ - 2]$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [14] , Micheletti and Pistoia [11][12] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [10] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.2).

The organization of this paper is as following. In section 2, we have a concern with a relation between multiplicity of solutions and source terms of a nonlinear elliptic equation when the nonlinearity crosses eigenvalues. We investigate the uniqueness and multiplicity of solutions for the single nonlinear elliptic equation. In section 3, we investigate the uniqueness and the existence of multiple solutions $u(x,t)$ for the elliptic system with Dirichlet boundary condition when the nonlinearity $(\mu + 2\nu)(Bu^+ - Au^-)$ crosses the eigenvalues of the elliptic operator.

2. A SINGLE NONLINEAR ELLIPTIC EQUATION

We have a concern with a relation between multiplicity of solutions and source terms of a nonlinear elliptic equation

$$Lu + bu^+ - au^- = f \quad \text{in } L^2(\Omega). \tag{2.1}$$

Here we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses eigenvalues. We consider three cases: The nonlinearity crosses no eigenvalue; the nonlinearity crosses the eigenvalue λ_1; the nonlinearity crosses the eigenvalues λ_1, λ_2.

Let us denote an element u, in H_0, as $u = \sum h_j \phi_j$ and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_j| h_j^2 < \infty\}.$$

Then this is a complete normed space with a norm $\|u\| = (\sum |\lambda_{mn}| h_{mn}^2)^{\frac{1}{2}}$. If $f \in H_0$ and a, b are not eigenvalues of L, then every solution in H_0 of $Lu + bu^+ - au^- = f$ belongs to H (cf. [2]).

Case 1) The nonlinearity crosses no eigenvalue

We suppose that the nonlinearity $-(bu^+ - au^-)$ crosses no eigenvalue, that is, $a, b < \lambda_1$. By the contraction mapping principle we have the following uniqueness theorem.

Theorem 2.1. *Let $a, b < \lambda_1$ and $f \in H_0$. Then equation (2.1) has a unique solution in H.*

Case 2) The nonlinearity crosses the eigenvalues λ_1, λ_2.

We suppose that the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues λ_1, λ_2, i.e., $a < \lambda_1 < \lambda_2 < b < \lambda_3$. We have a concern with a relation between multiplicity of solutions and source terms of a nonlinear elliptic equation

$$Lu + bu^+ - au^- = f \quad \text{in } L^2(\Omega). \tag{2.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_1 and ϕ_2.

Let V be the two dimensional subspace of $L^2(\Omega)$ spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in $L^2(\Omega)$. Let P be an orthogonal projection $L^2(\Omega)$ onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$. Hence equation (2.1) is equivalent to a system

$$Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \tag{2.2}$$

$$Lv + P(b(v + w)^+ - a(v + w)^-) = s_1\phi_1 + s_2\phi_2. \tag{2.3}$$

Lemma 2.1. *For fixed $v \in V$,* (2.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v.

The proof of the lemma is similar to that of Lemma 2.1 of [3].

By Lemma 2.1, the study of multiplicity of solutions of (2.1) is reduced to that of an equivalent problem

$$Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = s_1\phi_1 + s_2\phi_2 \tag{2.4}$$

defined on the two dimensional subspace V spanned by $\{\phi_1, \phi_2\}$.

We note that if $v \geq 0$ or $v \leq 0$ then $\theta(v) \equiv 0$.

Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and $\phi_1(x) > 0$ in Ω, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 : c_1 \geq 0, |c_2| \leq kc_1\}$$

for some $k > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 : c_1 \leq 0, |c_2| \leq k|c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

We define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V. \tag{2.5}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma (cf. Lemma 2.2 of [3]).

Lemma 2.2. $\Phi(cv) = c\Phi(v)$ *for* $c \geq 0$ *and* $v \in V$.

Lemma 2.2 implies that Φ maps a cone with vertex 0 onto a cone with vertex 0. We set the cones C_2, C_4 as follows

$$C_2 = \{c_1\phi_1 + c_2\phi_2 : c_2 \geq 0,\ c_2 \geq k|c_1|\},$$

$$C_4 = \{c_1\phi_1 + c_2\phi_2 : c_2 \leq 0,\ c_2 \leq -k|c_1|\}.$$

Then the union of four cones C_i ($1 \leq i \leq 4$) is the space V.

We investigate the images of the cones C_1 and C_3 under Φ. First we consider the image of the cone C_1. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\begin{aligned}
\Phi(v) &= L(v) + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\
&= -c_1\lambda_1\phi_1 - c_2\lambda_2\phi_2 + b(c_1\phi_1 + c_2\phi_2) \\
&= c_1(b - \lambda_1)\phi_1 + c_2(b - \lambda_2)\phi_2.
\end{aligned}$$

Thus the images of the rays $c_1\phi_1 \pm kc_1\phi_2 (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b - \lambda_1)\phi_1 \pm kc_1(b - \lambda_2)\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} : d_1 \geq 0, |d_2| \leq k\left(\frac{b - \lambda_2}{b - \lambda_1}\right) d_1 \right\}.$$

The cone R_1 is in the right half-plane of V and the restriction $\Phi|_{C_1} : C_1 \to R_1$ is bijective.

We determine the image of the cone C_3. If $v = -c_1\phi_1 + c_2\phi_2 \leq 0$, we have

$$\begin{aligned}
\Phi(v) &= L(v) + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\
&= Lv + P(av) \\
&= -c_1(-\lambda_1 + a)\phi_1 + c_2(-\lambda_2 + a)\phi_2.
\end{aligned}$$

Thus the images of the rays $-c_1\phi_{00} \pm kc_1\phi_2$ ($c_1 \geq 0$) can be explicitly calculated and they are

$$-c_1(-\lambda_1 + a)\phi_1 \pm kc_1(-\lambda_2 + a)\phi_2 \qquad (c_1 \geq 0).$$

Thus Φ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 : d_1 \geq 0,\ d_2 \leq k|\frac{\lambda_2 - a}{\lambda_1 - a}||d_1| \right\}.$$

The cone R_3 is in the right half-plane of V and the restriction $\Phi|_{C_3} : C_3 \to R_3$ is bijective. We note that $R_1 \subset R_3$ since $a < \lambda_1 < \lambda_2 < b < \lambda_3$.

Theorem 2.2. *If f belongs to R_1, then equation (2.1) has a positive solution and a negative solution.*

Lemma 2.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R_2' = \left\{ d_1\phi_1 + d_2\phi_2 : d_1 \geq 0, -k\left|\frac{\lambda_2 - a}{\lambda_1 - a}\right| d_1 \leq d_2 \leq k\left|\frac{\lambda_2 - b}{\lambda_1 - b}\right| d_1 \right\},$$

$$R_4' = \left\{ d_1\phi_1 + d_2\phi_2 : d_1 \geq 0, -k\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right) d_1 \leq d_2 \leq k\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right) d_1 \right\}.$$

We note that all the cones R_2', R_3, R_4' contains R_1. R_3 contain R_1, R_2', R_4'.

To investigate a relation between multiplicity of solutions and source terms in the nonlinear equation

$$Lu + bu^+ - au^- = f \quad \text{in} \qquad H, \tag{2.6}$$

we consider the restrictions $\Phi|_{C_i}(1 \leq i \leq 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, i.e.,

$$\Phi_i : C_i \to V.$$

For $i = 1, 3$, the image of Φ_i is R_i and $\Phi_i : C_i \to R_i$ is bijective.

Lemma 2.3. *For every $v = c_1\phi_1 + c_2\phi_2$, there exists a constant $d > 0$ such that*

$$(\Phi(v), \phi_1) \geq d|c_2|.$$

For the proof see [2].

From now on, our goal is to find the image of C_i under Φ_i for $i = 2, 4$. Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ is on the ray $c_1(b - \lambda_1)\phi_1 + kc_1(b - \lambda_2)\phi_2, c_1 \geq 0$ (a boundary ray of R_1) and the image of the other end point of γ under Φ is on the ray $-c_1(-\lambda_1 + a)\phi_1 + kc_1(-\lambda_2 + a)\phi_{10}, c_1 \geq 0$ (a boundary ray of R_3). Since Φ is continuous, $\Phi(\gamma)$ is a path in V. By Lemma 2.2, $\Phi(\gamma)$ does not meet the origin. Hence the path $\Phi(\gamma)$ meets all rays (starting from the origin) in R_2'.

Therefore it follows from Lemma 2.3 that the image $\Phi(C_2)$ of C_2 contains R_2'.

Similarly, we have that the image $\Phi(C_4)$ of C_4 contains R_4'.

If a solution of (2.1) is in $IntC_1$, then it is positive. If a solution of (2.1) is in $IntC_3$, then it is negative. If it is in $Int(C_2 \cup C_4)$, then it has both signs. Therefore we have the main theorem of this section.

Theorem 2.3. *Let $a < \lambda_1 < \lambda_2 < b < \lambda_3$. Let $v = c_1\phi_1 + c_2\phi_2$. Then we have the followings.*
(i) If $f \in Int\ R_1$, then equation (2.1) has a positive solution, a negative solution, and at least two solutions changing sign.
(ii) If $f \in\ \partial R_1$, then equation (2.1) has a positive solution, a negative solution, and at least one solution changing sign.
(iii) If $f \in Int(R_3\backslash R_1)$, then equation (2.1) has a negative solution and at least one solution changing sign.
(iv) If $f \in\ \partial R_3$, then equation (2.1) has a negative solution.

Case 2) The nonlinearity crosses the eigenvalue λ_1
We suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalues λ_1, i.e., $a < \lambda_1 < b < \lambda_2$. Then it is easy to prove the following theorem.

Theorem 2.4. *Let $a < \lambda_1 < b < \lambda_2$ and $f = \alpha\phi_1$. Then we have the followings.*
(i) If $\alpha > 0$, then equation (2.1) has a positive solution and a negative solution.
(ii) If $\alpha < 0$, then equation (2.1) has no solution.

3. MULTIPLE SOLUTIONS FOR THE ELLIPTIC SYSTEM

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1\leq i,j\leq n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}),$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$. In this section we investigate the existence of solutions $u(x,t)$ for perturbations of the elliptic system with Dirichlet boundary condition

$$\begin{aligned} &L\xi + \mu g(\xi + 2\eta) = f \quad \text{in} \quad \Omega, \\ &L\eta + \nu g(\xi + 2\eta) = f \quad \text{in} \quad \Omega, \\ &\xi = 0, \eta = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

where $g(u) = Bu^+ - Au^-$, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, μ, ν are nonzero constants and the nonlinearity $(\mu + 2\nu)g(u)$ crosses the eigenvalues of the elliptic operator L.
Here we assume that $-7 < \mu - \nu < -3$.
We suppose that the nonlinearity $(\mu + 2\nu)g(u)$ crosses no eigenvalue of L, that is, $(\mu + 2\nu)A, (\mu + 2\nu)B < \lambda_1$. By the contraction mapping principle we have the following uniqueness theorem.

Theorem 3.1. *Let μ, ν be nonzero constants and $2 + \frac{\mu}{\nu} \neq 0$. Assume that $(\mu + 2\nu)A, (\mu + 2\nu)B < \lambda_1$. and $f \in H_0$. Then elliptic system (3.1) has a unique solution (ξ, η).*

Proof. From problem (3.1) we get that $L\xi - f = \frac{\mu}{\nu}(L\eta - f)$. By Theorem 2.1, for any $F \in H_0$ the problem

$$\begin{aligned} &Lu = F \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.2}$$

has a unique solution. If $u_{1-\frac{\mu}{\nu}}$ is a solution of $L(\xi - \frac{\mu}{\nu}\eta) = (1 - \frac{\mu}{\nu})f$, then the solution (ξ, η) of problem (3.1) satisfies

$$\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}. \tag{A}$$

On the other hand, from problem (3.1) we get the equation

$$\begin{aligned} &L(\xi + 2\eta) + (\mu + 2\nu)g(\xi + 2\eta) = 3f \quad \text{in} \quad \Omega, \\ &\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.3}$$

Put $w = \xi + 2\eta$. Then the above equation is equivalent to

$$\begin{aligned} &Lw + (\mu + 2\nu)g(\xi + 2\eta) = 3f \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.4}$$

When $(\mu+2\nu)A, (\mu+2\nu)B < \lambda_1$, by Theorem 2.1 the above equation has a unique solution, say w_1. Hence we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}, \\ &\xi + 2\eta = w_1. \end{aligned} \tag{3.5}$$

Since $2 + \frac{\mu}{\nu} \neq 0$, system (3.5) has a unique solution (ξ, η). □

Theorem 3.2. *Let μ, ν be nonzero constants and $2 + \frac{\mu}{\nu} \neq 0$. Assume that $\mu + 2\nu)A < \lambda_1 < \lambda_2 < (\mu + 2\nu)B < \lambda_3$ and $f = c_1\phi_1 + c_2\phi_2$. Then we have the followings.*
(i) If $f \in Int\ R_1$, then system (3.1) has a positive solution, a negative solution, and at least two solutions changing sign.
(ii) If $f \in \partial R_1$, then system (3.1) has a positive solution, a negative solution, and at least one solution changing sign.
(iii) If $f \in Int(R_3 \backslash R_1)$, then system (3.1) has a negative solution and at least one solution changing sign.
(iv) If $f \in \partial R_3$, then system (3.1) has a negative solution.

Proof. (i) From problem (3.1) we get that $L\xi - f = \frac{\mu}{\nu}(L\eta - f)$. By Theorem 2.1, for any $F \in H_0$ the problem

$$\begin{aligned} &Lu = F \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.6}$$

has a unique solution. If $u_{1-\frac{\mu}{\nu}}$ is a solution of $L(\xi - \frac{\mu}{\nu}\eta) = (1 - \frac{\mu}{\nu})f$, then the solution (ξ, η) of problem (3.1) satisfies

$$\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}. \tag{A}$$

On the other hand, from problem (3.1) we get the equation

$$\begin{aligned} &L(\xi + 2\eta) + (\mu + 2\nu)g(\xi + 2\eta) = 3f \quad \text{in} \quad \Omega, \\ &\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.7}$$

Put $w = \xi + 2\eta$. Then the above equation is equivalent to

$$\begin{aligned} &Lw + (\mu + 2\nu)g(w) = 3f \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.8}$$

where $g(w) = Bw^+ - Aw^-$ and the nonlinearity $(\mu + 2\nu)g(w)$ crosses the eigenvalues of the elliptic operator L. When $(\mu + 2\nu)A < \lambda_1 < \lambda_2 < (\mu + 2\nu)B < \lambda_3$ and $f \in Int\ R_1$, by Theorem 2.3 (i) the above equation has a positive solution w_p, a negative solution w_n, and at least two solutions changing sign w_{c_1}, w_{c_2}.

Hence we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\ &\xi + 2\eta = w_p \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\ &\xi + 2\eta = w_n \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\ &\xi + 2\eta = w_1 \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\ &\xi + 2\eta = w_1. \end{aligned} \tag{3.12}$$

Since $2 + \frac{\mu}{\nu} \neq 0$, system (3.9) has a unique solution (ξ_1, η_1) with $\xi_1 + 2\eta_1 > 0$. From (3.10) we get the solution (ξ_2, η_2) with $\xi_2 + 2\eta_2 < 0$. From (3.11), (3.12) we get the solution (ξ_3, η_3), (ξ_4, η_4), where $\xi_i + 2\eta_i (i = 1, 2)$ are changing sign.

Therefore system(3.1) has at least four solutions.

By using the similar method as in the proof of (i), we have (ii), (iii), (iv). □

We suppose that the nonlinearity $(\mu+2\nu)g(u)$ crosses the eigenvalues λ_1, i.e., $(\mu+2\nu)A < \lambda_1 < (\mu + 2\nu)B < \lambda_2$. By using the similar method as in the proof of Theorem 3.2, we have the following theorem.

Theorem 3.3. *Let μ, ν be nonzero constants and $2 + \frac{\mu}{\nu} \neq 0$. Assume that $(\mu + 2\nu)A < \lambda_1 < (\mu + 2\nu)B < \lambda_2$ and $f = \alpha\phi_1$. Then we have the followings.*
(i) If $\alpha > 0$, then system (3.1) has at least two solutions (ξ_1, η_1), (ξ_2, η_2) with $\xi_1 + 2\eta_1 > 0$, $\xi_2 + 2\eta_2 > 0$.
(ii) If $\alpha < 0$, then system (3.1) has no solution.

ACKNOWLEDGMENTS

The authors appreciate very much the referee's kind corrections.

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