ON SIZE-BIASED POISSON DISTRIBUTION AND ITS USE IN ZERO-TRUNCATED CASES

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ABSTRACT. A size-biased Poisson distribution is defined. Its characterization by using a recurrence relation for first order negative moment of the distribution is obtained. Different estimation methods for the parameter of the model are also discussed. R-Software has been used for making a comparison among the three different estimation methods.

1. INTRODUCTION

The probability function of the Poisson distribution is given as

$$P\left(X=x\right)=e^{-\alpha}\alpha^{x}/x!, \qquad x=0,1,2..... \qquad (1)$$

David and Johnson [5] defined the decapitated Poisson distribution with probability function as

$$P_{1}\left(X=x\right)=e^{-\alpha}\alpha^{x}/x!(1-e^{-\alpha}), \qquad x=1,2..... \qquad (2)$$

Murakami [9] discussed the maximum likelihood estimator of the model (2). David and Johnson [5] studied the estimator of the model (2) based on the sample moments. They also derived the maximum likelihood estimator (MLE) of α, its asymptotic variance and efficiency by the method of moments. Placket [10] put forward a similar estimate of α in order to show that it is highly efficient. Tate and Goen [12] obtained minimum variance unbiased estimation and Cohen ([3],[4]) provided the estimation of the model (2) from the sample that are truncated on the right. Ayesha and Ahmad [1] studied the inverse ascending factorial moments and the estimation of the parameter of hyper-Poisson distribution using negative moments. Munir and Roohi [8] have discussed the characterization of the Poisson distribution. A brief list of authors and their substantial works can be seen in Johnson and Kotz [6] and Johnson, Kotz and Kemp [7].

In this paper, the size-biased Poisson distribution (SBPD) is defined and the characterization of the model is obtained by using a recurrent relation for its first order negative moment. The

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estimates have been obtained by employing the moments, maximum likelihood and Bayesian method of estimation. In order to make a comparative analysis among the three estimation methods for the parameter of the size-biased Poisson distribution (SBPD), one of the standard software packages R- Software is used which is meant for data analysis and graphics. It is freely available on internet.Its resemblance with the S-PLUS software makes it more useful. (See http://cran-project.org and Bates [2]).

2. SIZE-BIASED POISSON DISTRIBUTION(SBPD)

The size-biased Poisson distribution is obtained by taking the weights of the Poisson distribution (1) as x. Then,we have $\sum_{x=0}^{\infty} xP(X = x) = \alpha$, which gives the probability function of size-biased Poisson distribution as

$$P_2\left(X = x\right) = e^{-\alpha}\alpha^{x-1}/\left(x - 1\right)!, \qquad \alpha > 0, x = 1, 2....., \tag{3}$$

The moment generating function of the distribution (3) is given by

$$M_x\left(t\right) = e^{-\alpha}e^{\left(t+e^t\alpha\right)} \tag{4}$$

By using the relation (4), the mean and variance of the distribution are given as

$$\mu_1^{/} = 1 + \alpha \tag{5}$$

$$\mu_2 = \alpha \tag{6}$$

3. RECURRENCE RELATION

In this section, we use a property of hyper-geometric series function and give an alternate method of deriving the recurrence relation for the negative moment of size-biased Poisson distribution.

Theorem 1. *: Suppose X has a size-biased Poisson distribution with parameter α , then for A>1 the relation*

$$E(X + A)^{-1} = \frac{1}{\alpha} - \frac{A}{\alpha}E(X + A - 1)^{-1} \tag{7}$$

holds.

Proof. Since X is a size-biased Poisson variate with parameter α , then

$$\begin{aligned} E(X + A)^{-1} &= \sum_{x=1}^{\infty}\frac{1}{(x + A)}P_2\left(X = x\right) \\ &= e^{-\alpha}\left(A + 1\right)^{-1}{}^1F_1\left[A + 1; A + 2; \alpha\right] \end{aligned} \tag{8}$$

where

$${}^1F_1\left[A + 1; A + 2; \alpha\right] = 1 + \frac{(A + 1)}{(A + 2)}\alpha + \frac{(A + 1)(A + 2)}{(A + 2)(A + 3)}\alpha^2.......$$

Replacing A by A-1, we get

$$E(X + A - 1)^{-1} = e^{-\alpha} A^{-1} \, {}^1F_1 [A; A + 1; \alpha] \tag{9}$$

The R.H.S of the above relation gives the negative moment for simple Poisson model (1). Using the identity (see Rainville [11] page 124)

$$b \, {}^1F_1 [a; b; x] = b \, {}^1F_1 [a - 1; b; x] + x \, {}^1F_1 [a; b + 1; x] \,,$$

for $a = A + 1, b = A + 1$,and $x = \alpha$, we get

$$(A + 1) \, {}^1F_1 [A + 1; A + 1; \alpha] = (A + 1) \, {}^1F_1 [A; A + 1; \alpha] + \alpha \, {}^1F_1 [A + 1; A + 2; \alpha] \tag{10}$$

Also,

$${}^1F_1 [A + 1; A + 1; \alpha] = e^{\alpha} \tag{11}$$

Using (8),(9) and (11) in (10), we get the result. $\square$

4. CHARACTERIZATION

In this section, the recurrence relation derived in theorem 1 is used for the characterization of the size-biased Poisson distribution.

Theorem 2. : *If X is a random variable taking the positive-integer values and the relation $E(X + A)^{-1} = \frac{1}{\alpha} - \frac{A}{\alpha} E(X + A - 1)^{-1}$ for A>1 is true, then X is characterized by a size-biased Poisson distribution.*

Proof. Since for A>1, we have

$$E(X + A)^{-1} = \frac{1}{\alpha} - \frac{A}{\alpha} E(X + A - 1)^{-1}$$

$$\sum_{x=1}^{\infty} \frac{1}{(x + A)} P_2 (X = x) = \frac{1}{\alpha} - \frac{A}{\alpha} \sum_{x=1}^{\infty} \frac{1}{(x + A - 1)} P_2 (X = x)$$

$$= \frac{1}{\alpha} - \frac{1}{\alpha} P_2 (X = 1) - \frac{A}{\alpha} \sum_{x=2}^{\infty} \frac{1}{(x + A - 1)} P_2 (X = x)$$

$$= \frac{1}{\alpha} - \frac{1}{\alpha} P_2 (X = 1) - \frac{A}{\alpha} \sum_{x=1}^{\infty} \frac{1}{(x + A)} P_2 (X = x + 1).$$

By simple computation, we get

$$\alpha \sum_{x=1}^{\infty} \frac{1}{(x + A)} P_2 (X = x) = 1 - P_2 (X = 1) - A \sum_{x=1}^{\infty} \frac{1}{(x + A)} P_2 (X = x + 1). \tag{12}$$

Since $\sum\limits_{x=1}^{\infty} P_2\left(X=x\right)=1$, which gives

$$\sum_{x=2}^{\infty} P_2\left(X=x\right)=1-P_2\left(X=1\right)=\sum_{x=1}^{\infty} P_2\left(X=x+1\right),$$

(12) becomes

$$\alpha\sum_{x=1}^{\infty}\frac{1}{(x+A)}P_2\left(X=x\right) = \sum_{x=1}^{\infty}\frac{x}{(x+A)}P_2\left(X=x+1\right),$$

$$\sum_{x=1}^{\infty}\frac{\alpha P_2\left(X=x\right)-xP_2\left(X=x+1\right)}{(x+A)} = 0.$$

Since $\alpha P_2\left(X=x\right)-xP_2\left(X=x+1\right)$ is either $>=$ or <0, then in each case, we get

$$\alpha P_2\left(X=x\right)/\left(x+A\right)=xP_2\left(X=x+1\right)/\left(x+A\right),$$

thus

$$P_2\left(X=x+1\right)=\frac{\alpha}{x}P_2\left(X=x\right).$$

Putting $x=1,2,3.....x-1$, we get

$$P_2\left(X=2\right)=\alpha P_2\left(X=1\right),$$

$$P_2\left(X=3\right)=\frac{\alpha}{2}P_2\left(X=2\right)=\frac{\alpha^2}{2!}P_2\left(X=1\right),......$$

$$P_2\left(X=x\right)=\frac{\alpha^{x-1}P_2\left(X=1\right)}{(x-1)!}$$

From equation(3) for $x=1$, $P_2\left(X=1\right)=e^{-\alpha}$.
Therefore,

$$P_2\left(X=x\right)=\alpha^{x-1}e^{-\alpha}/\left(x-1\right)!,$$

which is the probability function of size-biased Poisson distribution. This completes the proof.

$\square$

5. ESTIMATION METHODS

In this section, we discuss the basic three estimation methods for the parameter of the size-biased Poisson distribution and verify their efficiencies.

5.1. **METHOD OF MOMENTS.** In the method of moments, replacing the population mean $\mu_1^{/}=1+\alpha$ by the corresponding sample mean $\overline{x}=\frac{1}{n}\sum\limits_{i=1}^{n}x_i$, we get

$$\widehat{\alpha}=\overline{x}-1 \tag{13}$$

5.2. **METHOD OF MAXIMUM LIKELIHOOD.** Let X_1, X_2......X_n be a random sample from the size-biased Poisson distribution, then the corresponding likelihood function is given as

$$L \quad = \quad e^{-n\alpha}\alpha^{\sum\limits_{i=1}^{n} x_i - n} \Big/ \prod_{i=1}^{n} (x_i - 1)!, \qquad x = 1, 2... \tag{14}$$

$$= \quad e^{-n\alpha}\alpha^{y-n} \Big/ \prod_{i=1}^{n} (x_i - 1)!, \text{ where } y = \sum_{i=1}^{n} x_i. \tag{15}$$

The log likelihood function of (15) can be written as

$$\log L = -n\alpha + (y - n)\log\alpha - \sum_{i=1}^{n} \log((x_i - 1)!)$$

The corresponding likelihood equation is given as

$$\frac{\partial \log L}{\partial \alpha} = -n + \frac{(y - n)}{\alpha}$$

On equating the above derivative equal to zero,we get the maximum likelihood estimate as

$$\widehat{\alpha} = \overline{x} - 1.$$

This coincides with the moment estimate.

5.3. **BAYESIAN METHOD OF ESTIMATION.** We assume that before the observations were made, our knowledge about the parameter α was only a vague one. Consequently, the non-informative vague prior $g(\alpha)$ proportional to $\frac{1}{\alpha}$ is applicable to a good approximation. Thus

$$g(\alpha) = \frac{1}{\alpha}, \alpha > 0. \tag{16}$$

The posterior distribution from (15) and (16) is given as

$$\Pi\left(\alpha/y\right) = \frac{\alpha^{y-n-1}\, e^{-n\alpha}}{\int\limits_{0}^{\infty} \alpha^{y-n-1}\, e^{-n\alpha} d\alpha}$$

The Bayes estimator of α becomes

$$\widetilde{\alpha} \;=\; \int\limits_{0}^{\infty} \alpha \Pi \left(\alpha/y \right) d\alpha$$

$$= \; \frac{\int\limits_{0}^{\infty} \alpha^{y-n} \, e^{-n\alpha} d\alpha}{\int\limits_{0}^{\infty} \alpha^{y-n-1} \, e^{-n\alpha} d\alpha}$$

$$= \; y - n/n = \overline{x} - 1,$$

which coincides with mle and moment estimate.

In order to find out the more general estimate of α, we consider the more general prior of α which is given by the gamma distribution with known hyper-parameters a, b>0 having the density function as

$$f\left(\alpha \right) = a^{b} e^{-a\alpha} \alpha^{b-1} / \Gamma b; \alpha \geq 0 \qquad (17)$$

Using (15) and (17), the Bayes estimator of α comes out to be

$$\widetilde{\alpha} = y + b - n/n + a \qquad (18)$$

For a=b=0, the estimator coincides with mle and moment estimate. This shows that the Baye's estimate $\widetilde{\alpha}$ serves as a general estimate which can be used for fitting purposes to a real life data.

6. NUMERICAL EXPERIMENTS AND DISCUSSIONS

It is very difficult to compare the theoretical performances of different estimators proposed in the previous section. Therefore, we perform extensive simulations to compare the performances of different methods of estimation mainly with respect to their biases and the mean squared errors (MSE's), for different sample sizes and different parametric values. Regarding the choice of values of (a, b) in Baye's estimator $(\widetilde{\alpha})$, there was no information about their values except that they are real and positive numbers. Therefore, 25 combinations of values of (a, b) were considered for $a, b = 1, 2, 3, 4, 5$ and those values of a, b were selected for which the Baye's estimator has minimum variance. It was found that for a=b=2, the Baye's estimator has minimum variance and χ^2 values between the simulated sample frequencies and the estimated Baye's frequencies were least.

6.1. **AVERAGE RELATIVE ESTIMATES AND AVERAGE RELATIVE MEAN SQUARED ERRORS OF** α**.** For the sample sizes $n = 15, 20, 30, 50, 100$ and different values of the parameter $\alpha = 0.5, 1.0, 2.0, 2.5$ and for each combination of n and α, we generate a sample of size n from the size-biased Poisson distribution and estimate α by different methods of estimation. We report the average values of $\left(\frac{\widehat{\alpha}}{\alpha} \right)$ and the corresponding average MSE's within brackets.. All the reported results are based on $10,000$ replications. The results are presented

in table 1. From the table it is clear that the average biases and the average MSE's decrease as sample size increases. It indicates that all the methods of estimation provide the asymptotically unbiased and the consistent estimators. It is also observed that the average biases and the average MSE's of $\left(\frac{\widehat{\alpha}}{\alpha}\right)$ depend on α. On comparing the performances of all the methods, it is clear that as far as the minimum bias is concerned the Baye's works the best in almost all the cases.

TABLE 1. AVERAGE RELATIVE ESTIMATES AND AVERAGE RELATIVE MEAN SQUARED ERRORS OF α

n	Method	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 2.0$	$\alpha = 2.5$
15	$Baye's$	1.180(0.230)	1.221(0.435)	1.331(0.857)	1.356(0.97)
	MLE	1.364(0.678)	1.383(0.871)	1.455(1.624)	1.465(1.4166)
20	$Baye's$	1.132(0.141)	1.161(1.88)	1.223(0.343)	1.244(0.424)
	MLE	1.314(0.448)	1.285(0.479)	1.317(0.675)	1.338(0.791)
30	$Baye's$	1.084(0.075)	1.100(0.100)	1.129(0146)	1.145(0.178)
	MLE	1.218(0.248)	1.191(0.242)	1.197(0.293)	1.213(0.357)
50	$Baye's$	1.048(0.038)	1.054(0.045)	1.077(0.068)	1.082(0.077)
	MLE	1.134(0.131)	1.112(0.117)	1.123(0.145)	1.125(0.157)
100	$Baye's$	1.022(0.016)	1.027(0.020)	1.035(0.027)	1.038(0.030)
	MLE	1.065(0.060)	1.056(0.053)	1.058(0.061)	1.061(0.066)

6.2. **FITTING OF SIZE-BIASED POISSON DISTRIBUTION MODEL.** The two different varieties of Mulberry Ichinose and Kokuso-20 having different leaf spot disease intensity were chosen for the study in a local Kashmir Sericulture division. Three trees of each variety were selected at random. From each tree, three branches were selected randomly and then from each branch, the spots were recorded from all the leaves. The leaves with no spot were referred as disease free and named as grade zero (0 grade). The leaves having 1 to 5, 6 to 10, 11 to 15, 16 to 20 and more than 20 spots were graded as 1,2,3,4 and 5 grades respectively. In our study, the leaves of zero grades were not found. The data for two varieties of Mulberry Ichinose and Kokuso-20 are listed in tables 2 and 3, respectively. A comparison is made between different methods of estimation for the parameter of the size-biased Poisson distribution and it was found that the Baye's estimator constitutes a better fit against MLE or moment estimator.

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TABLE 2

| Leaf Spot Grade | Obseved Frquency | Expected Frequency | |
		MLE $\widehat{\alpha}$	Baye's $\widetilde{\alpha}$
1	18	17.5	17.9
2	15	14.7	14.95
3	10	9.86	9.91
4	14	13.94	13.99
5	13	14	13.25
Total	70	70	70
χ^2		0.094	0.0063

TABLE 3

| Leaf Spot Grade | Obseved Frquency | Expected Frequency | |
		MLE $\widehat{\alpha}$	Baye's $\widetilde{\alpha}$
1	37	36.42	36.92
2	16	15.92	15.97
3	15	14.93	14.96
4	8	7.64	7.91
5	8	9.09	8.24
Total	84	84	84
χ^2		0.142	0.0083

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