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A FINITE ELEMENT METHOD USING SINGULAR FUNCTIONS FOR HELMHOLTZ EQUATIONS: PART I

SEOKCHAN KIM[1], JAE-HONG PYO[2†], AND JONGSIK LEE[1]

[1]DEPARTMENT OF APPLIED MATHEMATICS, CHANGWON NATIONAL UNIVERSITY, CHANGWON, 641-773, KOREA
E-mail address: sckim@sarim.changwon.ac.kr, changzo@hanmail.net

[2]DEPARTMENT OF MATHEMATICS, KANGWON NATIONAL UNIVERSITY, CHUNCHEON, 200-701, KOREA
E-mail address: jhpyo@kangwon.ac.kr

ABSTRACT. In [7, 8], they proposed a new singular function(NSF) method to compute singular solutions of Poisson equations on a polygonal domain with re-entrant angles. Singularities are eliminated and only the regular part of the solution that is in H^2 is computed. The stress intensity factor and the solution can be computed as a post processing step. This method was extended to the interface problem and Poisson equations with the mixed boundary condition. In this paper, we give NSF method for the Helmholtz equations $-\Delta u + Ku = f$ with homogeneous Dirichlet boundary condition. Examples with a singular point are given with numerical results.

1. INTRODUCTION

Assume that $\Omega \subset \mathcal{R}^2$ is an open, bounded polygonal domain with one reenterent corner. For a given function $f \in L^2(\Omega)$, consider the Helmholtz equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} Lu := -\Delta u + Ku &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma = \partial\Omega, \end{cases} \tag{1}$$

where Δ stands for the Laplacian operator. Solution of (1) has singular behavior near corners even when f is very smooth. Such corner singularity affects the accuracy of the finite element method throughout the whole domain. There are two main approaches to overcome this deficit. One is the method of local grid refinement (see, e.g., [1, 2]). This method also has the advantage that it does not require the knowledge of the exact forms of the singular functions. It only needs

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† Corresponding author.

the knowledge of the *exponents* of the singular functions. Another approaches are the so-called Singular Function Method (SFM) (see, e.g., [13]) and Dual Singular Function Method (DSFM) (see, e.g., [11, 5, 6, 12]). Both methods are based on the fact that solutions of the boundary value problems on polygonal domains have a singular function representation: a linear combination of singular functions and the regular part of the solution. Coefficients of singular functions in this representation are called the stress intensity factors.

Recently in [7] they used this property in order to calculate accurate finite element approximations to both the solution and the stress intensity factors. The loss of standard finite element approximation accuracy for elliptic boundary value problems with corner singularities is due to the non-smoothness of the solution. Therefore, it is natural to try to approximate first the regular part of the solution, and then compute the stress intensity factors and the solution. In [7], they considered the Poisson Problem and, using the dual singular functions and a particularly chosen cut-off function, deduced a well-posed variational problem for the regular part of the solution.

The purpose of this paper is to extend results for the Poisson equations with Dirichlet boundary conditions in [7] to the Helmholtz problem with homogeneous Dirichlet boundary conditions.

It can be easily checked that the singular function decomposition for Helmholtz problem is the same as that for Poisson problem. A little modified type of extraction form for Helmholtz problem is given in section 2. A detail proof of the existence and uniqueness of the variational problem for the regular part is given in section 3. In section 4, we introduce a finite element approximation and estimate its error bound. Finally, in section 5, we present the numerical results for examples with a singular point.

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively. $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$.

2. SINGULAR FUNCTION REPRESENTATIONS

Let ω be internal angles of Ω satisfying $\pi < \omega < 2\pi$ and denote by $\mathbf{v}$ the corresponding vertices. Let the polar co-ordinates $(r,\,\theta)$ be chosen at the vertex $\mathbf{v}$ so that the internal angle ω is spanned *counterclockwise* by two half-lines $\theta = 0$ and $\theta = \omega$. Let the singular function $s(r,\,\theta)$ and its dual singular function $s_-(r,\,\theta)$ to be defined as

$$s(r,\,\theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega} \tag{2}$$

and

$$s_-(r,\theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}. \tag{3}$$

Now we set

$$B(t_1;t_2) = \{(r,\theta_j) :\ t_1 < r < t_2 \text{ and } 0 < \theta_j < \omega\} \cap \Omega \quad \text{and} \quad B(t_1) = B(0;t_1),$$

and define a family of cut-off functions of r, $\eta_\rho(r)$, as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\frac{1}{2}\rho R), \\ \frac{1}{32}\left\{16 - 35p_\rho(r) + 35p_\rho(r)^3 - 21p_\rho(r)^5 + 5p_\rho(r)^7\right\} & \text{in } \bar{B}(\frac{1}{2}\rho R;\, \rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases}$$

with $p_\rho(r) = \frac{4r}{\rho R} - 3$ where ρ is a parameter in $(0,\, 2]$ and $R \in \mathcal{R}$ is a fixed number so that the $\eta_2 s$ has the same boundary condition as u.

The singular function representation for the solution of Poisson problem is well known ([3, 10, 11, 14]). Similarly we can show that the solution of (1) also has the following singular function representation([14, 16]):

$$u = w + \lambda\, \eta_\rho s(r,\, \theta) \tag{4}$$

where $w \in H^2(\Omega) \bigcap H_0^1(\Omega)$ is the regular part of the solution and $\lambda \in \mathcal{R}$ are the stress intensity factors that can be expressed in terms of w by the following Lemma.

Lemma 2.1. *There is a cut-off function η_ρ such that*

$$\lambda = \frac{1}{B}(f, \eta_2 s_-) + \frac{1}{B}(w, -L(\eta_2 s_-)), \tag{5}$$

with $B := \pi + K(\eta_\rho s, s_-) \neq 0$.

Proof. Multiplying $\eta_2 s_-$ to (1) and integrating gives

$$-(\Delta w, \eta_2 s_-) - \lambda(\Delta \eta_\rho, \eta_2 s_-) + (Kw, \eta_2 s_-) + \lambda(K\eta_\rho s, \eta_2 s_-) = (f, \eta_2 s_-),$$

which implies the lemma with the facts $(\Delta\eta_\rho, \eta_2 s_-) = -\pi$ and $(\Delta w, \eta_2 s_-) = (w, \Delta\eta_2 s_-)$ (see [7]). Here we note that ρ can be chosen so that $B = \pi + K(\eta_\rho s, s_-) = \pi + K \cdot \frac{41}{288}\omega(\rho R)^2 \neq 0$. □

Moreover, the following regularity estimate holds:

$$\|w\|_2 + |\lambda| \leq C_R \|f\|. \tag{6}$$

In the remainder of this section, we derive a well-posed problem for w. Using (4) and substituting (5) into the Helmholtz equation, we obtain an integro-differential equation for w:

$$Lw - \frac{1}{B}(w, -L(\eta_2 s_-))_{B(2R)} L(\eta_\rho s) = f - \frac{1}{B}(f, \eta_2 s_-)_{B(2R)} L(\eta_\rho s) \quad \text{in } \Omega.$$

Multiplying the above equation by a test function $v \in H_0^1(\Omega) = \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma\}$, integrating over Ω, and using integration by parts lead to the following variational problem: finding $w \in H^2(\Omega) \bigcap H_0^1(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall v \in H_0^1(\Omega), \tag{7}$$

where the bilinear form $a(\cdot,\cdot)$ and linear form $g(\cdot)$ are defined by

$$\begin{aligned} a(w,v) &= a^s(w,v)+b(w,v), \quad a^s(w,v)=(\nabla w,\nabla v)+K(w,v),\\ b(w,v) &= -\tfrac{1}{B}\,(w,L(\eta_2 s_-))_{B(2R)}(L(\eta_\rho s),v)_{B(\rho R)}, \end{aligned} \tag{8}$$

and

$$g(v)=(f,v)-\frac{1}{B}(f,\eta_2 s_-)_{B(2R)}(L(\eta_\rho s),v)_{B(\rho R)}. \tag{9}$$

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that $w\in H^2(\Omega)$ for $f\in L^2(\Omega)$. Note also that the bilinear forms $a(\cdot,\cdot)$ are not symmetric.

3. WELL-POSEDNESS

Lemma 3.1. *For any $0<\rho$, there are $C_i, i=1,2,3,$ such that*

$$\|\eta_\rho s\|_{B(\rho R)}\le C_1(\rho R)^{\frac{\pi}{\omega}+1},\quad \|\eta_2 s_-\|_{B(2R)}\le C_1(2R)^{-\frac{\pi}{\omega}+1},\quad \|\eta_\rho s\|_1\le C_2(\rho R)^{\frac{\pi}{\omega}}, \tag{10}$$

$$\|\Delta(\eta_\rho s)\|_{B(\frac{\rho R}{2};\rho R)}\le C_3(\rho R)^{\frac{\pi}{\omega}-1}\quad \textit{and}\quad \|\Delta(\eta_2 s_-)\|_{B(R;2R)}\le C_3(2R)^{-\frac{\pi}{\omega}-1}. \tag{11}$$

Proof. This lemma can be established by an elementary calculation. □

Lemma 3.2.

$$\|L(\eta_\rho s)\|\le C_3(\rho R)^{\frac{\pi}{\omega}-1}+|K|C_1(\rho R)^{\frac{\pi}{\omega}+1}\le C_4 \tag{12}$$

and

$$\|L(\eta_2 s_-)\|\le C_3(2R)^{-\frac{\pi}{\omega}-1}+|K|C_1(2R)^{-\frac{\pi}{\omega}+1}\le C_5. \tag{13}$$

Note that the L^2-norms of $L(\eta_\rho s)$ and $L(\eta_2 s_-)$ are bounded for fixed value of $\rho>0$ and that C_4 increases to infinity as ρ approaches to 0. In computation, we will choose $\frac{1}{2}<\rho\le 1$ so that the coefficients C_4 and C_5 are bounded numbers. We will need the following well-known Poincaré-Friedrichs inequality:

$$\|v\|\le C_\Omega\|\nabla v\| \qquad \forall v\in H_0^1(\Omega),$$

where C_Ω is a positive constant depending only on the domain Ω.

Now, in a similar fashion as in [7], we can prove the coercivity and continuity of the bilinear form $a(\cdot,\cdot)$ and the well posed-ness of problem (7).

Lemma 3.3. *For $0<\rho\le 1$, the bilinear forms $a(\cdot,\cdot)$ are continuous and coercive in $H_0^1(\Omega)$; i.e. there exist positive constants α, K_1, and K_2 such that*

$$\alpha\|\phi\|_1^2\le a(\phi,\phi)+K_1\|\phi\|^2 \tag{14}$$

for all $\phi\in H_0^1(\Omega)$ and that

$$a(\phi,\psi)\le K_2\|\phi\|_1\|\psi\|_1 \tag{15}$$

for all ϕ and ψ in $H_0^1(\Omega)$.

Proof. By the Cauchy-Schwarz inequality and Lemma 3.2, we have

$$|-\frac{1}{B}\,(\phi, L(\eta_2 s_-))_{B(R;2R)}(L(\eta_\rho s), \psi)_{B(\frac{1}{2}\rho R;\,\rho R)}| \leq \frac{C_4C_5}{B}\|\phi\|\|\psi\|. \tag{16}$$

The inequality (15) is an immediate consequence of the Cauchy-Schwarz inequality and (16) with $K_2 = 1 + C_4C_5/B$. By using (16) with $\psi = \phi$, we have

$$a(\phi,\phi) \geq \|\nabla\phi\|^2 + K\|\phi\|^2 - \frac{C_4C_5}{B}\|\phi\|^2,$$

which gives (14) together with the Poincaré-Friedrichs inequality with $\alpha = 1/(1 + C_\Omega^2)$ and $K_1 = C_4C_5/B - K$. □

To get the the well-posedness of our variational problem (7), we will make use of the Fredholm alternative (see, eq., [15]). For this, consider the following bilinear form:

$$a_\mu(w,v) = a(w,v) + \mu(w,v)$$

for $\mu > 0$.

Theorem 3.1. *For* $0 < \rho \leq 1$*, we have that*
(1) *if* $f \in L^2(\Omega)$*, then problem* (7) *has a unique solution* $w \in H^2(\Omega)\bigcap H_0^1(\Omega)$.
(2) *there exists a positive constant* γ *such that*

$$\gamma\|\phi\|_1 \leq \sup_{\psi\in H_0^1(\Omega)} \frac{a(\phi,\psi)}{\|\psi\|_1} \tag{17}$$

for any $\phi \in H_D^1(\Omega)$.

Proof. Let $T_\mu : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the corresponding operator of the bilinear form $a_\mu(\cdot,\cdot)$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ with the standard dual norm denoted by $\|\cdot\|_{-1}$. First we will see that $T_0w = 0$ has only trivial solution, then the rest parts of the proof will follow in a similar way as in [7].

Note $T_0w = 0$ in $H^{-1}(\Omega)$ means that $\forall v \in H_0^1$,

$$0 = (T_0w, v) = a(w,v) = (\nabla w, \nabla v) + K(w,v) - \frac{1}{B}\,(w, L(\eta_2 s_-))(L(\eta_\rho s), v),$$

which implies

$$-\Delta w + Kw + \frac{1}{B}\,(w, L(\eta_2 s_-))\Delta(\eta_\rho s) - \frac{1}{B}\,(w, L(\eta_2 s_-))K\eta_\rho s \equiv 0 \text{ in } \overline{\Omega},$$

since it satisfies the Poisson equation with zero data, or equivalently,

$$-\Delta(w - \frac{1}{B}\,(w, L(\eta_2 s_-))\eta_\rho s) + K(w - \frac{1}{B}\,(w, L(\eta_2 s_-))\eta_\rho s) \equiv 0 \text{ in } \overline{\Omega}.$$

Since $w - \frac{1}{B}\,(w, L(\eta_2 s_-))\eta_\rho s$ satisfies the Helmholtz equation with input function $f = 0$ and the homogeneous Dirichlet boundary condition, we observe that w is a scalar multiple of $\eta_\rho s$, or $w = c_0\eta_\rho s$, with $c_0 = \frac{1}{B}(w, L(\eta_2 s_-))$. Thus, by the definition of the cut-off function and the constant B (see Lemma 5) we have

$$\begin{aligned} c_0\eta_\rho s &= w = \tfrac{1}{B}(w, L(\eta_2 s_-))\eta_\rho s = \tfrac{c_0}{B}(\eta_\rho s, -\Delta(\eta_2 s_-))\eta_\rho s + \tfrac{c_0}{B}(\eta_\rho s, K\eta_2 s_-)\eta_\rho s \\ &= 0 + \tfrac{c_0}{B}K(\eta_\rho s, s_-)\eta_\rho s = \tfrac{c_0}{B}(B-\pi)\eta_\rho s = c_0\eta_\rho s - \tfrac{c_0\pi}{B}\eta_\rho s, \end{aligned} \tag{18}$$

which implies $c_0 = 0$, i.e., $w \equiv 0$, the trivial solution. Now, using similar methods in [7] with the above result, we have the theorem. □

Corollary 3.1. *Let w and λ be the solution of* (7) *and the stress intensity factors defined in* (5)*, respectively. For* $0 < \rho \le 1$,

$$u = w + \lambda\eta_\rho s \tag{19}$$

is the solution of (1).

4. FINITE ELEMENT APPROXIMATION

This section presents standard finite element approximation on a quasi-uniform grid for w based on the variational problem in (7). Approximations to the stress intensity factors and the solution of problem (1) can then be calculated according to (5) and (4), respectively. Error estimates are established in Theorem 4.1.

Let $\mathcal{T}_h$ be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \bigcup_{K\in\mathcal{T}_h} K$ with $h = \max\{\text{diam}\, K : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is regular. Denote continuous piecewise linear finite element space by

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \text{ is linear } \forall\ K \in \mathcal{T}_h \text{ and } \phi_h = 0 \text{ on } \Gamma\} \subset H_0^1(\Omega).$$

It is well known that

$$\inf_{\phi_h\in V_h}(\|\phi - \phi_h\| + h|\phi - \phi_h|_1) \le C_A h^{1+t}\|\phi\|_{1+t,\Omega} \tag{20}$$

for any $\phi \in H_0^1(\Omega)\bigcap H^{1+t}(\Omega)$ and $0 \le t \le 1$. The finite element approximation to problem (7) is to find $w_h \in V_h$ such that

$$a(w_h, v) = g(v) \qquad \forall\ v \in V_h. \tag{21}$$

Approximations to the λ and the solution are calculated as follows:

$$\lambda^h = \frac{1}{B}(f, \eta_2 s_-)_{B(2R)} + \frac{1}{B}(w_h, -L(\eta_2 s_-))_{B(2R)} \tag{22}$$

and

$$u_h = w_h + \lambda^h\,\eta_\rho s. \tag{23}$$

In order to establish the error bound in the L^2 norm, we consider the following adjoint problem of (7) with a simplified linear form: find $z \in H_0^1(\Omega)$ such that

$$a(v, z) = (w - w_h, v) \quad \forall v \in H_0^1(\Omega). \tag{24}$$

The next lemma establishes the well-posedness of problem (24) and provides the regularity estimate for z.

Lemma 4.1. *For $0 < \rho \le 1$, problem* (24) *has a unique solution z in $H_0^1(\Omega)$. Moreover, there is a singular function representation*

$$z = w_z + \lambda^z \eta_\rho s, \tag{25}$$

where $w_z \in H^2(\Omega) \bigcap H_0^1(\Omega)$ and $\lambda^z \in R$ satisfy the regularity estimate

$$\|w_z\|_2 + |\lambda^z| \le C_R' \|w - w_h\|. \tag{26}$$

Proof. Similar to Theorem 3.1, the adjoint problem in (24) has a unique solution in $H_0^1(\Omega)$ and that there exists a positive constant γ' such that

$$\gamma' \|\psi\|_1 \le \sup_{\phi \in H_0^1(\Omega)} \frac{a(\phi,\, \psi)}{\|\phi\|_1} \quad \forall\, \psi \in H_0^1(\Omega).$$

Let z be the solution of (24), by the Cauchy-Schwarz inequality we then have that

$$\|z\|_1 \le \frac{1}{\gamma'} \sup_{\phi \in H_0^1(\Omega)} \frac{a(\phi,\, z)}{\|\phi\|_1} = \frac{1}{\gamma'} \sup_{\phi \in H_0^1(\Omega)} \frac{(w - w_h,\, \phi)}{\|\phi\|_1} \le \frac{1}{\gamma'} \|w - w_h\|. \tag{27}$$

It is easy to check that the solution, $z \in H_0^1(\Omega)$, of problem (24) satisfies

$$\Delta z = Kz - \frac{1}{B}(L(\eta_\rho s), z) L(\eta_2 s_-) - (w - w_h) \quad \text{in } \Omega. \tag{28}$$

Since the right-hand side of the above equation is at least in $L^2(\Omega)$, so is Δz. Therefore, z has the singular function representation

$$z = w_z + \lambda^z \, \eta_\rho \, s,$$

where $w_z \in H^2(\Omega) \bigcap H_0^1$ and

$$\|w_z\|_2 + |\lambda^z| \le C_R \|\Delta z\|.$$

Now, the regularity bound in (26) follows from the triangle and Cauchy-Schwarz inequalities, (27), and Lemma 3.1 that

$$\begin{aligned} \|w_z\|_2 \;+\; & |\lambda^z| \le C_R \|\Delta z\| \\ \le\; & C_R(|K| \|z\| + \tfrac{1}{|B|} |(L(\eta_\rho s), z)_{B(\rho R)}| \cdot \|L(\eta_2 s_-)\|_{B(2R)} + \|w - w_h\|) \\ \le\; & C_R(\tfrac{|K|}{\gamma'} + \tfrac{C_4 C_5}{|B|\gamma'} + 1) \|w - w_h\|. \end{aligned}$$

This proves the inequality in (26) with

$$C_R' = C_R(\frac{|K|}{\gamma'} + \frac{C_4 C_5}{|B|\gamma'} + 1)$$

and, hence, the lemma.

Now we are ready to establish error bounds for the finite element approximations.

Theorem 4.1. (i) *For $0 < \rho \leq 1$, there exists a positive constant h_0 such that for all $h \leq h_0$* (21) *has a unique solution w_h in V_h. Moreover, let $w \in H^2(\Omega) \bigcap H_0^1(\Omega)$ be the solution of* (7)*, then we have the following error estimates:*

$$\|w - w_h\|_1 \leq C_6 h\|f\| \quad \textit{and} \quad \|w - w_h\| \leq C_7 h^{1+\frac{\pi}{\omega}}\|f\|. \tag{29}$$

(ii) *Let λ and λ^h be defined in* (5) *and* (22)*, respectively. Then*

$$|\lambda - \lambda^h| \leq \frac{C}{\pi} R^{-\frac{\pi}{\omega}-1}\|w - w_h\| \leq C_8 R^{-\frac{\pi}{\omega}-1} h^{1+\frac{\pi}{\omega}}\|f\|. \tag{30}$$

(iii) *Let u be the solution of* (1) *and u_h be its approximation defined in* (23)*, then we have the following error estimates:*

$$\|u - u_h\|_1 \leq C_9 h\|f\| \quad \textit{and} \quad \|u - u_h\| \leq C_{10} h^{1+\frac{\pi}{\omega}}\|f\|. \tag{31}$$

Proof. (i) We first establish error bounds in (29) for any solution to problem (21) that may exist. Then, for $f \equiv 0$, the uniqueness of the solution to problem (7) and the error bound in (29) imply that $w_h \equiv 0$. Hence, (21) has a unique solution w_h in V_h since it is a finite dimensional problem with the same number of unknowns and equations.

To establish error bounds, note first the orthogonality property

$$a(w - w_h,\, v) = 0 \quad \forall\, v \in V_h. \tag{32}$$

By choosing $v = w - w_h$ in (24) and using the orthogonality property in (32) and the continuity bound in (15), we have that

$$\|w - w_h\|^2 = a(w - w_h,\, z) = a(w - w_h,\, z - I_h z) \leq K_2\|w - w_h\|_1\|z - I_h z\|_1, \tag{33}$$

where $I_h z \in V_h$ is the nodal interpolant of z. From the triangle inequality, approximation property (20), the fact that (see [4])

$$\|\eta_\rho s - I_h(\eta_\rho s)\|_1 \leq C h^{\frac{\pi}{\omega}},$$

and Lemma 4.1, one has

$$\begin{aligned} \|z - I_h z\|_1 &\leq \|w_z - I_h w_z\|_1 + |\lambda^z|\; \|\eta_\rho s - I_h(\eta_\rho s)\|_1 \\ &\leq C h\|w_z\|_2 + C h^{\frac{\pi}{\omega}}|\lambda^z| \leq C_D h^{\frac{\pi}{\omega}}\|w - w_h\|. \end{aligned}$$

Substituting this into (33) and dividing $\|w - w_h\|$ on both sides give

$$\|w - w_h\| \leq K_2 C_D h^{\frac{\pi}{\omega}}\|w - w_h\|_1. \tag{34}$$

Now, it follows Lemma 3.3, orthogonality property (32), and inequality (34) that for any $v \in V_h$

$$\begin{aligned} \alpha\|w - w_h\|_1^2 &\leq a(w - w_h,\, w - w_h) + K_1\|w - w_h\|^2 \\ &= a(w - w_h,\, w - v) + K_1\|w - w_h\|^2 \\ &\leq K_2\|w - w_h\|_1\|w - v\|_1 + K_1(K_2 C_D h^{\frac{\pi}{\omega}})^2\|w - w_h\|_1^2, \end{aligned}$$

which, together with approximation property (20), implies the validity of the first error bound in (29) with $C_6 = 2\alpha^{-1}K_2C_AC_R$ for all $h \le h_0$. Here,

$$h_0 = (\frac{\alpha}{2K_1(K_2C_D)^2})^{\frac{\omega}{2\pi}}.$$

The second error bound in (29) is then a direct consequence of (34) with $C_7 = C_6K_2C_D$.

(ii) Note from (5) and (22) that

$$\lambda - \lambda^h = \frac{1}{B}(w - w_h,\, -L(\eta_2 s_-))_{B(2R)}.$$

Hence, (30) follows from the Cauchy-Schwarz inequality, Theorem 4.1(i), and Lemma 3.1 that

$$|\lambda - \lambda^h| \le \frac{1}{|B|}\|w - w_h\| \cdot \| - L(\eta_2 s_-)\|_{B(2R)} \le C_8 h^{1+\frac{\pi}{\omega}}\|f\|$$

with $C_8 = \frac{C_7C_5}{|B|}$.

(iii) It follows from (4) and (23) that

$$u - u_h = (w - w_h) + (\lambda - \lambda^h)\eta_\rho s.$$

By using the triangle inequality, Lemma 3.1, (29), and (30), we have that

$$\begin{aligned}\|u - u_h\|_1 &\le \|w - w_h\|_1 + |\lambda - \lambda^h|\; \|\eta_\rho s\|_{1)} \\ &\le C_6 h\|f\| + C_8(\rho R)^{\frac{\pi}{\omega}} C_2 h^{1+\frac{\pi}{\omega}}\|f\|.\end{aligned}$$

Therefore, the first inequality of (31) is valid with $C_9 = C_6 + C_8(\rho R)^{\frac{\pi}{\omega}} C_2 h^{\frac{\pi}{\omega}}$. In a similar fashion, by Lemma 3.1, (29), and (30), we may prove the validity of the second inequality of (31) with $C_{10} = C_7 + C_8C_1(\rho R)^{1+\frac{\pi}{\omega}}$. This completes the proof of the theorem.

5. NUMERICAL RESULTS

In this section, we carry out several numerical experiments with known solution to check performance of the algorithm (7) for the Helmholtz equations.

Example 1.[Dirichlet boundary problem on Γ shape domain] We consider a polygonal Γ shape domain $([-1,1]\times[-1,1]) - ([0,1]\times[-1,0])$. Then the equation (1) has singular function of the form

$$s = r^{\frac{2}{3}} sin(\frac{2}{3}\theta).$$

If we choose forcing function f to be exact solution $u = \eta_4 s$, then the solution u can be rewritten by

$$u = w + \lambda\eta_1 s$$

with $w = (\eta_4 - \eta_1)s \in H_0^2(\Omega)$ and $\lambda = 1$. We fix $R = \frac{1}{8}$ and note numerical results are not depend on $R < 1$. As we proved in previous sections, the algorithm (8) and (9) displays optimal error decays on Table 1 for $K = 1$ and on Table 2 for $K = -100$.

	1/32	1/64	1/128	1/256	1/512	1/1024
$\|u-u_h\|_0$	0.00200058	0.000478165	0.000122343	3.06834e-05	7.67792e-06	1.92022e-06
	Order	2.064838	1.966577	1.995401	1.998671	1.999444
$\|u-u_h\|_\infty$	0.0114182	0.00320753	0.000853718	0.000218068	5.45498e-05	1.36581e-05
	Order	1.831801	1.909631	1.968982	1.999132	1.997817
$\|u-u_h\|_1$	0.261026	0.126853	0.0643733	0.0322795	0.0161514	0.00807715
	Order	1.041036	0.978623	0.995844	0.998959	0.999741
$\|\lambda-\lambda_h\|$	0.00697096	0.00126885	6.2799e-05	2.75485e-06	1.05483e-06	1.11547e-07
	Order	2.457836	4.336636	4.510696	1.384963	3.241287

TABLE 1. The case $\rho = 1$ and $R = 0.125$ on Γ shape domain with $K = 1$

	1/32	1/64	1/128	1/256	1/512	1/1024
$\|u-u_h\|_0$	0.00483411	0.000907137	0.000282004	6.99397e-05	1.73996e-05	4.378e-06
	Order	2.413858	1.685605	2.011532	2.007057	1.990710
$\|u-u_h\|_\infty$	0.0114836	0.00319709	0.000849624	0.000216805	5.42273e-05	1.35786e-05
	Order	1.844744	1.911863	1.970426	1.999307	1.997685
$\|u-u_h\|_1$	0.264778	0.127094	0.0644238	0.0322856	0.0161521	0.00807725
	Order	1.058887	0.980230	0.996703	0.999169	0.999786
$\|\lambda-\lambda_h\|$	0.0285257	0.00656459	0.00103929	0.000238555	6.04036e-05	1.47783e-05
	Order	2.119485	2.659107	2.123204	1.981616	2.031154

TABLE 2. The case $\rho = 1$ and $R = 0.125$ on Γ shape domain with $K = -100$

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