ENERGY OF PLANAR GRAPHS

S. PIRZADA[†]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KASHMIR, SRINAGAR, INDIA
E-MAIL address: sdpirzada@yahoo.co.in

ABSTRACT. The energy of a graph is the sum of the absolute values of its eigen values. We obtain some bounds for the energy of planar graphs in terms of its vertices, edges and faces.

1. INTRODUCTION

We consider simple graphs, that is, graphs which do not contain loops or parallel edges. Let G be a graph with n vertices and m edges, and let A be the adjacency matrix of G. Clearly, A is symmetric (0,1)-matrix with its (i,j)-entry as 1 if ith and jth vertices of G are adjacent and zero otherwise.

An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, the so-called π-electron energy E. For a molecule with n=2k atoms, the total π-electron energy can be shown to be $E_\pi = 2\sum_{i=1}^{k} \lambda_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the top k eigen values of the adjacency matrix of the graph of the molecule. For a bipartite graph, because of the symmetry of the spectrum, we can write $E_\pi = \sum_{i=1}^{n} |\lambda_i|$, and this has motivated the following definition.

For any graph G, the energy of the graph is defined as $E_G = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of the adjacency matrix A of G. Further details on the energy of graphs can be found in [2,4,5,7]. Let G be a graph with n vertices and m edges and let E(G) be the energy of G.

The following upper bound for the energy of a graph can be found in [8].

$$E(G) \leq \sqrt{2mn}. \tag{1}$$

A lower bound for the energy of a graph solely in terms of its number of vertices is [7]

$$E(G) \geq 2\sqrt{n-1},$$ (2)

with equality if and only if G is the star $K_{1,n-1}$.

The energy of a graph as a function only of its number of edges satisfies [3]

$$E(G) \geq 2\sqrt{m}.$$ (3)

The following upper bound on energy is from [8].

$$E(G) \leq \frac{2m}{n}\sqrt{(n-1)\left(2m-\left(\frac{2m}{n}\right)^2\right)},$$ (4)

with equality if and only if G is either $\left(\frac{n}{2}\right)K_2$, the complete graph K_n, or certain strongly regular graphs.

An improvement of the bound for energy in terms of its vertices is [8]

$$E(G) \leq \frac{n}{2}\left(1+\sqrt{n}\right).$$ (5)

2. ENERGY OF PLANAR GRAPHS

A graph G_p is said to be planar if there exists some geometric representation of G_p which can be drawn on a plane such that no two of its edges intersect. A plane representation of a graph divides the plane into regions also called faces. A region is characterized by the set of edges(or the set of vertices) forming its boundary.

Let G_p be a planar graph with n vertices and m edges. Denote the energy of G_p by $E(G_p)$. Clearly,

$$m \leq 3n-6.$$ (6)

Theorem 2.1. *For a planar graph Gp ,*

$$2\sqrt{\frac{m+3}{3}} \leq E(G_p) \leq \sqrt{6n(n-1)}.$$

Proof. If G_p is a planar graph with n vertices and m edges, then (6) holds.
Using (6) in (1), we get

$$E(G_p) \leq \sqrt{2(3n-6)n} = \sqrt{6n(n-1)}$$ (7)

Using (6) in (2), we get

$$E(G_p) \geq 2\sqrt{\frac{m+6}{3}-1} = 2\sqrt{\frac{m+3}{3}}$$ (8)

Combining (7) and (8), the result follows.   □

Theorem 2.2. *If Gp is a connected planar graph, then*

$$2\sqrt{\frac{n+2}{3}} \leq E(G_p) \leq \sqrt{6n(n-1)}$$

Proof. For a connected planar graph, $m \geq n-1$ and using this in (8), we get

$$E(G_p) \geq 2\sqrt{\frac{n-1+3}{3}-1} = 2\sqrt{\frac{n+2}{3}} \quad .$$

Together with (7), we obtain

$$2\sqrt{\frac{n+2}{3}} \leq E(G_p) \leq \sqrt{6n(n-1)} \quad . \qquad \square$$

A polyhedron is a solid bounded by surfaces, called faces, each of which is a plane. A polyhedron is said to be convex if any two of its interior points can be joined by a straight line lying entirely within the region. The vertices and edges of a polyhedron, which form a skeleton of the solid, give a simple graph in three dimensional space. For a convex polyhedron this graph is planar.

A simple connected plane graph G_p is called polyhedral if $d(v) \geq 3$ for each vertex v of G_p and $d(\phi) \geq 3$ for every face ϕ of G_p.

Theorem 2.3. *If Gp is a polyhedral graph, then*

$$\sqrt{2(n+2)} \leq G_p \leq \sqrt{6n(n-1)}$$

.

Proof. As G_p is polyhedral, $d(\phi) \geq 3$ for every vertex v of G_p. So,

$$\sum d(\phi) \geq 3,$$

or

$$2m \geq 3n,$$

or

$$m \geq \frac{3n}{2}. \tag{9}$$

Using (9) in (8), we get

$$E(G_p) \geq 2\sqrt{\frac{3n+6}{6}} = \sqrt{2(n+2)} \quad .$$

Togather with (7), we get the required result. $\qquad \square$

If G is a bipartite graph with n>2 vertices, then [9]

$$E(G) \leq \frac{n}{\sqrt{8}}(\sqrt{n}+\sqrt{2}), \tag{10}$$

and

$$E(G) \leq 2\left(\frac{2m}{n}\right) + \sqrt{(n-2)\left(2m-2\left(\frac{2m}{n}\right)^2\right)}. \tag{11}$$

Theorem 2.4. *If Gp is a planar bipartite graph, then*

$$E(Gp) \leq \frac{3m-4}{8} \left(\sqrt{3m-4} + 2 \right).$$

Proof. Since G_p is planar bipartite,

$$m \geq \frac{2}{3}(n+2) \qquad \text{or} \qquad n \leq \frac{3m-4}{2}. \tag{12}$$

Using (12) in (10), the result follows. □

A polyhedron is called regular if it is convex and its faces are congruent regular polygons. We know, the only regular polyhedra are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

Balakrishman[1] showed that if G is a regular graph of degree k, then

$$E(G) \leq k + \sqrt{k(n-1)(n-k)}. \tag{13}$$

By using (13), we see that the tetrahedron, the cube and the dodecahedron are not hyperenergetic, since they are regular graphs of degree 3.

This can also be verified by using, the result of I.Gutman et al [6] as under.

In these three regular plyhedras, $m = \frac{3n}{2}$ and $m = \frac{3n}{2} = 2n-n- \frac{n-4}{2} < 2n-2$, and

hence the argument.

Now by using (13), E(Octahedron)$\leq 4 + \sqrt{40}$ and E(Icosahedron) $\leq 5 + \sqrt{385}$.

For the planar graph G_p, with n vertices, m edges and f faces, we have the well known Euler's formula

n-m + f   = 2.

Together with (6), we get

$$m \geq \frac{3f-12}{2}, \tag{14}$$

and

$$n \geq \frac{f+4}{2}. \tag{15}$$

With the help of (14) and (15), the bounds for the energy of a planar graph solely in terms of its faces f can be obtained as under.

$$E(Gp) \geq \left(\frac{f+4}{2} \right) \sqrt{\frac{f+2}{2}}, \tag{16}$$

$$E(Gp) \geq \sqrt{\frac{3}{2}f^2 - 16}, \tag{17}$$

$$E(Gp) \geq \sqrt{2(f+2)}. \tag{18}$$

Further, the smallest non-planar graphs are K_5 and $K_{3,3}$ and we see that both are not hyperenergetic.

From (10),

$$E(k_{3,3}) \leq \frac{6}{\sqrt{8}}(\sqrt{6} + \sqrt{2}) < 2n - 2 = 10.$$

REFERENCES

[1]. R. Balakrishman, *The energy of a graph*, Linear Algebra Appl., **387** (2004), 287-295.
[2]. R. B. Bapat and S. Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc., **1** (2004), 129-132.
[3]. G. Caprossi, D. Cvetkovic, I. Gutman and B. Hansen, *Variable neighborhood graphs, Finding graphs with extreme energy*, J. Chem. Inf. Comput. Sci., **39** (1999), 984-986.
[4]. D. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs,* Academic Press, New York, 1979.
[5]. I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungszentrum Graz., **103** (1978), 1-22.
[6]. I. Gutman, Y. Hou, H. B. Walikar, H. S. Ramane and P. R. Hamphiholi, No Huckel *graph is hyperenergetic*, J. Serb. Chem. Soc., **65** (11) (2000), 799 –801.
[7]. I. Gutman, The energy of a graph: old and new results, *Algebraic Combinatorics and Applications*, Springer-Verlag, Berlin, 2001, 196-211.
[8]. J. Koolen and V. Moulton, *Maximal energy graphs*, Advances in Appld. Math., **26** (2001), 47-52.
[9]. J. Koolen and V. Moulton, *Maximal energy bipartite graphs*, Graphs and Comb., **19** (2003)131-135.