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SIGN CHANGING PERIODIC SOLUTIONS OF A NONLINEAR WAVE EQUATION

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Abstract. We seek the sign changing periodic solutions of the nonlinear wave equation $u_{tt} - u_{xx} = a(x,t)g(u)$ under Dirichlet boundary and periodic conditions. We show that the problem has at least one solution or two solutions whether $\frac{1}{2}g(u)u - G(u)$ is bounded or not.

1. Introduction

In this paper we seek the sign changing solutions of the following nonlinear wave equation

$$u_{tt} - u_{xx} = a(x,t)g(u), \tag{1.1}$$

under Dirichlet boundary condition and periodic condition:

$$u(0,t) = u(\pi,t) = 0,$$

$$u(x,t+T) = u(x,t),$$

where $a : [0,\pi]\times\mathrm{R}\to\mathrm{R}$ is a continuous function which changes sign such that $a(x,t) = -a\left(x, t+\frac{T}{2}\right)$, and the open sets

$$\{(x,t) \mid a(x,t) > 0\}, \qquad \{(x,t) \mid a(x,t) < 0\}$$

are both nonempty. We shall write $a = a^+ - a^-$, where $a^+ = a \cdot \chi_{\Omega^+}$ and $a^- = -a \cdot \chi_{\Omega^-}$. In what follows we assume systematically that T is a rational multiple of π. We assume that g satisfies the following conditions:

$(g1)$ $g \in C(\mathrm{R},\mathrm{R})$,

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$(g2)$ $g(u) = o(u)$,
$(g3)$ there exists a constant $\mu > 2$ such that

$$g(u)u \geq \mu \int_0^u g(s)ds > 0,$$

$(g4)$ there exist constants $a_1, a_2 > 0$ and $p > 1$ such that

$$|g(u)| \leq a_1|u|^p + a_2 \qquad \text{for all } u,$$

where $G(u) = \int_0^x g(t)dt$.

Integrating condition $(g3)$ shows that there exist constants $a_3, a_4 > 0$ such that

$$G(u) \geq a_3|u|^\mu - a_4. \tag{1.2}$$

The purpose of this paper is to show the existence of solutions of the problem (1.1) when $\frac{1}{2}g(u)u - G(u)$ is bounded or $\frac{1}{2}g(u)u - G(u)$ is not bounded. Our main results are as follows:

Theorem 1.1. *Assume that g satisfies $(g1)-(g4)$ and $\frac{1}{2}g(u)u-G(u)$ is bounded. Then the problem (1.1) has at least one bounded solution provided that p in $(g4)$ is further restricted by $p + 1 < \mu$.*

Theorem 1.2. *Assume that g satisfies $(g1) - (g4)$, $\frac{1}{2}g(u)u - G(u)$ is not bounded. We also assume that there exists a small $\epsilon > 0$ such that $\int_\Omega a^-(x,t) < \epsilon$. Then for each T the problem (1.1) has at least two solutions, (1)one of which is bounded and (2) the other is a large norm solution such that for each real number M,*

$$\max_{\substack{x\in[0,\pi]\\ t\in[0,T]}} |u(x,t)| > M$$

provided that p in $(g4)$ is further restricted by $p + 1 < \mu$.

Theorem 1.1 and Theorem 1.2 will be proved in Section 3 and 4 via variational methods.

An outline of this paper is as follows: in Section 2 we introduce a subspace H of functions satisfying some symmetry properties, stable by $A(Au = u_{tt} - u_{xx})$, g such that the intersection of H with the kernel of A is reduced to 0. The search of a solution of the problem (1.1) in the space H reduces the problem to a situation where A^{-1} is a compact operator. In Section 3 we prove Theorem1.1 and 1.2(1). We introduce a functional I whose critical points and weak solutions of (1.1) possess one-to-one correspondence. Next we prove that $I \in C^1(E,\mathrm{R})$ and satisfies the Palais-Smale condition. Then, we show that there exist $\rho > 0, \delta > 0$, and $u_0 \in E$ satisfying $\|u_0\| > \rho$ such that if $\|u\| = \rho$, then $I(u) \geq \delta$, and $I(u_0) \leq 0$. By critical point theorem for indefinite functionals (cf. [3]) there exists at least one solution of (1.1) which is bounded. In Section 4, we prove Theorem1.2(2) by the method of Rabinowitz (cf. [13]). We introduce a functional J such that large critical values of J induce large critical values of I.

2. Invariant spaces

Let $\Omega = (0, \pi) \times (0, T)$; T is a rational multiple of π, that is, $T = \frac{2\pi b}{a}$, where a and b are coprime integers. Let $\mathcal{A}$ be the operator defined by

$$\mathcal{A}u = u_{tt} - u_{xx}$$

and D(A) be a collection of functions which belongs to the domain of an operator A and which satisfies some boundary conditions. Let A be the adjoint of $\mathcal{A}$ in $L^2(\Omega)$. We investigate solutions of

$$Au = a(x,t)g(u).$$

We note that the eigenvalues of A are $j^2 - (\frac{2\pi k}{T})^2, j = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ and the corresponding eigenfunctions are

$$\sin jx \sin \frac{2\pi kt}{T} \qquad and \qquad \sin jx \cos \frac{2\pi kt}{T}.$$

We also note that the set of functions $\sin jx \sin \frac{2\pi kt}{T}, \sin jx \cos \frac{2\pi kt}{T}$ is an orthogonal base for $L^2(\Omega)$. Let u is a function of $L^2(\Omega)$. Then there exists one and only one function of $L^2([0, \pi] \times R)$ which is T periodic in t and equals u on Ω. We shall again denote this function by u. Let us denote an element u, in $L^2(\Omega)$, as

$$u = \sum_{\substack{j>0 \\ k}} u_{j,k} \sin jx \exp ik\frac{a}{b}t$$

with $u_{j,k} = \overline{u}_{j,-k}$. We assume that b is even and a is odd. Let H be the closed subspace of $L^2(\Omega)$ defined by

$$H = \{u \in L^2(\Omega) | u(x,t) = -u\left(x, t + \frac{T}{2}\right) \qquad a.e.\ x \in (0, \pi), t \in R\}.$$

Then H is invariant under shifts: Let $u \in H$ and τ be a real number. If $v(x,t) = u(x, t+\tau)$, then $v \in H$. H is invariant by g: Let $u \in H$ such

that $g(u) \in L^2(\Omega)$. Then $g(u) \in H$.
Let $\tilde{u}(x,t) = u(x, t + \frac{T}{2})$. Then

$$\tilde{u} = \sum_{\substack{j>0 \\ k}} u_{j,k}(-1)^k \sin jx \exp ik\frac{a}{b}t.$$

Therefore

$$u \in H \Longleftrightarrow u_{j,k} = 0 \qquad \text{for any even } k. \tag{2.1}$$

Let A_1 be the linear operator of H defined by

$$D(A_1) = D(A) \cap H$$

$$A_1 u = Au \quad \text{for} \quad \text{every} \quad u \in H.$$

Then it follows from (2.1) that A_1 is self adjoint in H.
We claim that $H \cap N(A) = \{0\}$,
where $N(A)$ is the kernel of A. In fact, let $u \in H \cap N(A)$. Then

$$u = \sum u_{j,k} \sin jx \exp ik\frac{a}{b}t,$$

$$j^2 - \frac{k^2a^2}{b^2} \neq 0 \Longrightarrow u_{j,k} = 0.$$

Let j and k be such that

$$j^2 - \frac{k^2a^2}{b^2} = 0.$$

Since b is even and a is odd, k is even. Using (2.1) we have $u_{j,k} = 0$ and therefore $H \cap N(A) = \{0\}$.

3. Proof of Theorem 1.1 and Theorem 1.2(1)

To prove Theorem 1.1 we shall show that the corresponding functional $I(u)$ of the problem (1.1) satisfies the geometric assumptions of the critical point theorem for indefinite functionals (cf. [3]). Then, by critical point theorem we shall seek solutions of (1.1). Now, we are going to seek a function u in H such that

$$A_1 u = a(x,t)g(u). \tag{3.1}$$

The eigenvalues of A_1 are $j^2-(\frac{2\pi k}{T})^2$, where j is odd and k is even. Given $u \in H$, we write

$$u = \sum_{\substack{j>0 \\ j\ odd \\ k\ even}} u_{j,k} \sin jx \exp i\frac{2\pi kt}{T}$$

with $u_{j,k} = \overline{u}_{j,-k}$. Let

$$E = \{u \in H \mid \sum_{j,k} |j^2 - \frac{a^2k^2}{b^2}| \cdot |u_{j,k}|^2 < +\infty\},$$

$$(u,v) = \sum_{j,k} |j^2 - \frac{a^2k^2}{b^2}| u_{j,k} \cdot \overline{v_{j,k}} \text{ for } u, v \in E,$$

where (,) is a scalar product on E. With this scalar product E is a Hilbert space with a norm

$$\|u\| = (u,u)^{\frac{1}{2}}, \qquad u \in E.$$

Let

$$\|u\|_r = \left(\int_\Omega |u|^r\right)^{\frac{1}{r}}, \qquad r \geq 1.$$

By the classical theorem of Riesz (cf. [9, p525]), we have

$$\|u\|_r \leq \left(\frac{\pi T}{2}\right)^{\frac{1}{r}} \left(\sum_{j,k} |u_{j,k}|^{r'}\right)^{\frac{1}{r'}}, \qquad r \geq 2, \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

Since for every $\epsilon > o$

$$\sum_{\substack{j\ odd \\ k\ even}} \frac{1}{|j^2 - \frac{a^2k^2}{b^2}|^{1+\epsilon}} < \infty,$$

it follows that for every $r \in [2, +\infty)$ there is $c_r \in$R such that

$$\|u\|_r \leq c_r \|u\|. \tag{3.2}$$

Let

$$E_+ = \{u|\ u \in E,\ u_{j,k} = 0 \text{ if } j^2 - \frac{a^2k^2}{b^2} < 0\},$$

$$E_- = \{u|\ u \in E,\ u_{j,k} = 0 \text{ if } j^2 - \frac{a^2k^2}{b^2} > 0\}.$$

Then $E = E_+ \oplus E_-$, for $u \in E$, $u = u^+ + u^- \in E_+ \oplus E_-$. Let P_+ be the orthogonal projection on E_+ and P_- be the orthogonal projection on E_-. We can write $P_+u = u^+$, $P_-u = u^-$, for $u \in E$. We consider the following functional associated with (1.1),

$$I(u) = \frac{1}{2}\int_\Omega [-|u_t|^2 + |u_x|^2]dx\,dt - \int_\Omega a(x,t)G(u)dx\,dt. \tag{3.3}$$
$$= \frac{1}{2}\left(\|P_+u\|^2 - \|P_-u\|^2\right) - \int_\Omega a(x,t)G(u)dx\,dt,$$

where

$$G(u) = \int_0^u g(s)ds.$$

From($g4$) and (3.2), I is well defined. The solutions of (1.1) coincide with the nonzero critical points of $I(u)$. The following proposition shows that $I(u) \in C^1(E,\mathrm{R})$ (For the proof, refer to [3]).

Proposition 1. *Assume that g satisfies $(g1) - (g4)$. Then $I(u)$ is continuous and Fréchet differentiable in E with Fréchet derivative*

$$I'(u)h = \int_\Omega [-u_t \cdot h_t + u_x \cdot h_x - a(x,t)g(u)h]dx\,dt \tag{3.4}$$
$$= (P_+u, P_+h) - (P_-u, P_-h) - \int_\Omega a(x,t)g(u)hdx\,dt$$

for all $h \in E$. Moreover if we set

$$F(u) = \int_\Omega a(x,t)G(u)dx\,dt,$$

then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact, and

$$F'(u)h = \int_\Omega a(x,t)g(u)hdx\,dt \qquad \text{for all } \ h \in E$$

. This implies that $I \in C^1(E,R)$ and $F(u)$ is weakly continuous.

The following proposition shows that $I(u)$ satisfies (PS) condition when $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) < \epsilon$.

Proposition 2. *Assume that g satisfies $(g1) - (g4)$. We also assume that $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t)dx\,dt < \epsilon$. Then $I(u)$ satisfies the Palais-Smale condition provided that p in $(g4)$ is restricted by $p+1 < \mu$: If for a sequence $(u_m), I(u_m)$ is bounded from above and $I'(u_m) \to 0$ as $m \to \infty$, then (u_m) is bounded.*

Proof. Suppose that (u_m) is a sequence with $I(u_m) \le M$ and $I'(u_m) \to 0$ as $m \to \infty$. Then, by $(g3)$, $(g4)$, (3.2), (1.2) and the Hölder inequality, we have: for large m with $u = u_m$,

$$
\begin{aligned}
M + \frac{1}{2}\|u\| &\ge I(u) - \frac{1}{2}I'(u)u = \int_\Omega \frac{1}{2}a(x,t)g(u)u - a(x,t)G(u) \\
&= \int_\Omega a^+(x,t)[\frac{1}{2}g(u)u - G(u)] - \int_\Omega a^-(x,t)[\frac{1}{2}g(u)u - G(u)] \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x,t)\cdot G(u) \\
&\quad - \max_\Omega |\frac{1}{2}g(u)u - G(u)| \int_{\Omega^-} a^-(x,t)dx\,dt \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x,t)\cdot (a_3|u|^\mu - a_4) \\
&\quad - \max_\Omega |\frac{1}{2}g(u)u - G(u)| \int_{\Omega^-} a^-(x,t)dx\,dt
\end{aligned}
$$

Thus if $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) < \epsilon$, then we have

$$
1 + \|u\| \ge M_1 \int_\Omega |u|^\mu \ge M_2 \left(\int_\Omega |u|^2 dx\,dt\right)^{\frac{1}{2}\cdot\mu}. \tag{3.5}
$$

Moreover since

$$
|I'(u_m)\varphi| \le \|\varphi\| \tag{3.6}
$$

for large m and all $\varphi \in E$, choosing $\varphi = u_m^+ \in E_+$ gives

$$\begin{aligned}
\|u_m^+\|^2 &= \int_\Omega ((u_m)_{tt} - (u_m)_{xx}) \cdot u_m^+ \\
&\leq \int_\Omega a(x,t) g(u_m) u_m^+ + \|u_m^+\| \\
&\leq \int_\Omega |a(x,t)||g(u_m)||u_m| + \|u_m\| \\
&\leq \|a\|_\infty \int_\Omega \big(a_1 |u_m|^{p+1} + a_2 |u_m|\big) + \|u_m\| \\
&\leq C_1 \int_\Omega |u_m|^{p+1} + C_2 \|u_m\|_{L^2(\Omega)} + \|u_m\| \\
&\leq C_1 \int_\Omega |u_m|^{p+1} + C_2' \|u_m\|.
\end{aligned}$$

Taking $\varphi = -u_m^-$ in (3.6) yields

$$\begin{aligned}
\|u_m^-\|^2 &= \int_\Omega ((u_m)_{tt} - (u_m)_{xx}) \cdot (-u_m^-) \\
&\leq \int_\Omega a(x,t) g(u_m) \cdot (-u_m^-) + \| - u_m^-\| \\
&\leq \int_\Omega |a(x,t)||g(u_m)||u_m| + \|u_m\| \\
&\leq \|a\|_\infty \int_\Omega (a_1 |u_m|^{p+1} + a_2 |u_m|) + \|u_m\| \\
&\leq C_3 \int_\Omega |u_m|^{p+1} + C_4 \|u_m\|_{L^2(\Omega)} + \|u_m\| \\
&\leq C_3 \int_\Omega |u_m|^{p+1} + C_4' \|u_m\| + \|u_m\|.
\end{aligned}$$

Thus, by (3.5), if $p+1 \leq \mu$, we have

$$\begin{aligned}
\|u_m\|^2 = \|u_m^+\|^2 + \|u_m^-\|^2 &\leq M_3 \int_\Omega |u_m|^{p+1} + M_4 \|u_m\| \\
&\leq M_3 \int_\Omega |u_m|^{\mu} + M_4 \|u_m\| \\
&\leq M_5 1 + \|u_m\|) + M_4 \|u_m\| \leq M_6 (1 + \|u_m\|),
\end{aligned}$$

from which the boundedness of (u_m) follows. Thus (u_m) converges weakly in E. Since $P_\pm I'(u_m) = \pm P_\pm u_m + P_\pm \tilde{\mathcal{P}}(u_m)$ with $\tilde{\mathcal{P}}$ compact and the weak convergence of $P_\pm u_m$ imply the strong convergence of $P_\pm u_m$ and hence $(P.S.)$ condition holds. □

Next, we will prove that $I(u)$ satisfies one of geometrical assumptions of the critical point theorem of indefinite functional $I(u)$.

PROPOSITION 3. *Assume that g satisfies $(g1)-(g4)$. Then there exist a small real number $\rho > 0$, $\delta > 0$, $u_0 \in E$ satisfying $\|u_0\| > \rho$ such that (1) if $\|u\| = \rho$, then*

$$I(u) \geq \delta \text{ and} \tag{3.7}$$

(2) $I(u_0) \leq 0$.

Proof. (1) By $(g4)$, (1.2), (3.2) and the Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|P_+u\|^2 - \frac{1}{2}\|P_-u\|^2 - \int_\Omega a(x,t)G(u) \\ &\geq \frac{1}{2}\|P_+u\|^2 - \frac{1}{2}\|P_-u\|^2 - \|a\|_\infty \int_\Omega C_1|u|^{p+1} \\ &\geq \frac{1}{2}\|P_+u\|^2 - \frac{1}{2}\|P_-u\|^2 - \|a\|_\infty C_1'\|u\|^{p+1} \end{aligned}$$

for C_1, $C_1' > 0$. Since $p+1 > 2$, there exist $\rho > o$ and $\delta > o$ such that if $\|u\| = \rho$, then $I(u) \geq \delta$.
(2) If we choose $\psi \in E$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and support$(\psi) \subset \Omega^+$, then we have

$$\begin{aligned} I(t\psi) &\leq \frac{1}{2}\|P_+(t\psi)\|^2 - \frac{1}{2}\|P_-(t\psi)\|^2 - \int_{\Omega^+} a(x,t)\,(a_3t^\mu\psi^\mu - a_4) \\ &\leq \frac{1}{2}\|t\psi\|^2 - \int_{\Omega^+} (a_3t^\mu\psi^\mu - a_4) \\ &= \frac{1}{2}t^2 - \int_{\Omega^+} a(x,t)\,(a_3t^\mu\psi^\mu - a_4) \end{aligned}$$

for all $t > 0$. Since $\mu > 2$, for t_0 great enough, $u_0 = t_0\psi$ is such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. □

Proof of Theorem 1.1 and Theorem 1.2(1)

By Proposition 3.1 and 3.2 $I(u) \in C^1(E,\mathrm{R})$ and satisfies the Palais-Smale condition. By Proposition 3.3 there exist $\rho > 0$, $\delta > 0$, $u_0 \in E$ satisfying $\|u_0\| > \rho$ such that if $\|u\| = \rho$, then $I(u) \geq \delta$, and $I(u_0) \leq 0$. By the critical point theorem for indefinite functional, $I(u)$ has a critical value $b \geq \delta$ given by

$$b = \inf_{\gamma\in\Gamma} \max_{[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0 \text{ and } \ \gamma(1) = u_0\}$.
We denote by $\tilde{u}$ a critical point of I such that $I(\tilde{u}) = b$. We claim that there exists a constant $C > 0$ such that

$$\|a^+(x,t)^{\frac{1}{\mu}}\tilde{u}\|_{L^2(\Omega)} \leq C\left(1 + L\int_{\Omega^-} a^-(x,t)dx\,dt\right)^{\frac{1}{\mu}},$$

where $L = \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$.
In fact, we have

$$b \leq \max I(tu_0), \qquad 0 \leq t \leq 1,$$

and

$$\begin{aligned}
I(tu_0) &= t^2\left(\frac{1}{2}\|P_+u_0\|^2 - \frac{1}{2}\|P_-u_0\|^2\right) - \int_\Omega a(x,t)G(tu_0)dx\,dt \\
&\leq t^2\|u_0\|^2 - \int_\Omega a^+(x,t)G(tu_0)dx\,dt + \int_\Omega a^-(x,t)G(tu_0)dx\,dt \\
&\leq t^2\|u_0\|^2 - a_3t^\mu\int_\Omega a^+(x,t)u_0^\mu + a_4\int_\Omega a^+(x,t) + \\
&\quad a_5t^{p+1}\int_\Omega a^-(x,t)u_0^{p+1} \\
&= Ct^2 - Ct^\mu + C + C't^{p+1}.
\end{aligned}$$

Since $0 \leq t \leq 1$, b is bounded: $b < \tilde{C}$.
We can write

$$\begin{aligned}
b &= I(\tilde{u}) - \frac{1}{2} I'(\tilde{u})\tilde{u} \\
&= \int_\Omega a(x,t) \left(\frac{1}{2} g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx\, dt \\
&= \int_\Omega a^+(x,t) \left(\frac{1}{2} g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx\, dt \\
&\qquad - \int_\Omega a^-(x,t) \left(\frac{1}{2} g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx\, dt \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_\Omega a^+(x,t) g(\tilde{u})\tilde{u} - \max_\Omega |\frac{1}{2} g(\tilde{u})\tilde{u} - G(\tilde{u})| \int_{\Omega^-} a^-(x,t) dx\, dt \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \mu \int_\Omega a^+(x,t) \left(a_3 |\tilde{u}|^\mu - a_4 \right) - L \int_{\Omega^-} a^-(x,t) dx\, dt,
\end{aligned}$$

where $L = \max_\Omega |\frac{1}{2} g(\tilde{u})\tilde{u} - G(\tilde{u})|$. Thus we have

$$\begin{aligned}
C \left(1 + L \int_{\Omega^-} a^-(x,t) dx\, dt \right) &\geq \int_\Omega a^+(x,t) |\tilde{u}|^\mu \\
&\geq \left[\int_\Omega \left(a^+(x,t)^{\frac{1}{\mu}} |\tilde{u}| \right)^2 \right]^{\frac{\mu}{2}}, \qquad (3.8)
\end{aligned}$$

from which we can conclude that $\tilde{u}$ is bounded. In fact, suppose that $\tilde{u}$ is not bounded. Then for any $R > 0$, $|\tilde{u}| \geq R$. Thus we have

$$\int_\Omega a^+(x,t) |\tilde{u}|^\mu \geq R^\mu \int_\Omega a^+(x,t) dxdt$$

for any R, which contradicts to the fact (3.8) and the proof of Theorem 1.1 is complete. On the other hand, by Proposition 3.2, if $\frac{1}{2} g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) dxdt < \epsilon$, then $I(u)$ satisfies the Palais-Smale condition. Proposition 3.3 and the critical point theorem for indefinite functional show that $I(u)$ has a critical value b with critical point $\tilde{u}$ such that $I(\tilde{u}) = b$. If $\int_{\Omega^-} a^-(x,t) dx\, dt$ is sufficiently small, by (3.8), we have

$$\int_\Omega a^+(x,t) |\tilde{u}|^\mu \leq C$$

for $C > 0$, from which we can conclude that $\tilde{u}$ is bounded and the proof of Theorem 1.2(1) is complete.

4. Proof of Theorem 1.2(2)

In this section we assume that $\frac{1}{2}g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x,t) < \epsilon$. Then $I \in C^1(E,\mathrm{R})$ and satisfies the Palais-Smale condition (cf. Proposition 3.1 and 3.2). Now, we define a functional J

$$J(u) = \frac{1}{2}\left(\|P_+u\|^2 - \|P_-u\|^2\right) - \|a\|_\infty \int_\Omega \frac{a_1}{p+1}|u|^{p+1}.$$

Then $J \in C^1(E,\mathrm{R})$ and satisfies the Palais-Smale condition, and

$$J(u) - \|a\|_\infty a_2 \pi T \leq I(u).$$

Let $(E_i)_{i\geq 0}$ be a sequence of subspaces of E such that there exist an odd integer j_i and an even integer k_i such that
(1) E_i is spanned by $\sin j_i x \sin k_i \frac{a}{b}t, \sin j_i x \cos k_i \frac{a}{b}t$.
(2) $i \leq i' \Rightarrow \left(k_i \frac{a}{b}\right)^2 - (j_i)^2 \leq \left(k_{i'} \frac{a}{b}\right)^2 - (j_{i'})^2$,
(3) $E = \oplus_{i\in N} E_i$.
Let $V_m = \oplus_{i\leq m} E_i \oplus E_-$.
From Proposition 3.3, there exists an $R_m > 0$ such that

$$J(u) - \|a\|_\infty a_2\pi T \leq I(u) \leq 0 \qquad \text{for } u \in \left(V_m \cap E^+\right) \setminus B_{R_m}.$$

For $u \in E$, $\theta \in [0,T]$ set:

$$s_\theta u(x,t) = u(x, t+\theta).$$

If $u \in E$, $s_\theta u \in E$ and $I(u) = I(s_\theta u)$, $J(u) = J(s_\theta u)$.
Let

$$F = \{u \in E \mid u \text{ is independent of } t\}.$$

We have

$$F = \{u \in E \mid s_\theta u = u \qquad \forall \theta \in [0,T]\}.$$

We remark that

$$F \subset E_+.$$

We call a subset B of E an invariant set if for all $u \in B$, $s_\theta u \in B$ for all $\theta \in [0,T]$. Let $C(B,\mathrm{E})$ be the set of continuous functions from B into E. If B is an invariant set we say $h \in C(B,\mathrm{E})$ is an equivariant map if $h(s_\theta u) = s_\theta h(u)$ for all $\theta \in [0,T]$ and $u \in B$. Let

$$\varepsilon = \{B \mid B \subset E\setminus\{0\}, B \text{ is closed and invariant }\}.$$

In [10] it is proved that there is an index theory i.e., a mapping $i : \varepsilon \to N \cup \{\infty\}$ such that if B, $B_1 \in \varepsilon$,

(1) $i(B) \le i(B_1)$ if there is $\varphi \in C(B, B_1)$ with φ equivariant.
(2) $i(B \cup B_1) \le i(B) + i(B_1)$.
(3) If $B \subset E \backslash F$ and B is compact, $i(B) < +\infty$ and there is a $\delta > 0$ such that $i(N_\delta(B)) = i(B)$ where $N_\delta(B) = \{x \mid |x - B| \le \delta\}$.
(4) If $S \subset E \backslash F$ is a $2n$ dimensional invariant sphere,

$$i(S) = n.$$

Let G_m denote the class of mapping $h \in C(D_m,\mathrm{E})$ which satisfy the following properties
(1) h is equivariant
(2) $h(u) = u$ for all $u \in (\partial B_{R_m} \cap V_m) \cup F$.
(3) $Ph(u) = \alpha(u)Pu + \Psi(u)$ where Ψ is compact and $\alpha \in C(D_m, [1, \overline{\alpha}])$, $\overline{\alpha}$ depending on h.
Let

$$\Gamma_j = \{\overline{h(D_m \setminus Y)} \mid m \ge j, h \in G_m, Y \in \varepsilon and i(Y) \le m - j\}, \tag{4.1}$$

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B} I(u), \tag{4.2}$$

$$b_j = \inf_{B \in \Gamma_j} \sup_{u \in B} J(u). \tag{4.3}$$

As in [13] we have the following lemma.

LEMMA 4.1. *b_j is a critical value of J,*

$$b_j - a_2\|a\|_\infty \pi T \le c_j, \tag{4.4}$$

$$\textit{if } c_j \ge \delta, \textit{then } c_j \textit{ is a critical value of } I, \tag{4.5}$$

where δ is defined as in [13], i.e.,

$$\delta = \sup_{E_0} \left(a_4 \int_\Omega a^+(x,t) + \frac{c}{p+1} \int_\Omega a^-(x,t)|u|^{p+1} \right),$$

where $c = \max\{a_1, a_2\} > 0$ and E_0 is the null space of A.

Proof of Theorem 1.2(2)

We note that

$$b_j \ge \sup_\rho \left(\inf_{u \in V_{j-1}^\perp} J(\rho u) \right). \tag{4.6}$$

If $u \in V_{j-1}^\perp$, by (3.2), there exists ϵ_j with

$$\lim_{j\to\infty} \epsilon_j = 0.$$

such that $\|u\|_{p+1} \leq \epsilon_j \|u\|$.
If $u \in V_{j-i}^{\perp}$ and $\|u\| = 1$, by $(g3)$, $(g4)$,

$$J(\rho u) \geq \frac{\rho^2}{2} - \epsilon_j^{p+1} \frac{a_1}{p+1} \rho^{p+1} \|a\|_\infty. \tag{4.7}$$

Thus if $j \to \infty$, then $J(\rho u) \geq \frac{\rho^2}{2}$. Using (4.6) we have

$$\lim_{j\to\infty} b_j = \infty. \tag{4.8}$$

Using (4.8), (4.4), and (4.5) we see that for j large enough c_j is a critical value of I and

$$\lim_{j\to\infty} c_j = +\infty. \tag{4.9}$$

Note that $A_1 u = a(x,t)g(u)$ and $\max_{\substack{x\in[0,\pi]\\ t\in[0,T]}} |u(x,t)| \leq K$ imply

$$I(u) \leq \left(\max_{|s|<K} \frac{1}{2} s g(s) - \min_{|s|<K} G(s) \right) \int_\Omega a^+(x,t) dx\, dt.$$

We conclude the proof using (4.9).

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