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CONVECTION IN A STEADY STATE CHIMNEY

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Abstract. We present an axisymmetric model containing only one chimney to study how convection affects the flow in a steady state chimney. We find that the mass fraction of solid in a mush and the depth of a mush when the strength of convection is given. We use the knowledge of the variables in the mush to find the fluid flow in the chimney Our procedure employs the von Kármán-Pohlhausen technique for determining chimney flow and makes use of the fact that the radius of the chimney is much less than the thickness of the mush.

1. Introduction

When binary mixtures are cooled and directionally solidified, a mushy layer comprising a solid matrix of crystals with interstitial liquid commonly forms between the melt and solid regions (Copley et al.1970). These mushy layers are the result of a supercooling induced morphological instability of the solidification front (Mullins & Sekerka 1964).

Copley et al.(1970) reported experiments in which they had cooled and crystallized from below aqueous solutions of ammonium chloride. The authors found that convection of buoyant fluid from the interstices of the mushy layer, which formed as crystals of ammonium chloride grew at the base of the container, took the form of narrow, vertical plumes rising through crystal-free vents or 'chimneys' in the dendritic matrix. They suggested that these convectively formed chimneys are the cause of the 'freckles' that are often observed in castings of steel and binary alloy systems such as aluminum-copper, lead-tin and nickel-aluminum.

Freckles are imperfections that interrupt the uniformity of the microstructure of a casting, causing areas of mechanical weakness. Both because of the metallurgical importance and the pure scientific interest,

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a prized goal is a thorough understanding of the phenomena observed in aqueous NH_4Cl (Huppert, 1990). Furthermore, there are possible applications to the formation of the solid inner core of the Earth (Roberts & Loper, 1983), the evolution of magma chambers due to the pipes in geological systems (Tait & Jaupart, 1991), and the sulphide mineral zonation of black smoker chimney walls (Haymon, 1983).

Some previous numerical computations of convection in mushy layers have been conducted based on a single Darcy-Brinkman formulation for the combined mushy, liquid and chimney regions (Felicelli et al, 1991). Though chimneys have been found in these studies, they have not been highly resolved numerically. An alternative approach is to concentrate on a steady-state systems and to treat the mushy and the liquid regions in separate domains. The fluid motion within the mushy region is modeled by Darcy's equation, while that in the liquid region is modeled in general by the Navier-Stokes equations. It is then necessary to match the thermodynamic and fluid-mechanical variables across the mush-liquid interfaces, the position of which also need to be determined as part of the overall solution. Moreover, the chimney wall is itself a free boundary, which must be determined.

Hills, Loper & Roberts (1983) developed a set of governing equations for a mushy zone, based on principles of diffusive mixture theory, and solved a one-dimensional freezing problem. We use simplified version of this complete model for a mushy zone. Roberts $ Loper (1983) used lubrication theory to model the flow through a chimney of prescribed uniform cross-section and analyzed some aspects of the flow in the mushy layer around the chimney. Worster (1991) analyzed the structure of a convecting mushy layer by seeking solutions in the asymptotic regime $R_m >> 1$, where R_m is an appropriately defined Rayleigh number. In his analysis, he estimated the volume flux through a chimney by using a type of Pohlhausen method, and related the flux to the vertical downflow of the mushy layer by global mass conservation. Therefore, he achieved an expression for the vertical flow W of the mush in terms of the single parameter $\mathcal{F}$ which denotes the strength of convection. He showed that increasing the strength of convection reduces the depth of the mush and increases the solid fraction.

We also parameterize the strength of convection and obtain similar results; the thickness of the mush decreases and the solid fraction increases when the strength of convection increases. Unlike Worster (1991), we analyze first the structure of the mush, and we find solutions of the temperature, the solid fraction, and the pressure in the chimney wall. Our pressure expression shows that the pressure in the chimney can be large when the permeability of the chimney wall is small.

The purpose of this study is to analyze the flow of a mush-chimney system by developing a simple model having convection with horizontal divergence. We study how convection affects the mass fraction of the solid, the thickness of the mush, the radius of a chimney and the number density of chimneys. With knowledge of composition, velocity, and pressure obtained from the mush, we find the radius of the chimney and analyze the fluid flow in a chimney by using the von Karman Pohlhausen technique.

2. Mathematical formulation for a mush-chimney system

We consider an axisymmetric model of a mush-chimney system containing only one chimney. We assume the system to be steady in a frame fixed to the mush-solid interface, which moves upward relative to the solid with a prescribed constant speed V. The liquid region has fixed temperature T_∞ and composition ξ_∞ of light constitutent as $z \to \infty$, where z measures vertical displacement in the moving frame. The temperature decreases downward, and we consider the case in which a mushy zone separates a completely solid region from a completely liquid region. In this model problem we assume that the eutectic front, at which the temperature is equal to the eutectic temperature T_e and below which the system is completely solid, can be maintained at the fixed position $z = 0$. The mush-liquid interface $z = h$ is a free boundary to be determined as part of the solution. We nondimensionalize the governing equations by choosing a thermal length scale κ/V and thermal time scale κ/V^2, where κ is the thermal diffusivity $\kappa = k/\rho_r c_p$, c_p is the specific heat, k is the thermal conductivity, and ρ_r is a reference density. Specifically, put $\mathbf{x} = (\kappa/V)\mathbf{x}^*$, $\mathbf{w} = V\mathbf{w}^*$, $p = \kappa\eta/\gamma_o p^*$, $\gamma = \gamma_o\gamma^*$, $T - T_r = (T_r - T_e)T^*$, $\xi - \xi_\infty = (\xi_e - \xi_\infty)\xi^*$, where γ_o is a reference value of the permeability of the mush, η is the dynamic viscosity of the liquid, T_r is the liquidus temperature of ξ_∞ and ξ_e is the eutectic composition of light constitutent. Dropping the asterisks, conservation of total mass, conservation of a constitutent in the liquid phase and energy, the liquid momentum equation and the liquid relation, respectively, become

$$\nabla \cdot \mathbf{w} = 0, \tag{1}$$

$$\mathbf{w}.\nabla\xi = \frac{\partial \xi}{\partial z} - \frac{\partial(\phi\xi)}{\partial z} - C\frac{\partial \phi}{\partial z}, \tag{2}$$

$$\mathbf{w} \cdot \nabla T = \nabla^2 T + \frac{\partial T}{\partial z} - S\frac{\partial \phi}{\partial z}, \tag{3}$$

$$\frac{\mathbf{w}}{\gamma(\phi)(1-\phi)^2} + \nabla p + R_a T\mathbf{z} = 0, \tag{4}$$

$$T = -\xi, \tag{5}$$

where ϕ is the mass fraction of solid, $\mathbf{z}$ is the unit upward vector. The parameters are a Stefan number $S = L/c_p(T_r - T_e)$, which represents the ratio of the latent heat needed to melt the solid and the heat needed to warm the solid from its eutectic temperature to the reference temperature T_r, the ratio of composition $C = \xi_\infty/(\xi_e - \xi_\infty)$, which denotes the compositional contrast between solid and liquid phases compared to the typical variations of concentration within the liquid (Worster, 1991), and a Rayleigh number $R_a = \gamma_o\rho_r(\beta - \alpha\Gamma)g(T_r - T_e)/V\eta\Gamma$, which will act to drive buoyancy induced convection in the mush if it is large enough, where L is the latent heat, Γ is the liquidus slope, g is the gravity, α and β, are constant coefficients of thermal and compositional expansion. Let (r, θ, z) be cylindrical coordinates with z upwards. We assume within the main body of the mush that the vertical velocity w, the temperature T, the mass fraction of light constitutent ξ and the mass fraction of solid ϕ, depend on z only. Then the governing equations for the mush are

$$\frac{\partial(ru_m(r,z))}{\partial r} - \frac{\partial(rw_m(z))}{\partial z} = 0 \tag{6}$$

$$-w_m(z)T_m'(z) = T_m'(z) - (\phi_m(z)T_m(z))' + C\phi_m'(z) \tag{7}$$

$$-w_m(z)T_m'(z) = T_m''(z) + T_m'(z) - S\phi_m'(z) \tag{8}$$

$$\frac{u_m(r,z)}{F(\phi_m(z))} + \frac{\partial p_m(r,z)}{\partial r} = 0, \tag{9}$$

$$\frac{-w_m(z)}{F(\phi_m(z))} + \frac{\partial p_m(r,z)}{\partial z} + R_a T_m(z) = 0, \tag{10}$$

$$T_m(z) = -\xi_m(z), \tag{11}$$

where the subscripts 'm' represents the mush, prime $'$ denotes the derivative with respect to z, and $F(\phi_m) = (1-\phi_m)^5$ for Worster's choice (1991). Note that very large Lewis number $Le = \kappa/D_o$ is assumed in equation (23), where D_o is the compositional diffusivity in the liquid. From the above equations, we obtain the set of equations involving variables $T_m(z)$, $\phi_m(z)$ and $w_m(z)$:

$$T'_m = (C + S - T_m)\phi_m + H, \tag{12}$$

$$\phi'_m = \frac{T'_m}{T_m - C}(1 + w_m - \phi_m), \tag{13}$$

$$w'_m = WF(\phi_m), \tag{14}$$

$$p_m(r,z) = p_a(z) + p_b(r), \tag{15}$$

with the boundary conditions

$$T_m(h_0) = 0, \qquad \phi_m(h_0) = 0, \qquad T_m(0) = -1, \qquad w_m(0) = 0, \tag{16}$$

where h_0 is a constant mush-liquid interface away from the chimney wall, as suggested by the experiments of ammonium chloride solution (Roberts & Loper (1983), Chen & Chen (1991)), $H = T'(h_0)$ measures the amount of superheat and $W = w'(h_0)$.

We now introduce the governing equations for a chimney, and nondimensionalize them by using the same scales as in the mush. We rewrite the equations for a steady state in a reference frame moving steadily upward with the speed V and rescale the velocity in the chimney by balancing fluxes of the mush and the chimney. We assume that the thickness of the mush $h_0 = 0(1)$ and the Prandtl number $\sigma >> R^2$ since $\sigma \sim 10$ (Emms & Fowler (1994) and Hellawell *et. al.* (1993)) for NH_4Cl-H_2O and R is less than or equal to h_0 (Roberts & Loper, 1983). We derive simplified equations for a model of the chimney with the approximation $a(z) << h_0$ and $R^2/a_h^2 >> 1$, where a_h is the radius of the top of the chimney.

The governing equations in the chimney are

$$\nabla \cdot \mathbf{u_c} = 0, \tag{17}$$

$$\frac{DT_c}{Dt} = \kappa \nabla^2 T_c, \tag{18}$$

$$\frac{D\xi_c}{Dt} = D_0\nabla^2\xi_c, \tag{19}$$

$$\rho_r\frac{D\mathbf{u_c}}{Dt} = -\nabla p_c + (\rho^l - \rho_r)\hat{\mathbf{z}} + \eta\nabla^2\mathbf{u_c}, \tag{20}$$

$$\rho^l = \rho_r[1 - \alpha(T_c - T_r) - \beta(\xi_c - \xi_r)], \tag{21}$$

where the subscripts $'c'$ represents the chimney, κ and D_0 are the thermal and material diffusivity, respectively, $D/Dt = \partial/\partial t + \mathbf{u_c}\cdot\nabla$ is the material derivative following the fluid element, ρ_r is the reference density, η is the dynamic viscosity of the liquid, p_c is the nonhydrostatic pressure field, α and β are the constant coefficients of thermal and compositional expansion.

As before, we nondimensionalize the governing equations in the chimney by using a thermal length scale κ/V, thermal time scale κ/V^2, and velocity scale V. We put $t = \kappa/V^2t^*$, $\mathbf{x} = \kappa/V\mathbf{x}^*$, $\mathbf{u_c} = V\mathbf{u_c}^*$, $p_c = \kappa\eta/\gamma_0 p_c^*$, $T_c - T_r = (T_r - T_e)T_c^*$, and $\xi_c - \xi_\infty = (\xi_e - \xi_\infty)\xi_c^*$, where T_r is the liquidus temperature of ξ_∞, and T_e and ξ_e are the eutectic temperature and composition. Dropping the asterisks,

$$\nabla\cdot\mathbf{u_c} = 0, \tag{22}$$

$$\frac{DT_c}{Dt} = \nabla^2 T_c, \tag{23}$$

$$\frac{D\xi_c}{Dt} = \frac{1}{Le}\nabla^2\xi_c, \tag{24}$$

$$\frac{1}{\sigma}\frac{D\mathbf{u_c}}{Dt} = -\frac{\nabla p_c}{\delta} + (R_\alpha T_c + R_\beta\xi_c)\hat{\mathbf{z}} + \nabla^2\mathbf{u_c}, \tag{25}$$

where

$$\delta = \frac{\gamma_0 V^2}{\kappa^2}$$

is a measure of dendritic spacing,

$$\sigma = \frac{\eta}{\rho_r\kappa}$$

is the Prandtl number,

$$Le = \frac{\kappa}{D_0}$$

is the Lewis number,

$$R_\alpha = \frac{\alpha(T_r - T_e)g\gamma_0\rho_r}{\eta V\delta}$$

and

$$R_\beta = \frac{\beta(\xi_e - \xi_\infty)g\gamma_0\rho_r}{\eta V\delta}$$

are the thermal and compositional Rayleigh number, respectively.

We rewrite (23), (24) and (25) in a reference frame moving steadily upward with the solidification speed V as follows:

$$\mathbf{u_c}\cdot\nabla T_c = \frac{\partial T_c}{\partial z} + \nabla^2 T_c, \tag{26}$$

$$\mathbf{u_c}\cdot\nabla \xi_c = \frac{\partial \xi_c}{\partial z} + \frac{1}{Le}\nabla^2 \xi_c, \tag{27}$$

$$\frac{1}{\sigma}[-\frac{\partial \mathbf{u_c}}{\partial z} + (\mathbf{u_c}\cdot\nabla)\mathbf{u_c}] = -\frac{\nabla p_c}{\delta} + R_\beta\xi_c\hat{\mathbf{z}} + \nabla^2\mathbf{u_c}. \tag{28}$$

Note that in the chimney, since the compositional buoyancy dominates the thermal buoyancy, in (28) we have neglected the thermal contribution to the buoyancy force. Therefore, the energy equation (26) is not important in our analysis.

We balance the fluxes of the mush and the chimney. Then we have $\bar{w}_c a_h^2 = w_m R^2$, where $\bar{w}_c$ is the average velocity over z-cross section.
If we assume that $w_m = O(1)$, we obtain $\bar{w}_c = O(\frac{R^2}{a_h^2})$.
We scale the radius and the velocity of the chimney as the following:

$$r = a_h s, \quad u_c = \frac{R^2}{a_h}u_s, \quad w_c = \frac{R^2}{a_h^2}w_s,$$

where $\mathbf{u_c} = (u_c, w_c)$.
We now write equations (22), (27) and (28) in a scaled form.

$$\frac{1}{s}\frac{\partial}{\partial s}(su_s) + \frac{\partial w_s}{\partial z} = 0, \tag{29}$$

$$\frac{R^2}{a_h^2}D_s\xi_c = \frac{\partial \xi_c}{\partial z} + \frac{1}{a_h^2 Le}[\frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial \xi_c}{\partial s}) + \frac{\partial^2 \xi_c}{\partial z^2}], \tag{30}$$

$$\frac{R^4}{a_h^3}\frac{1}{\sigma}(-\frac{a_h^2}{R^2}\frac{\partial u_s}{\partial z} + D_s u_s) = -\frac{1}{a_h\delta}\frac{\partial p_c}{\partial s} + \frac{R^2}{a_h^3}\nabla_s^2 u_s, \tag{31}$$

$$\frac{R^4}{a_h^4}\frac{1}{\sigma}(-\frac{a_h^2}{R^2}\frac{\partial w_s}{\partial z}+D_s w_s) = -\frac{1}{\delta}\frac{\partial p_c}{\partial z} + R_\beta \xi_c + \frac{R^2}{a_h^4}\nabla_s^2 w_s, \tag{32}$$

where

$$D_s = u_s\frac{\partial}{\partial s} + w_s\frac{\partial}{\partial z}, \qquad \nabla_s^2 = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial}{\partial s}) + a_h^2\frac{\partial^2}{\partial z^2}.$$

We obtain our model by utilizing the fact that $a_h << h_0$ and $R^2 >> a_h^2$, and assuming $h_0 = O(1)$. Also, we assume $\sigma >> R^2$. Then we may neglect the inertia term compared to the viscous term in (31) and (32). Furthermore, if we use the data of the ammonium-chloride experiment, $\delta \sim 10^{-4}$ (Emms & Fowler, 1994) and $R \sim 10a_h$ (Hellawell *et. al.*, 1993), we obtain from (31)

$$p_c = p_c(z). \tag{33}$$

Also, equations (30) and (32) simplify to

$$u_s\frac{\partial \xi_c}{\partial s} + w_s\frac{\partial \xi_c}{\partial z} = 0, \tag{34}$$

$$0 = -\frac{1}{\delta}\frac{dp_c}{dz} + R_\beta \xi_c + \frac{R^2}{a_h^4}[\frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial w_s}{\partial s})]. \tag{35}$$

Note that we used in (34) the fact that $Le >> 1$ in most liquids. We analyze our model consisting of (29), (33), (34) and (35) with boundary conditions on the chimney wall

$$< p >= 0, \quad < \xi >= 0, \quad < u >= 0, \quad w_c = 0. \tag{36}$$

where, $' <> '$ represents a jump condition on the chimney wall.

3. Convection in a steady-state chimney

In this section, we find that the mass fraction of solid in a mush and the depth of a mush when the strength of convection is given. Also, we analyze the fluid flow in a chimney and find the equation for the radius of the chimney by using the von Karman Pohlhausen technique.

From equation (12), we know that $dT/dz > 0$ Now we let $T' = X(T)$, $\phi = Y(T)$, and $w = Z(T)$. From equations (12), (13) and (14), we obtain

$$\frac{dY}{dT} = \frac{1+Z-Y}{T-C} \tag{37}$$

(38)
$$\frac{dZ}{dT} = \frac{W(1-Y)^5}{(C+S-T)Y+H}$$

with conditions

(39)
$$Y(0) = 0, \qquad Z(-1) = 0.$$

We have five parameters; $w_h = w(h_0)$, $V = w'(h_0)$, measures of convection; S, a Stefan number ; C, the ratio of composition ; $H = T'(h_0)$, a measure of superheat. From the linear liquidus equation $T = T_r - \Gamma(\xi - \xi_\infty)$, we have $S = L(1+C)/c_p \mathit{\Gamma} \xi_e$, and $H = (T_\infty - T_r)(1+C)(1+w_h)/\mathit{\Gamma}\xi_e$. Both C and S are experimentally controllable parameters, H depends on C and w_h, while H, w_h and V are internal parameters. But, since H depends on w_h, we actually have two internal parameters in our mathematical system. Given w_h, the value of V is determined numerically by stopping condition (39.1) and therefore, we can solve the equations (37) and (38). For example, by the NH_4Cl-74wt%H_2O experiment, we get $C = 12.3$ and $S = 3.3$. If $w_h = 1.0$, then $V = 1.27$, and the solid fraction at the bottom of the mush is 0.2.

Now, we analyze the fluid flow in a chimney. We introduce a Stokes stream function ψ such that

(40)
$$(u_s, w_s) = (-\frac{1}{s}\frac{\partial \psi_s}{\partial z}, \frac{1}{s}\frac{\partial \psi_s}{\partial s}).$$

where ψ satisfies the following boundary conditions

(41)
$$\psi_s(s_a, z) = \frac{1}{2} w_m(z),$$

(42)
$$(\frac{\partial \psi_s}{\partial s})_{s=s_a} = 0, \qquad \psi_s(0,z) = 0, \qquad \psi_s(s,0) = 0,$$

$s_a(z)$ is the scaled form of the radius of the chimney, *i.e.*, $s_a = a(z)/a_h$.

We express the z-momentum equation (35) by using the stream function ψ_s.

(43)
$$0 = -\frac{1}{\delta}\frac{\partial p_c}{\partial z} + R_\beta \xi_c + \frac{R^2}{a_h^4}\frac{1}{s}\frac{\partial}{\partial s}[s\frac{\partial}{\partial s}(\frac{1}{s}\frac{\partial \psi_s}{\partial s})].$$

If we substitute (40) into (29), we obtain

(44)
$$-\frac{1}{s}\frac{\partial \psi_s}{\partial z}\frac{\partial \xi_c}{\partial s} + \frac{1}{s}\frac{\partial \psi_s}{\partial s}\frac{\partial \xi_c}{\partial z} = 0.$$

Equation (44) expresses the fact that ξ_c is constant along streamlines, so that

$$\xi_c = \xi_c(\psi_s). \tag{45}$$

If we use (45), together with the boundary conditions (42), we obtain

$$(\frac{\partial \xi_c}{\partial s})_{s=s_a} = 0, \qquad \xi_c(0,z) = 1, \qquad \xi_c(s,0) = 1. \tag{46}$$

Also, the continuity of composition (36.2) at the chimney wall requires

$$\xi_c(s_a, z) = \xi_w, \tag{47}$$

where ξ_w is a known function from the solution of the mush.

We now begin the von Karman Pohlhausen technique by introducing trial functions for ψ_s and ξ_c

$$\psi_s = \psi_w P_1(x) + \alpha(z)P_2(x) + \beta(z)P_3(x), \tag{48}$$

$$\xi_c = 1 - (1-\xi_w)P_1(x) - \lambda(z)P_2(x), \tag{49}$$

where $\alpha(z)$, $\beta(z)$ and $\lambda(z)$ are to be determined,

$$P_1(x) = (2x^2 - x^4), \qquad P_2(x) = x^2(1-x^2)^2, \qquad P_3(x) = x^2(1-x^2)^3,$$

and

$$x = \frac{s}{s_a} = \frac{r}{a(z)}.$$

We substitute (48) and (49) into (43). We then have

$$1-(1-\xi_w)P_1-\lambda P_2 = \frac{1}{R_\beta \delta}\frac{dp_m}{dz} + A(z)[\psi_w + \alpha(2-6x^2) + \beta(3-18x^2+18x^4)],$$

or

$$1-(1-\xi_w)P_1-\lambda P_2 = X + A(z)[\alpha(2-6x^2)+\beta(3-18x^2+18x^4)], \tag{50}$$

where

$$A(z) = \frac{16}{R_\beta {s_a}^4}\frac{R^2}{a_h^4} \tag{51}$$

represents the ratio of viscous and buoyancy forces, and

$$X = \frac{1}{R_\beta \delta}\frac{dp_m(a,z)}{dz} + A(z)\psi_w(z). \tag{52}$$

Note that the continuity of the pressure in (36.1) on the wall was used in (50). The pressure term in (52) is known by (15), (10), and (11);

$$\frac{dp_m(a,z)}{dz} = \frac{w_m(z)}{F(\phi_m(z))} + R_a \xi_m(z) + \frac{V}{2s_a}\frac{ds_a}{dz}, \tag{53}$$

where a/R^2 has been neglected compared to $1/a$ in the last term of (53).

We satisfy (50) and (44) on the axis, on the wall, and in an integrated form over a z-cross section of the chimney. We then obtain from (50)

$$1 = X + A(2\alpha + 3\beta), \tag{54}$$

$$\xi_w = X + A(-4\alpha + 3\beta), \tag{55}$$

$$\frac{1}{2} - \frac{1-\xi_w}{3} - \frac{\lambda}{24} = \frac{X - A\alpha}{2}. \tag{56}$$

Since the requirements of (44) on the axis and on the wall are equivalent to the first two conditions in (46), respectively, we have

$$\frac{d}{dz}\left(\int_0^1 \xi_c \frac{\partial \psi_s}{\partial x}\, dx\right) = \xi_w \frac{d\psi_w}{dz}. \tag{57}$$

If we replace ξ_c and ψ_s in (57) by (48) and (49), we obtain

$$\frac{d}{dz}\left[\psi_w - (1-\xi_w)\left(\frac{\psi_w}{2} - \frac{\alpha}{10} - \frac{\beta}{15}\right) - \frac{\lambda\psi_w}{10} + \frac{\lambda\beta}{210}\right] = \frac{d}{dz}(\xi_w\psi_w) - \psi_w\frac{d\xi_w}{dz}$$

or

$$\frac{d}{dz}\left[(1-\xi_w)\left(\frac{\psi_w}{2} + \frac{\alpha}{10} + \frac{\beta}{15}\right) - \frac{\lambda\psi_w}{10} + \frac{\lambda\beta}{210}\right] = \psi_w\frac{dT_m}{dz}, \tag{58}$$

where the liquidus relation (5) was used.

We now solve (54), (55) and (56) for $\alpha(z)$, $\beta(z)$ and $\lambda(z)$. Then we have

(59)

$$\alpha(z) = \frac{1}{6A}(1-\xi_w), \quad \beta(z) = \frac{1}{9A}(2+\xi_w-3X), \quad \lambda(z) = 6(1+\xi_w-2X)$$

By integrating (58), we also find an expression for the radius of the chimney $s_a(z)$.

$$(1-\xi_w)\left(\frac{\psi_w}{2}+\frac{\alpha}{10}+\frac{\beta}{15}\right) - \frac{\lambda}{10}\left(\psi_w - \frac{\beta}{21}\right) - \frac{\lambda(0)\beta(0)}{210} = \int_0^z \psi_w \frac{dT_m}{dz}\, dz. \tag{60}$$

Note that we have used in (60) $\xi_w(0) = 1$, $\psi_w(0) = 0$. We replace $\alpha(z)$, $\beta(z)$ and $\lambda(z)$ in (60) by (59). We then obtain an expression for $A(z)$

$$21(1-\xi_w)(360\psi_w+\frac{22-7\xi_w-15X}{A})-48(1+\xi_w-2X)(189\psi_w-\frac{2+\xi_w-3X}{A})$$

$$= 15120[\frac{4}{210A(0)}(1-X(0))^2+\int_0^z \psi_w \frac{dT_m}{dz}\,dz], \tag{61}$$

or if we substitute (52) into (61) after replacing dp_m/dz by (53), since $V/R_a << 1$ and (from the ammonium chloride experiment) the radius of the chimney does not change rapidly, we obtain a simple expression for $A(z)$

$$w_m^2 A^2 - [\frac{105}{64}\int_{-1}^{T_m} w_m\,dT_m + \frac{721}{3072}(1-\xi_w)w_m - \frac{65}{32}\frac{w_m^2}{R_a F}]A$$

$$+\frac{31}{256}(1-\xi_w)^2 - \frac{217}{1536}(1-\xi_w)\frac{w_m}{R_a F} + (\frac{w_m}{4R_a F})^2 = 0, \tag{62}$$

If we solve the above equation for $A(z)$ by utilizing the knowledge of w_m, T_m, ξ_w and ϕ obtained from analyzing the mush, then we get the stream functionψ_s from the equation (48).

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