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COVERING $(L,\odot)$-UNIFORMITIES AND HUTTON $(L,\otimes)$-UNIFORMITES

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Abstract. In strictly two-sided, commutative quantale, we introduce the notion of Hutton $(L,\otimes)$-uniform spaces and covering $(L,\odot)$-uniform spaces and investigate the properties of them.

1. Introduction

Uniformities in fuzzy sets, have the entourage approach of Lowen [16] and Höhle [6-7] based on powersets of the form $L^{X\times X}$, the uniform covering approach of Kotzé [14] and the uniform operator approach of Rodabaugh [18] as generalization of Hutton [10] based on powersets of the form $(L^X)^{(L^X)}$. For a fixed basis L, algebraic structures in L (cqm-lattices, quantales, MV-algebras) are extended for a completely distributive lattice L or the unit interval or t-norms. Recently, Gutiérrez García et al.[5] introduced L-valued Hutton unifomity on GL-monoid and Kim et al. [11-13] studied Hutton $(L,\otimes)$-unifomity and $(L,\odot)$-uniformity in a sense of the entourage approach on stsc-quantale.

In this paper, for a stsc-quantale $(L,\odot)$ as a somewhat different aspect in [4,5], we introduce the notion of $(L,\odot)$-covering uniformities in a sense García et al. [4-5] and Kotzé [14] based on coverings of L^X and Hutton $(L,\otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [18]. We investigate the relationship between Hutton $(L,\otimes)$-uniformities and covering $(L,\odot)$-covering uniformities.

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2. Preliminaries

Definition 2.1 [8,17]. A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, \vee, \wedge, \top, \bot)$ is a completely distributive lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L, \odot)$ is a commutative semigroup;

(L3) $a = a \odot \top$, for each $a \in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Definition 2.2. Let $\Omega(X)$ be a subset of $(L^X)^{(L^X)}$ such that if it satisfies, $\phi \in \Omega(X)$,

(O1) $\phi(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

(O2) $\lambda \leq \phi(\lambda)$ for all $\lambda \in L^X$.

Theorem 2.3 [11-13]. *For $\phi, \phi_1, \phi_2 \in \Omega(X)$, we define, for all $\lambda, \rho \in L^X$,*

$$\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),$$

$$\phi_1 \otimes \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\}.$$

Then the following properties hold:

(1) $\phi^{\leftarrow}(\rho) = \bigvee\{\lambda \in L^X \mid \phi(\lambda) \leq \rho\}$ *such that* $\phi^{\leftarrow}$ *is a right adjoint of* ϕ *with* $\phi \circ \phi^{\leftarrow}(\rho) \leq \rho$ *and* $\lambda \leq \phi^{\leftarrow} \circ \phi(\lambda)$.

(2) $\phi_1 \circ \phi_2 \in \Omega(X)$ *and* $\phi_1 \otimes \phi_2 \in \Omega(X)$.

(3) $\phi_1 \otimes \phi_2 \leq \phi_1$ *and* $\phi_1 \otimes \phi_2 \leq \phi_2$.

(4) $(\phi_1 \otimes \phi_2) \otimes \phi_3 = \phi_1 \otimes (\phi_2 \otimes \phi_3)$.

Lemma 2.4 [11-13]. *Let $f : X \to Y$ be a function. We define the image and preimage operators*

$$f^{\Rightarrow} : (L^X)^{(L^X)} \to (L^Y)^{(L^Y)}, \;\; f^{\Leftarrow} : (L^Y)^{(L^Y)} \to (L^X)^{(L^X)}$$

such that for each $\psi \in (L^Y)^{(L^Y)}$ for all $\mu, \mu_1, \mu_2 \in L^X, \rho_1, \rho_2 \in L^Y$,

$$f^{\Rightarrow}(\phi)(\rho) = (f^{\rightarrow} \circ \phi \circ f^{\leftarrow})(\rho) = f^{\rightarrow}(\phi(f^{\leftarrow}(\rho)),$$

$$f^{\Leftarrow}(\psi)(\mu) = (f^{\leftarrow} \circ \psi \circ f^{\rightarrow})(\mu) = f^{\leftarrow}(\psi(f^{\rightarrow}(\mu))).$$

For each $\psi, \psi_1, \psi_2 \in \Omega(Y)$ and $\phi_1, \phi_2 \in \Omega(X)$, we have the following properties.

(1) *The pair $(f^{\Rightarrow}, f^{\Leftarrow})$ is a Galois connection; i.e., $f^{\Rightarrow} \dashv f^{\Leftarrow}$.*

(2) *$f^{\rightarrow}(\mu_1 \odot \mu_2) \leq f^{\rightarrow}(\mu_1) \odot f^{\rightarrow}(\mu_2)$ with equality if f is injective and $f^{\leftarrow}(\rho_1 \odot \rho_2) = f^{\leftarrow}(\rho_1) \odot f^{\leftarrow}(\rho_2)$.*

(3) *$f^{\Leftarrow}(\psi) \in \Omega(X)$.*

(4) *If $\psi_1 \leq \psi_2$, then $f^{\Leftarrow}(\psi_1) \leq f^{\Leftarrow}(\psi_2)$.*

(5) *$f^{\Leftarrow}(\psi_1) \circ f^{\Leftarrow}(\psi_2) \leq f^{\Leftarrow}(\psi_1 \circ \psi_2)$ with equality if f is onto.*

(6) *$f^{\Leftarrow}(\psi_1) \otimes f^{\Leftarrow}(\psi_2) = f^{\Leftarrow}(\psi_1 \otimes \psi_2)$.*

Definition 2.5 [12]. A subset $\mathbb{T}$ of L^X is called an $(L,\odot)$-*topology* on X if it satisfies the following conditions:

(T1) $1_X, 1_\emptyset \in \mathbb{T}$.

(T2) If $\lambda_1, \lambda_2 \in \mathbb{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathbb{T}$.

(T3) If $\lambda_i \in \mathbb{T}$ for all $i \in \Gamma$, then $(\bigvee_{i\in\Gamma} \lambda_i) \in \mathbb{T}$

(TO) If $\lambda_1, \lambda_2 \in \mathbb{T}$, then $\lambda_1 \odot \lambda_2 \in \mathbb{T}$.

The pair $(X, \mathbb{T})$ is called an $(L,\odot)$ *-topological space.*

Let $(X, \mathbb{T}_1)$ and $(Y, \mathbb{T}_2)$ be $(L,\odot)$ -topological spaces. A function $f : (X, \mathbb{T}_1) \to (Y, \mathbb{T}_2)$ is *L-continuous* if $f^{\leftarrow}(\lambda) \in \mathbb{T}_1$, for every $\lambda \in \mathbb{T}_2$.

Theorem 2.6 [15,17]. *Let $(M, \leq)$ and $(N, \leq)$ be a partially ordered set and $\phi : M \to N$ join-preserving map,i.e; $\phi(\bigvee x_i) = \bigvee \phi(x_i)$. ϕ has a right adjoint $\psi : N \to M$ as follows*

$$\psi(y) = \bigvee\{x \in M \mid \phi(x) \leq y\}.$$

Moreover, $\phi(x) \leq y$ iff $x \leq \psi(y)$. Equivalently, $id_M \leq \psi \circ \phi$ and $\phi \circ \psi \leq id_N$.

3. Covering $(L,\odot)$-uniformities and Hutton $(L,\otimes)$-uniformites

We define a somewhat different aspect of uniformities in [4], we introduce the notion of $(L,\otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [18].

A function $\phi \in \Omega(X)$ is called symmetric if it satisfies

(S) $\phi(\lambda) \odot \mu \neq \overline{0}$ iff $\lambda \odot \phi(\mu) \neq \overline{0}$, for each $\lambda, \mu \in L^X$.

Definition 3.1. A nonempty subset $\mathbb{U}$ of $\Omega(X)$ is called a *Hutton $(L,\otimes)$-uniformity* on X if it satisfies the following conditions:

(U1) If $\phi \leq \psi$ with $\phi \in \mathbb{U}$ and $\psi \in \Omega(X)$, then $\psi \in \mathbb{U}$.

(U2) For each $\phi \in \mathbb{U}$, there exists $\psi \in \mathbb{U}$ such that $\psi \circ \psi \leq \phi$.

(U3) For each $\phi, \psi \in \mathbb{U}$, $\phi \otimes \psi \in \mathbb{U}$.

(U4) For each $\phi \in \mathbb{U}$, there exists a symmetric $\psi \in \mathbb{U}$ such that $\psi \leq \phi$.

The pair $(X, \mathbb{U})$ is called a *Hutton $(L,\otimes)$-uniform space*. Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be Hutton $(L,\otimes)$-uniformites on X. If $\mathbb{U}_1 \subset \mathbb{U}_2$, $\mathbb{U}_2$ is called *finer than* $\mathbb{U}_1$.

Let $(X, \mathbb{U}_1)$ and $(Y, \mathbb{U}_2)$ be Hutton $(L,\otimes)$-uniform spaces. A function $f : (X, \mathbb{U}_1) \to (Y, \mathbb{U}_2)$ is *H-uniformly continuous* if $f^{\Leftarrow}(\psi) \in \mathbb{U}_1$, for every $\psi \in \mathbb{U}_2$.

Theorem 3.2. *Let $\mathbb{U}$ be a Hutton $(L,\otimes)$-uniformity on X. We define a subset $\mathbb{T}_{\mathbb{U}}$ of L^X as follows:*

$$\mathbb{T}_{\mathbb{U}} = \{\rho \in L^X \mid \exists \phi \in \mathbb{U}, \phi(\rho) = \rho\}.$$

Then $\mathbb{T}_{\mathbb{U}}$ is an $(L,\odot)$-topology on X induced by $\mathbb{U}$.

Proof. (T1). Since $\phi(\overline{0}) = \overline{0}$ and $\phi(\overline{1}) = \overline{1}$ for all $\phi \in \mathbb{U}$, we have $\overline{0}, \overline{1} \in \mathbb{T}_{\mathbb{U}}$.

(T2) and (TO). Let $\lambda_i \in \mathbb{T}_{\mathbb{U}}$ for $i = 1, 2$. Then $\phi_i \in \mathbb{T}_{\mathbb{U}}$ such that $\phi_i(\lambda_i) = \lambda_i$. Since $\overline{1} \odot (\lambda_1 \wedge \lambda_2) = \lambda_1 \wedge \lambda_2$, we have

$$\begin{aligned}(\phi_1 \otimes \phi_2)(\lambda_1 \wedge \lambda_2) &\leq \phi_1(\lambda_1 \wedge \lambda_2) \wedge \phi_2(\lambda_1 \wedge \lambda_2)\\ &\leq \phi_1(\lambda_1) \wedge \phi_2(\lambda_2) = \lambda_1 \wedge \lambda_2,\end{aligned}$$

$$(\phi_1 \otimes \phi_2)(\lambda_1 \odot \lambda_2) \leq \phi_1(\lambda_1) \odot \phi_2(\lambda_2) \leq \lambda_1 \odot \lambda_2.$$

So, $\lambda_1 \wedge \lambda_2, \lambda_1 \odot \lambda_2 \in \mathbb{T}_{\mathbb{U}}$.

(T3) Let $\lambda_i \in \mathbb{T}_{\mathbb{U}}$ for $i \in \Gamma$. Then, for each $i \in \Gamma$, there exists $\phi_i \in \mathbb{U}$ such that $\phi_i(\lambda_i) = \lambda_i$.

Suppose that $\phi(\bigvee_{i\in\Gamma} \lambda_i) \not\leq \bigvee_{i\in\Gamma} \lambda_i$ for all $\phi \in \mathbb{U}$. Since $\phi(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi(\lambda_i)$, there exists $\{i_1, i_2, ... i_m\} \subset \Gamma$ such that

$$\phi(\vee_{k=1}^{m} \lambda_{i_k}) \not\leq \bigvee_{i\in\Gamma} \lambda_i.$$

Put $\phi = \otimes_{k=1}^m \phi_{i_k}$. Then

$$\otimes_{k=1}^m \phi_{i_k}(\vee_{k=1}^m \lambda_{i_k}) = \vee_{k=1}^m (\otimes_{k=1}^m \phi_{i_k})(\lambda_{i_k}) \le \vee_{k=1}^m \phi_{i_k}(\lambda_{i_k})$$
$$\le \vee_{k=1}^m \lambda_{i_k} \le \bigvee_{i\in\Gamma} \lambda_i.$$

It is a contradiction. Hence there exists $\phi \in \mathbb{U}$ such that $\phi(\bigvee_{i\in\Gamma}\lambda_i) = \bigvee_{i\in\Gamma}\lambda_i$. Hence $\bigvee_{i\in\Gamma}\lambda_i \in \mathbb{T}_{\mathbb{U}}$. □

EXAMPLE 3.3. Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ a quantale defined by $x \odot y = 0 \vee (x + y - 1)$. Define $\phi \in \Omega(X)$ as

$$\phi(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ 1_{\{x,y\}} & \text{if } \overline{0} \ne \lambda \le 1_{\{x,y\}}, \\ 1_{\{z\}}, & \text{if } \overline{0} \ne \lambda \le 1_{\{z\}}, \\ \overline{1} & \text{otherwise.} \end{cases}$$

where $1_{\{x,y\}}$ and $1_{\{z\}}$ are characteristic functions. We have $(\phi\otimes\phi) = \phi$, $\phi \circ \phi = \phi$ and ϕ is symmetric. Thus, $\mathbb{U} = \{\psi \in \Omega(X) \mid \phi \le \psi\}$ is a Hutton $(L,\otimes)$-uniformity on X. From Theorem 3.2, we obtain an $(L,\odot)$-topology on X as follows:

$$\mathbb{T}_{\mathbb{U}} = \{\overline{0}, \overline{1}, 1_{\{x,y\}}, 1_{\{z\}}\}.$$

THEOREM 3.4. *Let $(Y,\mathbb{U})$ be a Hutton $(L,\otimes)$-uniform space, X a set and $f : X \to Y$ a function. Define a subset $f^{\Leftarrow}(\mathbb{U})$ of $\Omega(X)$ as follows:*

$$f^{\Leftarrow}(\mathbb{U}) = \{\phi \in \Omega(X) \mid \exists \psi \in \mathbb{U},\ \ f^{\Leftarrow}(\psi) \le \phi\}.$$

Then we have the following properties.

(1) *If ψ is symmetric, $f^{\Leftarrow}(\psi)$ is symmetric.*

(2) *The structure $f^{\Leftarrow}(\mathbb{U})$ is the coarsest Hutton $(L,\otimes)$-uniformity on X for which f is H-uniformly continuous.*

(3) *A map $g : (Z,\mathbb{U}_1) \to (X, f^{\Leftarrow}(\mathbb{U}))$ is H-uniformly continuous iff $f \circ g : (Z,\mathbb{U}_1) \to (Y,\mathbb{U})$ is H-uniformly continuous.*

Proof. (1) For $f^{\Leftarrow}(\psi)(\lambda)\odot\mu\neq\bar{0}$, we have

$$\begin{aligned}\bar{0}\neq f^{\rightarrow}\Big(f^{\Leftarrow}(\psi)(\lambda)\odot\mu\Big) &\leq f^{\rightarrow}(f^{\Leftarrow}(\lambda))\odot f^{\rightarrow}(\mu)\\ &\leq \psi(f^{\rightarrow}(\lambda))\odot f^{\rightarrow}(\mu).\end{aligned}$$

By the symmetric of ψ, $f^{\rightarrow}(\lambda)\odot\psi(f^{\rightarrow}(\mu))\neq\bar{0}$, there exists $x\in X$ such that

$$f^{\rightarrow}(\lambda)(f(x))\odot\psi(f^{\rightarrow}(\mu))(f(x))\neq\bar{0}.$$

It implies $\lambda(x)\odot f^{\Leftarrow}(\psi)(\mu)(x)\neq 0$. Hence $\lambda\odot f^{\Leftarrow}(\psi)(\mu)\neq\bar{0}$. So, $f^{\Leftarrow}$ is symmetric.

(2) First, we will show that $f^{\Leftarrow}(\mathbb{U})$ is a Hutton $(L,\otimes)$-uniformity on X.

(U1) Obvious.

(U2) For each $\phi\in f^{\Leftarrow}(\mathbb{U})$, there exists $\psi\in\mathbb{U}$ with $f^{\Leftarrow}(\psi)\leq\phi$. For $\psi\in\mathbb{U}$, since $(Y,\mathbb{U})$ is a Hutton $(L,\otimes)$-uniform space, by (U2), there exists $\gamma\in\mathbb{U}$ with $\gamma\circ\gamma\leq\psi$. By Lemma 2.4(5), since

$$f^{\Leftarrow}(\gamma)\circ f^{\Leftarrow}(\gamma)\leq f^{\Leftarrow}(\gamma\circ\gamma)\leq f^{\Leftarrow}(\psi)\leq\phi,$$

then $f^{\Leftarrow}(\gamma)\in f^{\Leftarrow}(\mathbb{U})$.

(U3) If $\phi_i\in f^{\Leftarrow}(\mathbb{U})$, for $i=1,2$, there exists $\psi_i\in\mathbb{U}$ with $f^{\Leftarrow}(\psi_i)\leq\phi_i$. Since $f^{\Leftarrow}(\psi_1)\otimes f^{\Leftarrow}(\psi_2)=f^{\Leftarrow}(\psi_1\otimes\psi_2)\leq\phi_1\otimes\phi_2$ from Lemma 2.4(6), we have $\phi_1\otimes\phi_2\in f^{\Leftarrow}(\mathbb{U})$.

(U4) By (1), it is easily proved.

Second, by definition of $f^{\Leftarrow}(\mathbb{U})$, $f^{\Leftarrow}(\psi)\in f^{\Leftarrow}(\mathbb{U})$, for all $\psi\in\mathbb{U}$. Hence $f:(X,f^{\Leftarrow}(\mathbb{U}))\to(Y,\mathbb{U})$ is H-uniformly continuous.

Finally, let $f:(X,\mathbb{U}_1)\to(Y,\mathbb{U})$ be H-uniformly continuous. For each $\phi\in f^{\Leftarrow}(\mathbb{U})$, there exists $\psi\in\mathbb{U}$ with $f^{\Leftarrow}(\psi)\leq\phi$. Since $f^{\Leftarrow}(\psi)\in\mathbb{U}_1$, then $\phi\in\mathbb{U}_1$. Hence $f^{\Leftarrow}(\mathbb{U})\subset\mathbb{U}_1$.

(3) Necessity of the composition condition is clear since the composition of H-uniformly continuous maps is H-uniformly continuous.

If $\phi\in f^{\Leftarrow}(\mathbb{U})$, there exists $\psi\in\mathbb{U}$ such that $f^{\Leftarrow}(\psi)\leq\phi$. Since $f\circ g$ is H-uniformly continuous, for $\psi\in\mathbb{U}$,

$$(f\circ g)^{\Leftarrow}(\psi)=g^{\Leftarrow}\circ f^{\Leftarrow}(\psi)\in\mathbb{U}_1.$$

Since $g^{\Leftarrow}(\phi)\geq g^{\Leftarrow}\circ f^{\Leftarrow}(\psi)\in\mathbb{U}_1$, we have $g^{\Leftarrow}(\phi)\in\mathbb{U}_1$. □

□

A subset C of L^X is a cover of X if $\bigvee\{\lambda \mid \lambda \in C\} = 1_X$. For any cover C_1, C_2, we denote $C_1 \leq C_2$ if each $\lambda \in C_1$, there exists $\mu \in C_2$ such that $\lambda \leq \mu$. We denote $C(X)$ as the collections of all covering of X.

THEOREM 3.5. *Let $f : X \to Y$ be a function. For $C, C_1, C_2 \subset L^X$ and $\lambda, \mu \in L^X$, we define*

$$st(\lambda, C) = \bigvee\{\mu \in C \mid \mu \odot \lambda \neq \emptyset\},$$

$$st(C) = \{st(\lambda, C) \mid \lambda \in C\},$$

$$C_1 \odot C_2 = \{\lambda_1 \odot \lambda_2 \mid \lambda_i \in C_i, i = 1, 2\}.$$

Then we have the following properties:
(1) If C is a cover, then $\lambda \leq st(\lambda, C)$ and $C \leq st(C)$.
(2) If C_1 and C_2 are covers, then $C_1 \odot C_2$ and $C_1 \wedge C_2$ are covers.
(3) If $\lambda \leq \mu$, then $st(\lambda, C) \leq st(\mu, C)$.
(4) If $C_1 \leq C_2$, then $st(\lambda, C_1) \leq st(\lambda, C_2)$.
(5) $st(\lambda \odot \mu, C_1 \odot C_2) \leq st(\lambda, C_1) \odot st(\mu, C_2)$.
(6) $st(\bigvee \lambda_i, C) = \bigvee st(\lambda_i, C)$.
(7) $st(st(\lambda, C), C) \leq st(\lambda, st(C))$.
(8) $f^{\to}(st(\lambda, C)) \leq st(f^{\to}(\lambda), f^{\to}(C))$.
(9) $f^{\to}(st(C)) \leq st(f^{\to}(C))$.
(10) $st(f^{\leftarrow}(\lambda), f^{\leftarrow}(C))) \leq f^{\leftarrow}(st(\lambda, C))$.
(11) $st(f^{\leftarrow}(C)) \leq f^{\leftarrow}(st(C))$.

Proof. (5) Suppose $st(\lambda \odot \mu, C_1 \odot C_2) \not\leq st(\lambda, C_1) \odot st(\mu, C_2)$. By the definition of $st(\lambda \odot \mu, C_1 \odot C_2)$, there exist $\rho_i \in C_i$ for $i = 1, 2$, with $(\rho_1 \odot \rho_2) \odot (\lambda \odot \mu) \neq \overline{0}$ such that

$$\rho_1 \odot \rho_2 \not\leq st(\lambda, C_1) \odot st(\mu, C_2)$$

Since $(\rho_1 \odot \rho_2) \odot (\lambda \odot \mu) \neq \overline{0}$ implies $\rho_1 \odot \lambda \neq \overline{0}$ and $\rho_2 \odot \mu \neq \overline{0}$, we have $st(\lambda, C_1) \odot st(\mu, C_2) \geq \rho_1 \odot \rho_2$. It is a contradiction. Hence $st(\lambda \odot \mu, C_1 \odot C_2) \leq st(\lambda, C_1) \odot st(\mu, C_2)$.

(8) Since $\lambda \odot \rho \neq \overline{0}$ implies $f^{\rightarrow}(\lambda) \odot f^{\rightarrow}(\rho) \geq f^{\rightarrow}(\lambda \odot \rho) \neq \overline{0}$, we have

$$\begin{aligned} f^{\rightarrow}(st(\lambda, C)) &= f^{\rightarrow}(\bigvee\{\rho \mid \rho \odot \lambda \neq \overline{0}, \rho \in C\}) \\ &= \bigvee\{f^{\rightarrow}(\rho) \mid \rho \odot \lambda \neq \overline{0}, \rho \in C\} \\ &\leq \bigvee\{f^{\rightarrow}(\rho) \mid f^{\rightarrow}(\rho) \odot f^{\rightarrow}(\lambda) \neq \overline{0}, \rho \in C\} \\ &= st(f^{\rightarrow}(\lambda), f^{\rightarrow}(C)). \end{aligned}$$

Other cases follow from Proposition 3.2 in [4].

□

EXAMPLE 3.6. Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ a stsc-quantale defined by $a \odot b = 0 \vee (a + b - 1)$. Let $C = \{\rho_i \in [0,1]^X \mid i = 1, 2, 3\}$ be a cover where

$$\rho_1(x) = 0.3, \rho_1(y) = 1, \rho_1(z) = 0, \quad \rho_2(x) = 1, \rho_2(y) = 0.2, \rho_2(z) = 0,$$

$$\rho_3(x) = 0, \rho_3(y) = 0, \rho_3(z) = 1.$$

We obtain

$$C \odot C = \{\rho_1 \odot \rho_1 = 1_{\{y\}}, \rho_2 \odot \rho_2 = 1_{\{x\}}, \rho_1 \odot \rho_2, \rho_3\},$$

$$\begin{aligned} 1_{\{x,y\}} &= st(\rho_1 \odot \rho_2, C \odot C) \leq st(\rho_1, C) \odot st(\rho_2, C) \\ &= 1_{\{x,y\}} \odot 1_{\{x,y\}} = 1_{\{x,y\}}. \end{aligned}$$

Since $st(\rho_1, C) = st(\rho_2, C) = 1_{\{x,y\}}, st(\rho_3, C) = 1_{\{z\}}$, we obtain

$$st(C) = \{1_{\{x,y\}}, 1_{\{z\}}\}, \quad C \leq st(C). \square$$

DEFINITION 3.7. A nonempty family $\mathcal{U}$ of L-covers of X is called a *covering* $(L, \odot)$*-uniformity* on X if it satisfies the following conditions:

(UC1) If $C_1 \leq C_2$ and $C_1 \in \mathcal{U}$, then $C_2 \in \mathcal{U}$.

(UC2) For each $C_1, C_2 \in \mathcal{U}$, $C_1 \odot C_2 \in \mathcal{U}$.

(UC3) For each $C_1 \in \mathcal{U}$, there exists $C_2 \in \mathcal{U}$ such that $st(C_2) \leq C_1$.

The pair $(X, \mathcal{U})$ is said to be a *covering* $(L, \odot)$*-uniform space.*

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be covering $(L, \odot)$-uniformites on X. If $\mathcal{U}_1 \subset \mathcal{U}_2$, $\mathcal{U}_2$ is called *finer than* $\mathcal{U}_1$.

The pair $(X, \mathcal{B})$ is said to be a *covering* $(L, \odot)$*-uniform base* if it satisfies (UC2) and (UC3).

Let $(X, \mathcal{U}_1)$ and $(Y, \mathcal{U}_2)$ be covering $(L, \odot)$-uniform spaces. A function $f : (X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ is C*-uniformly continuous* if $f^{\leftarrow}(C) \in \mathcal{U}_1$, for every $C \in \mathcal{U}_2$.

THEOREM 3.8. *Let $(X, \mathcal{U})$ be a covering $(L, \odot)$-uniform space. We define $I_{\mathcal{U}} : L^X \to L^X$ as follows:*

$$I_{\mathcal{U}}(\lambda) = \bigvee \{\rho \in L^X \mid st(\rho, C) \leq \lambda, \ \exists C \in \mathcal{U}\}.$$

$$st(C) = \{st(\lambda, C) \mid \lambda \in C\}.$$

Then we have the following properties:

(1) $I_{\mathcal{U}}(\overline{1}) = \overline{1}$.

(2) $I_{\mathcal{U}}(\lambda) \leq \lambda$.

(3) $I_{\mathcal{U}}(I_{\mathcal{U}}(\lambda)) = I_{\mathcal{U}}(\lambda)$.

(4) $I_{\mathcal{U}}(\lambda \wedge \mu) \geq I_{\mathcal{U}}(\lambda) \wedge I_{\mathcal{U}}(\mu)$.

(5) $I_{\mathcal{U}}(\lambda \odot \mu) \geq I_{\mathcal{U}}(\lambda) \odot I_{\mathcal{U}}(\mu)$.

Proof. (5) Suppose $I_{\mathcal{U}}(\lambda \odot \mu) \not\geq I_{\mathcal{U}}(\lambda) \odot I_{\mathcal{U}}(\mu)$. By the definition of $I_{\mathcal{U}}(\lambda)$ and $I_{\mathcal{U}}(\mu)$, there exist $\rho_i \in C_i$ and $C_i \in \mathcal{U}$ for $i = 1, 2$ with $st(\rho_1, C_1) \leq \lambda$ and $st(\rho_2, C_2) \leq \mu$ such that

$$I_{\mathcal{U}}(\lambda \odot \mu) \not\geq \rho_1 \odot \rho_2$$

Since

$$st(\rho_1 \odot \rho_2, C_1 \odot C_2) \leq st(\rho_1, C_1) \odot st(\rho_1, C_2) \leq \lambda \odot \mu,$$

we have $I_{\mathcal{U}}(\lambda \odot \mu) \not\geq \rho_1 \odot \rho_2$. It is a contradiction. Hence $I_{\mathcal{U}}(\lambda \odot \mu) \geq I_{\mathcal{U}}(\lambda) \odot I_{\mathcal{U}}(\mu)$.

Other cases follows from Proposition 3.4 in [4]. □

THEOREM 3.9. *Let $\mathcal{U}$ be a covering $(L,\odot)$-uniformity on X. Then the following properties hold:*

(1) *We define a subset $\mathbb{T}_{\mathcal{U}}$ of L^X as follows:*

$$\mathbb{T}_{\mathcal{U}} = \{\rho \in L^X \mid \exists C \in \mathcal{U}, st(\rho, C) = \rho\}.$$

Then $\mathbb{T}_{\mathcal{U}}$ is an $(L,\odot)$-topology on X induced by $\mathcal{U}$.

(2) *We define a subset $\mathbb{T}_{I_{\mathcal{U}}}$ of L^X as follows:*

$$\mathbb{T}_{I_{\mathcal{U}}} = \{\rho \in L^X \mid I_{\mathcal{U}}(\rho) \geq \rho\}.$$

Then $\mathbb{T}_{I_{\mathcal{U}}}$ is an $(L,\odot)$-topology on X induced by $I_{\mathcal{U}}$.

(3) $\mathbb{T}_{I_{\mathcal{U}}} = \mathbb{T}_{\mathcal{U}}$.

Proof. (1) (T1) Since $st(\overline{0}, C) = \overline{0}$ and $st(\overline{1}, C) = \overline{1}$ for all $C \in \mathcal{U}$, we have $\overline{0}, \overline{1} \in \mathbb{T}_{\mathcal{U}}$.

(T2) and (TO). Let $\lambda_i \in \mathbb{T}_{\mathcal{U}}$ for $i = 1, 2$. Then $C_i \in \mathcal{U}$ such that $st(\lambda_i, C_i) = \lambda_i$. Since

$$st(\lambda_1 \wedge \lambda_2, C_1 \wedge C_2) \leq st(\lambda_1, C_1) \wedge st(\lambda_2, C_2) = \lambda_1 \wedge \lambda_2,$$
$$st(\lambda_1 \odot \lambda_2, C_1 \odot C_2) \leq st(\lambda_1, C_1) \odot st(\lambda_2, C_2) = \lambda_1 \odot \lambda_2$$

we have $\lambda_1 \wedge \lambda_2, \lambda_1 \odot \lambda_2 \in \mathbb{T}_{\mathcal{U}}$.

(T3) Let $\lambda_i \in \mathbb{T}_{\mathcal{U}}$ for $i \in \Gamma$. Then, for each $i \in \Gamma$, there exists $C_i \in \mathcal{U}$ such that $st(\lambda_i, C_i) = \lambda_i$.

Suppose that $st(\bigvee_{i\in\Gamma} \lambda_i, C) \not\leq \bigvee_{i\in\Gamma} \lambda_i$ for all $C \in \mathcal{U}$. Since $st(\bigvee \lambda_i, C) = \bigvee st(\lambda_i, C)$, there exists $\{i_1, i_2, ...i_m\} \subset \Gamma$ such that

$$st(\vee_{k=1}^m \lambda_{i_k}, C) \not\leq \bigvee_{i\in\Gamma} \lambda_i$$

Put $C = \odot_{k=1}^m C_{i_k}$. Then

$$\begin{aligned} st(\vee_{k=1}^m \lambda_{i_k}, \odot_{k=1}^m C_{i_k}) &= \vee_{k=1}^m st(\lambda_{i_k}, \odot_{k=1}^m C_{i_k}) \\ &\leq \vee_{k=1}^m st(\lambda_{i_k}, C_{i_k}) \\ &= \vee_{k=1}^m \lambda_{i_k} \leq \bigvee_{i\in\Gamma} \lambda_i. \end{aligned}$$

It is a contradiction. Thus, there exists $C \in \mathcal{U}$ such that $st(\bigvee_{i\in\Gamma}\lambda_i, C) = \bigvee_{i\in\Gamma}\lambda_i$. Hence $\bigvee_{i\in\Gamma}\lambda_i \in \mathbb{T}_{\mathcal{U}}$.

(2) (T1) Since $I_{\mathcal{U}}(\overline{1}) = \overline{1}$ and $I_{\mathcal{U}}(\overline{0}) = \overline{0}$, then $\overline{0}, \overline{1} \in \mathbb{T}_{I_{\mathcal{U}}}$.

(T2) If $\lambda_i \in \mathbb{T}_{I_{\mathcal{U}}}$ for each $i = 1, 2$, by Theorem 3.8 (4-5), $\lambda_1 \wedge \lambda_2, \lambda_1 \odot \lambda_2 \in \mathbb{T}_{I_{\mathcal{U}}}$.

(T3) Let $\lambda_i \in \mathbb{T}_{I_{\mathcal{U}}}$ for $i \in \Gamma$. Since

$$I_{\mathcal{U}}(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigvee_{i\in\Gamma} I_{\mathcal{U}}(\lambda_i) \geq \bigvee_{i\in\Gamma}\lambda_i$$

we have $\bigvee_{i\in\Gamma}\lambda_i \in \mathbb{T}_{I_{\mathcal{U}}}$.

(3) Let $\rho \in \mathbb{T}_{\mathcal{U}}$. Then $C \in \mathcal{U}$ with $st(\rho, C) = \rho$. So, $I_{\mathcal{U}}(\rho) = \rho$. Hence $\rho \in \mathbb{T}_{I_{\mathcal{U}}}$.

Let $\lambda \in \mathbb{T}_{I_{\mathcal{U}}}$. Then $I_{\mathcal{U}}(\lambda) \geq \lambda$. For all $st(\rho_i, C_i) \leq \lambda$, $\bigvee_{i\in\Gamma}\rho_i = \lambda$. Hence $\bigvee_{i\in\Gamma}\rho_i = \bigvee_{i\in\Gamma} st(\rho_i, C_i) = \lambda$. By a similar proof as in (1), there exists $C \in \mathcal{U}$ such that $st(\bigvee_{i\in\Gamma}\rho_i, C) = \bigvee_{i\in\Gamma}\rho_i = \lambda$. So, $\lambda \in \mathbb{T}_{\mathcal{U}}$. □

THEOREM 3.10. *Let $(Y, \mathcal{B})$ be a covering $(L,\odot)$-uniform base, X a set and $f : X \to Y$ a function. Define a subset $f^{\leftarrow}(\mathcal{B})$ of $C(X)$ as follows:*

$$f^{\leftarrow}(\mathcal{B}) = \{f^{\leftarrow}(C) \mid C \in \mathcal{B}\}.$$

Then we have the following properties.

(1) *The structure $f^{\Leftarrow}(\mathcal{B})$ is a covering $(L,\odot)$-uniform base on X.*

(2) *The structure $[f^{\Leftarrow}(\mathcal{B})] = \{C \in C(X) \mid f^{\Leftarrow}(B) \leq C, B \in \mathcal{B}\}$ is the coarsest covering $(L,\odot)$-uniform base on X for which f is C-uniformly continuous.*

(3) *A map $g : (Z, \mathcal{U}_1) \to (X, [f^{\Leftarrow}(\mathcal{U})]$ is C-uniformly continuous iff $f \circ g : (Z, \mathcal{U}_1) \to (Y, \mathcal{U})$ is C-uniformly continuous.*

Proof. (1) (UC2) It follows from $f^{\leftarrow}(C_1 \odot C_2) = f^{\leftarrow}(C_1) \odot f^{\leftarrow}(C_2)$.

(UC3) For each $f^{\leftarrow}(C) \in f^{\leftarrow}(\mathcal{B})$ with $C \in \mathcal{B}$, there exists $C_1 \in \mathcal{B}$ such that $st(C_1) \leq C$. Since $st(f^{\leftarrow}(C_1)) \leq f^{\leftarrow}(st(C_1)) \leq f^{\leftarrow}(C)$, $f^{\leftarrow}(\mathcal{B})$ is a covering $(L,\odot)$-uniform base on X.

(2) and (3) are similarly proved as in Theorem 3.4. □

Example 3.11. Let $Z = \{a,b,c,d\}$ and $X = \{x,y,z\}$ be a set and $([0,1],\odot)$ a stsc-quantale defined by $a \odot b = 0 \vee (a+b-1)$. Let $f : Z \to X$ be a function as $f(a) = x, f(b) = y, f(c) = f(d) = z$. Let $\mathcal{U} = \{C \in C(X) \mid C_1 = \{1_{\{x,y\}}, 1_{\{z\}}\} \leq C\}$ be a covering $(L,\odot)$-uniformity on X. Then $f^{\leftarrow}(\mathcal{U}) = \{f^{\leftarrow}(C) \in C(Z) \mid C \in \mathcal{U}\}$ is not a covering $(L,\odot)$-uniformity on X because, for $\rho(a) = \rho_1(b) = 1, \rho(c) = 0.3, \rho(d) = 0$,

$$f^{\to}(1_{\{x,y\}}) \leq \rho, \ \ \rho \notin f^{\leftarrow}(\mathcal{B}).$$

4. Covering $(L,\odot)$ uniformities and Hutton $(L,\otimes)$-uniformites

Theorem 4.1. *We define a mapping $\Delta : C(X) \to \Omega(X)$ as follows:*

$$\Delta(C)(\lambda) = st(\lambda, C) = \bigvee\{\mu \in C \mid \mu \odot \lambda \neq \emptyset\}.$$

Then we have the following properties:
(1) For each $C \in C(X)$, $\Delta(C) \in \Omega(X)$.
(2) $\Delta(C)$ has a right adjoint mapping $\Delta(C)^{\leftarrow}$ defined by

$$\Delta(C)^{\leftarrow}(\lambda) = \bigvee\{\rho \in L^X \mid st(\rho, C) \leq \lambda\}.$$

It follows $\Delta(C)^{\leftarrow} \circ \Delta(C) \geq 1_{L^X}$ and $\Delta(C) \circ \Delta(C)^{\leftarrow} \leq 1_{L^X}$. Furthermore, $\Delta(C)(\lambda) \leq \rho$ iff $\lambda \leq \Delta(C)^{\leftarrow}(\rho)$.
(3) Δ has a right adjoint mapping $\Sigma : \Omega(X) \to C(X)$ as follows:

$$\Sigma(\phi) = \{\phi(\lambda) \mid \lambda \odot \phi(\lambda) \neq \emptyset\}.$$

It implies $\Sigma \circ \Delta \geq 1_{C(X)}$ and $\Delta \circ \Sigma \leq 1_{\Omega(X)}$.

Proof. (1) It follows from: $\Delta(C)(\bigvee \lambda_i) = st(\bigvee \lambda_i, C) = \bigvee st(\lambda_i, C) = \bigvee \Delta(C)(\lambda_i)$ and $\Delta(C)(\lambda) = st(\lambda, C) \geq \lambda$.

(2) By (1) and Theorem 2.6, $\Delta(C)$ has a right adjoint mapping $\Delta(C)^{\to}$ as follows:

$$\Delta(C)^{\leftarrow}(\lambda) = \bigvee\{\rho \in L^X \mid st(\rho, C) \leq \lambda\}.$$

By Theorem 2.6, the results hold.

(6) Since $\Delta(\bigvee C_i)(\lambda) = st(\lambda, \bigvee C_i) = \bigvee st(\lambda, C_i) = \bigvee \Delta(C_i)(\lambda)$, we have $\Delta(\bigvee C_i) = \bigvee \Delta(C_i)$. By Theorem 2.6, Δ has a right adjoint mapping Σ as follows:

$$\begin{aligned}
\Sigma(\phi) &= \bigvee\{C \in C(X) \mid \Delta(C) \leq \phi\} \\
&= \bigvee\{C \in C(X) \mid \Delta(C)(\lambda) \leq \phi(\lambda)\} \\
&= \bigvee\{C \in C(X) \mid st(\lambda, C) \leq \phi(\lambda)\} \\
&= \bigvee\{C \in C(X) \mid \bigvee_i \mu_i \leq \phi(\lambda),\ \ \lambda \odot \mu_i \neq \overline{0},\ \mu_i \in C\} \\
&= \{\phi(\lambda) \in L^X \mid \lambda \odot \phi(\lambda) \neq \overline{0}\}.
\end{aligned}$$

By Theorem 2.6, others cases hold.

□

THEOREM 4.2. *Let $\mathcal{U}$ be a covering $(L,\odot)$-uniformity on X. Then $\mathbb{U}_{\mathcal{U}} = \{\phi \in \Omega(X) \mid \exists C \in \mathcal{U},\ \phi_C \leq \phi\}$ is a Hutton $(L,\otimes)$-uniformity on X where $\phi_C(\lambda) = \Delta(C)(\lambda) = st(\lambda, C)$.*

Proof. (U1) It is easy.

(U2) For each $\psi \in \mathbb{U}_{\mathcal{U}}$, there exists $C \in \mathcal{U}$ such that $\phi_C \leq \psi$. For $C \in \mathcal{U}$, there exists $C_1 \in \mathcal{U}$ such that $st(C_1) \leq C$. Since

$$\begin{aligned}
\phi_{C_1} \circ \phi_{C_1}(\lambda) &= \phi_{C_1}(st(\lambda, C_1)) \\
&= st(st(\lambda, C_1), C_1) \\
&\leq st(\lambda, st(C_1)) \ \ \text{(by Theorem 3.5(7))} \\
&\leq st(\lambda, C) = \phi_C(\lambda),
\end{aligned}$$

we have $\phi_{C_1} \circ \phi_{C_1} \leq \phi_C \leq \psi$.

(U3) For each $\psi_i \in \mathbb{U}_{\mathcal{U}}$ for $i = 1, 2$, there exist $C_i \in \mathcal{U}$ such that $\phi_{C_i} \leq \psi_i$. Since

$$\begin{aligned}
\phi_{C_1} \otimes \phi_{C_2}(\lambda) &= \bigwedge\{\phi_{C_1}(\lambda_1) \odot \phi_{C_2}(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\} \\
&= \bigwedge\{st(\lambda_1, C_1) \odot st(\lambda_2, C_2) \mid \lambda = \lambda_1 \odot \lambda_2\} \\
&\geq st(\lambda_1 \odot \lambda_2, C_1 \odot C_2) = \phi_{C_1 \odot C_2}(\lambda),
\end{aligned}$$

we have $\phi_{C_1\odot C_2} \le \phi_{C_1} \otimes \phi_{C_2} \le \psi_1 \otimes \psi_2$. Hence $\psi_1 \otimes \psi_2 \in \mathbb{U}_{\mathcal{U}}$.

(U5) For each $\psi \in \mathbb{U}_{\mathcal{U}}$, there exists $C \in \mathcal{U}$ such that $\phi_C \le \psi$. Since $\phi_C(\lambda) \odot \mu = st(\lambda, C) \odot \mu \ne \overline{0}$ iff there exists $\rho \in C$ such that $\rho \odot \lambda \ne \overline{0}$ and $\rho \odot \mu \ne \overline{0}$ iff $\lambda \odot \phi_C(\mu) \ne \overline{0}$. Hence ϕ_C is symmetric.

□

In Theorem 4.1(3), we define:

$$C_\phi = \begin{cases} \Sigma(\phi) - \{\overline{1}\} & \text{if } \bigvee \lambda_i = 1,\ \forall \lambda_i \in \Sigma(\phi) - \{\overline{1}\}, \\ \Sigma(\phi) & \text{otherwise.} \end{cases}$$

THEOREM 4.3. *Let $\mathbb{U}$ be a Hutton $(L, \otimes)$-uniformity on X satisfying the following condition*

(C) *for $\lambda, \mu \in C_\phi$ with $\lambda \odot \mu \ne \overline{0}$, we have $\lambda \le \phi(\mu)$.*

Then we have the following properties

(1) *$\mathcal{U}_{\mathbb{U}} = \{C \in C(X) \mid C_\phi \le C,\ \phi \in \mathbb{U}\}$ is a covering $(L, \odot)$-uniformity on X.*

(2) *$\mathbb{U} \subset \mathbb{U}_{\mathcal{U}_{\mathbb{U}}}$ and $\mathcal{U}_{\mathbb{U}_{\mathcal{U}}} = \mathcal{U}$.*

Proof. (1) (UC1) If $C_1 \le C_2$ and $C_1 \in \mathcal{U}_{\mathbb{U}}$, then there exist $\phi \in \mathbb{U}$ such that $C_\phi \le C_1 \le C_2$. So, $C_2 \in \mathcal{U}_{\mathbb{U}}$.

(UC2) For each $C_i \in \mathcal{U}_{\mathbb{U}}$ for $i = 1, 2$, there exist $\phi_i \in \mathbb{U}$ such that $C_{\phi_i} \le C_i$. For $\lambda \odot (\phi_1 \otimes \phi_2)(\lambda) \ne \overline{0}$, since

$$(\phi_1 \otimes \phi_2)(\lambda) = \bigwedge \{\phi_1(\lambda) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\},$$

$\lambda_1 \odot \lambda_2 \odot \phi_1(\lambda_1) \odot \phi_2(\lambda_2) \ne \overline{0}$ implies $\lambda_1 \odot \phi_1(\lambda_1) \ne \overline{0},\ \lambda_2 \odot \phi_2(\lambda_2) \ne \overline{0}$,

then $(\phi_1 \otimes \phi_2)(\lambda) \le \phi_1(\lambda_1) \odot \phi_2(\lambda_2)$. Hence $C_{\phi_1 \otimes \phi_2} \le C_{\phi_1} \odot C_{\phi_2} \le C_1 \odot C_2$. Thus $C_1 \odot C_2 \in \mathcal{U}_{\mathbb{U}}$.

(UC3) We only show that $st(C_{\phi_1}) \le C_{\phi_2}$ such that $\phi_1 \circ \phi_1 \le \phi_2$. For $st(\phi_1(\lambda), C_{\phi_1}) \in st(C_{\phi_1})$, since

$$\begin{aligned} st(\phi_1(\lambda), C_{\phi_1}) &= \bigvee \{\phi_1(\rho) \in C_{\phi_1} \mid \phi_1(\lambda) \odot \phi_1(\rho) \ne \overline{0}\} \\ &\le \bigvee \{\phi_1(\rho) \in C_{\phi_1} \mid \phi_1(\rho) \le \phi_1(\phi_1(\lambda))\} \text{ (by (C))} \\ &\le \phi_1(\phi_1(\lambda)) \le \phi_2(\lambda), \end{aligned}$$

$$\phi_1(\lambda) \le st(\phi_1(\lambda), C_{\phi_1}) \le \phi_1(\phi_1(\lambda)) \le \phi_2(\lambda),$$

and $\overline{0} \neq \lambda \odot \phi_1(\lambda) \leq \lambda \odot \phi_2(\lambda)$, then there exists $\phi_2(\lambda) \in C_{\phi_2}$ such that $st(\phi_1(\lambda), C_{\phi_1}) \leq \phi_2(\lambda)$. Thus, the results hold.

(2) Since $\Sigma \circ \Delta \geq 1_{C(X)}$ and $\Delta \circ \Sigma \leq 1_{\Omega(X)}$, we have

$$C \leq C_{\phi_C}, \;\; \phi_{C_\phi} \leq \phi.$$

It implies $\mathcal{U}_{\mathbb{U}_\mathcal{U}} \subset \mathcal{U}$ and $\mathbb{U} \subset \mathbb{U}_{\mathcal{U}_\mathbb{U}}$.

For each $B \in \mathcal{U}$, there exists $C \in \mathcal{U}$ such that $st(C) \leq B$. Let $\phi_C(\lambda) \in C_{\Phi_C}$ with $\phi_C(\lambda) \odot \lambda \neq \overline{0}$. Since $st(C) \leq B$, there exists $\rho \in B$ such that $\phi_C(\lambda) = st(\lambda, C) \leq \rho$. Hence $C_{\phi_C} \leq B$. So, $\mathcal{U} \subset \mathcal{U}_{\mathbb{U}_\mathcal{U}}$.

□

Example 4.4. Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ a quantale defined by $x \odot y = 0 \vee (x + y - 1)$. Define $\phi \in \Omega(X)$ as

$$\phi(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ 1_{\{x,y\}} & \text{if } \overline{0} \neq \lambda \leq 1_{\{x\}}, \\ 1_{\{z\}}, & \text{if } \overline{0} \neq \lambda \leq 1_{\{z\}}, \\ \overline{1} & \text{otherwise.} \end{cases}$$

Since $\phi \otimes \phi = \phi$, $\phi \circ \phi = \phi$ and ϕ is symmetric. Thus, $\mathbb{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$ is a Hutton $(L, \otimes)$-uniformity on X. From Theorem 4.1, we obtain

$$\Sigma(\phi) = \{1_{\{x,y\}}, 1_{\{z\}}, \overline{1}\},$$
$$C_\phi = \{1_{\{x,y\}}, 1_{\{z\}}\}.$$

Since $C_\phi \odot C_\phi = C_\phi$ and $st(C_\phi) = C_\phi$, we obtain a covering $(L, \odot)$-uniformity on X as follows

$$\mathcal{U}_\mathbb{U} = \{C \in C(X) \mid C_\phi \leq C\},$$

and a Hutton $(L, \otimes)$-uniformity $\mathbb{U} = \{\psi \in \Omega(X) \mid \phi_{C_\phi} \leq \psi\}$ where

$$\phi_{C_\phi}(\lambda) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0}, \\ 1_{\{x,y\}} & \text{if } \overline{0} \neq \lambda \leq 1_{\{x,y\}}, \\ 1_{\{z\}}, & \text{if } \overline{0} \neq \lambda \leq 1_{\{z\}}, \\ \overline{1} & \text{otherwise.} \end{cases}$$

Since $\phi_{C_\phi} \leq \phi$, we have $\mathbb{U} \subset \mathbb{U}_{\mathcal{U}_\mathbb{U}}$.

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