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ANOTHER COMPLETE DECOMPOSITION OF A SELF-SIMILAR CANTOR SET

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Abstract. Using informations of subsets of divergence points and the relation between members of spectral classes, we give another complete decomposition of spectral classes generated by lower(upper) local dimensions of a self-similar measure on a self-similar Cantor set with full information of their dimensions. We note that it is a complete refinement of the earlier complete decomposition of the spectral classes. Further we study the packing dimension of some uncountable union of distribution sets.

1. Introduction

Recently a complete decomposition of a self-similar Cantor set was investigated using the relation between the distribution subset and the subset by local dimension of a self-similar measure on the self-similar Cantor set. The distribution set gives full information of their Hausdorff dimensions and some information of their packing dimensions. More recently full information of their packing dimension was also given([4]) using the information([2]) of packing dimension of some subsets of the subsets in the decomposition class. We give another complete decomposition which is a refinement of the decomposition of [1] using the information of [7] which gives the essential information of accumulation points in the frequency sequence of a divergence point in the self-similar Cantor set. We note that a singular value([1]) derived from the contraction ratios of the self-similar Cantor set behaves an important role in the dual results to relate the distribution subsets to subsets by local

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dimension of a self-similar measure on the Cantor set. Further we compute the packing dimension of some uncountable union of distribution sets related to a coordinate (r_1, r_2) where $F[r_1, r_2]$ is a characteristic distribution set([3]).

2. Preliminaries

We denote F a self-similar Cantor set, which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0,1]$ with $a > 0$, $b > 0$ and $1-(a+b) > 0$. Let $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{1,2\}$ and $1 \le j \le k$. We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). If $x \in F$ and $x \in I_\sigma$ where $\sigma \in \{1,2\}^k$, $c_k(x)$ denotes I_σ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \cdots$. Let $p \in (0,1)$ and we denote γ_p a self-similar Borel probability measure on F satisfying $\gamma_p(I_1) = p$(cf. [5]). $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E([5]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E([5]). We denote $n_1(x|k)$ the number of times the digit 1 occurs in the first k places of $x = \sigma$(cf. [6]).
For $r \in [0,1]$, we define the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 1 in proportion r by

$$\underline{F}(r) = \{x \in F : \liminf_{k\to\infty} \frac{n_1(x|k)}{k} = r\},$$

$$\overline{F}(r) = \{x \in F : \limsup_{k\to\infty} \frac{n_1(x|k)}{k} = r\}.$$

We call $\{\underline{F}(r) : 0 \le r \le 1\}$ the lower distribution class and $\{\overline{F}(r) : 0 \le r \le 1\}$ the upper distribution class.

Similarly for $r_1, r_2 \in [0,1]$ with $r_1 \le r_2$, we define a distribution set $F[r_1, r_2]$ containing the digit 1 in proportion from r_1 to r_2 by

$$F[r_1, r_2] = \{x \in F : A(\frac{n_1(x|k)}{k}) = [r_1, r_2]\},$$

where $A(x_k)$ is the set of the accumulation points of the sequence (x_k).

We write $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$) for the set of points at which the lower(upper) local dimension of γ_p on F is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \{x : \liminf_{r\to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\},$$

$$\overline{E}_\alpha^{(p)} = \{x : \limsup_{r\to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\}.$$

We call $\{\underline{E}_\alpha^{(p)}(\neq \phi) : \alpha \in \mathbb{R}\}$ the spectral class generated by the lower local dimensions of a self-similar measure γ_p and $\{\overline{E}_\alpha^{(p)}(\neq \phi) : \alpha \in \mathbb{R}\}$ the spectral class generated by the upper local dimensions of a self-similar measure γ_p. We call α satisfying $\underline{E}_\alpha^{(p)}(\neq \phi)(\overline{E}_\alpha^{(p)}(\neq \phi))$ an associated lower(upper) local dimension of γ_p.

Similarly for $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \leq \alpha_2$, we define a subset $E_{[\alpha_1,\alpha_2]}^{(p)}$

$$E_{[\alpha_1,\alpha_2]}^{(p)} = \{x \in F : A(\frac{\log \gamma_p(B_r(x))}{\log r}) = [\alpha_1, \alpha_2]\},$$

where $A(f(r))$ is the set of the accumulation points of the function of $f(r)$ where $r > 0$.

In this paper, we assume that $0 \log 0 = 0$ for convenience. Let $p \in (0,1)$ and consider a self-similar measure γ_p on F. We define for $r \in [0,1]$

$$g(r,p) = \frac{r \log p + (1-r)\log(1-p)}{r \log a + (1-r)\log b}.$$

From now on we will use $g(r,p)$ as the above definition.

3. Main results

Proposition 1. ([7])*For* $0 \leq r_1 \leq r_2 \leq 1$, $A(\frac{n_1(x|k)}{k}) = [r_1, r_2]$ *for each* $x \in \underline{F}(r_1) \cap \overline{F}(r_2)$.

Proposition 2. ([2, 4, 7]) *For* $0 \leq r_1 \leq r_2 \leq 1$,

$$\dim(\underline{F}(r_1) \cap \overline{F}(r_2)) = \inf_{r_1 \leq r \leq r_2} g(r,r)$$

and

$$\mathrm{Dim}(\underline{F}(r_1) \cap \overline{F}(r_2)) = \sup_{r_1 \leq r \leq r_2} g(r,r).$$

Lemma 3. ([4]) *Let $p \in (0,1)$ and consider a self-similar measure γ_p on F and let $r \in [0,1]$. Then for a real number s satisfying $a^s + b^s = 1$*
(1) $\underline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ if $0 < p < a^s$,
(2) $\underline{F}(r) = \overline{E}^{(p)}_{g(r,p)}$ if $a^s < p < 1$,
(3) $\overline{F}(r) = \overline{E}^{(p)}_{g(r,p)}$ if $0 < p < a^s$,
(4) $\overline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ if $a^s < p < 1$.

Theorem 4. *If $0 < p < 1$ and $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$, then $E^{(p)}_{[\alpha_1,\alpha_2]} = \underline{F}(r_1) \cap \overline{F}(r_2)$ where r_i is the solution of the equation $\alpha_i = g(r_i, p)$ for each $i = 1, 2$. In particular, if $p = a^s$, $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$ for $\alpha_i = s$ for each $i = 1, 2$.*

Proof. It follows from the above Lemma. □

Corollary 5. *If $0 < p < a^s$ and $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$, then $E^{(p)}_{[\alpha_1,\alpha_2]} = \underline{E}^{(p)}_{g(r_1,p)} \cap \overline{E}^{(p)}_{g(r_2,p)}$ where r_i is the solution of the equation $\alpha_i = g(r_i, p)$ for each $i = 1, 2$. Similarly if $a^s < p < 1$ and $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$, then $E^{(p)}_{[\alpha_1,\alpha_2]} = \underline{E}^{(p)}_{g(r_2,p)} \cap \overline{E}^{(p)}_{g(r_1,p)}$ where r_i is the solution of the equation $\alpha_i = g(r_i, p)$ for each $i = 1, 2$.*

Proof. It follows from the above Theorem and the above Lemma. □

Corollary 6. *If $0 < p < 1$ and $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$, then $E^{(p)}_{[\alpha_1,\alpha_2]}$ has Hausdorff dimension $\min\{g(r_1, r_1), g(r_2, r_2)\}$.*

Proof. It follows from the above Lemma and Proposition. □

Corollary 7. *Let $0 < p < 1$ and $E^{(p)}_{[\alpha_1,\alpha_2]} \neq \phi$. If $s \in [r_1, r_2]$ where r_i is the solution of the equation $\alpha_i = g(r_i, p)$ for each $i = 1, 2$, then $E^{(p)}_{[\alpha_1,\alpha_2]}$ has packing dimension s. If $s \notin [r_1, r_2]$ where r_i is the solution of the equation $\alpha_i = g(r_i, p)$ for each $i = 1, 2$, then $E^{(p)}_{[\alpha_1,\alpha_2]}$ has packing dimension $\max\{g(r_1, r_1), g(r_2, r_2)\}$.*

Proof. It follows from the above Lemma and Proposition. □

Remark 1. The self-similar Cantor set F can be completely decomposed as the union of the subsets by the lower(upper) distribution. Further every lower(upper) distribution subset can be also completely decomposed as the union of the subsets by the upper(lower) distribution.

Precisely,

$$F = \cup_{0\le r_1\le 1}\underline{F}(r_1) = \cup_{0\le r_1\le 1}\cup_{r_1\le r_2\le 1} F[r_1,r_2].$$

Similarly

$$F = \cup_{0\le r_2\le 1}\overline{F}(r_2) = \cup_{0\le r_2\le 1}\cup_{0\le r_1\le r_2} F[r_1,r_2].$$

We also note that all the subsets $F[r_1,r_2]$ are mutually disjoint in the sense that $F[r_1,r_2]\cap F[r_3,r_4]=\phi$ if $(r_1,r_2)\neq(r_3,r_4)$.

Remark 2. The self-similar Cantor set F can be completely decomposed as the union of the subsets by the lower(upper) local dimensions of a self-similar measure on the self-similar Cantor set. Further every same lower(upper) local dimension subset can be also completely decomposed as the union of the subsets by the upper(lower) local dimension of the given self-similar measure also. Precisely,

$$F = \cup_{\alpha_1\in[\frac{\log(1-p)}{\log b},\frac{\log p}{\log a}]}\underline{E}^{(p)}_{\alpha_1} = \cup_{\alpha_1\in[\frac{\log(1-p)}{\log b},\frac{\log p}{\log a}]}\cup_{\alpha_2\in[\alpha_1,\frac{\log p}{\log a}]} E^{(p)}_{[\alpha_1,\alpha_2]}$$
$$\text{if } 0<p<a^s,$$

$$F = \cup_{\alpha_1\in[\frac{\log p}{\log a},\frac{\log(1-p)}{\log b}]}\underline{E}^{(p)}_{\alpha_1} = \cup_{\alpha_1\in[\frac{\log p}{\log a},\frac{\log(1-p)}{\log b}]}\cup_{\alpha_2\in[\alpha_1,\frac{\log(1-p)}{\log b}]} E^{(p)}_{[\alpha_1,\alpha_2]}$$
$$\text{if } a^s<p<1,$$

$$F = \cup_{\alpha_2\in[\frac{\log(1-p)}{\log b},\frac{\log p}{\log a}]}\overline{E}^{(p)}_{\alpha_2} = \cup_{\alpha_2\in[\frac{\log(1-p)}{\log b},\frac{\log p}{\log a}]}\cup_{\alpha_1\in[\frac{\log(1-p)}{\log b},\alpha_2]} E^{(p)}_{[\alpha_1,\alpha_2]}$$
$$\text{if } 0<p<a^s,$$

$$F = \cup_{\alpha_2\in[\frac{\log p}{\log a},\frac{\log(1-p)}{\log b}]}\overline{E}^{(p)}_{\alpha_2} = \cup_{\alpha_2\in[\frac{\log p}{\log a},\frac{\log(1-p)}{\log b}]}\cup_{\alpha_1\in[\frac{\log p}{\log a},\alpha_2]} E^{(p)}_{[\alpha_1,\alpha_2]}$$
$$\text{if } a^s<p<1.$$

We also note that all the subsets $E^{(p)}_{[\alpha_1,\alpha_2]}$ are mutually disjoint in the sense that $E^{(p)}_{[\alpha_1,\alpha_2]}\cap E^{(p)}_{[\alpha_3,\alpha_4]}=\phi$ if $(\alpha_1,\alpha_2)\neq(\alpha_3,\alpha_4)$.

The followings are the second part of our main results. We only consider $F[x,y]$ where $0\le x\le y\le 1$ since $F[x,y]=\phi$ if $x>y$. From now on, s is a real number satisfying $a^s+b^s=1$ and $\delta(r)=g(r,r)$ where $0\le r\le 1$. We recall a characteristic coordinate set([3])

$$\Delta = \{(r_1,r_2)\in[0,1]\times[0,1] : \delta(r_1)=\delta(r_2)\in[0,s), r_1<r_2\}\cup\{(a^s,a^s)\}.$$

THEOREM 8. *Let $0\le r_2<a^s$ and $A\subset[0,r_2]$. Then*

$$\mathrm{Dim}(\cup_{x\in A}F[x,r_2]) = \delta(r_2).$$

Proof. We note that $\delta(x)$ is an increasing function on $[0, a^s]$. Since $0 \leq x \leq r_2 < a^s$, $\delta(x) \leq \delta(r_2)$. So we clearly see that $\mathrm{Dim}(F[x, r_2]) = \delta(r_2)$. Hence $\mathrm{Dim}(\cup_{x \in A} F[x, r_2]) \geq \delta(r_2)$.

Since $F[x, r_2] \subset \overline{F}(r_2) = \overline{E}^{(r_2)}_{g(r_2, r_2)}$ if $x \leq r_2 < a^s$, $\mathrm{Dim}(\cup_{x \in A} F[x, r_2]) \leq \delta(r_2) = g(r_2, r_2)$ by the proposition 2.3 of [5] which is an essential result of Frostman's density theorem. $\square$

THEOREM 9. *Let $a^s < r_1 \leq 1$ and $B \subset [r_1, 1]$. Then*

$$\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) = \delta(r_1).$$

Proof. We note that $\delta(x)$ is a decreasing function on $[a^s, 1]$. Since $a^s < r_1 \leq y \leq 1$, $\delta(y) \leq \delta(r_1)$. So we clearly see that $\mathrm{Dim}(F[r_1, y]) = \delta(r_1)$. Hence $\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) \geq \delta(r_1)$.

Since $F[r_1, y] \subset \underline{F}(r_1) = \overline{E}^{(r_1)}_{g(r_1, r_1)}$ if $a^s < r_1 \leq y$, $\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) \leq \delta(r_1) = g(r_1, r_1)$ by the proposition 2.3 of [5] which is an essential result of Frostman's density theorem. $\square$

THEOREM 10. *Let $(r_1, r_2) \in \Delta$ which is the characteristic coordinate set. Then we have*
(1) $\mathrm{Dim}(\cup_{x \in A} F[x, r_2]) = \sup_{x \in A \cap [0, r_2]} \delta(x) = \delta(\inf(A \cap [0, r_2]))$ *if* $A \cap [0, a^s] = \phi$,
(2) $\mathrm{Dim}(\cup_{x \in A} F[x, r_2]) = s$ *if* $A \cap [0, a^s] \neq \phi$,
(3) $\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) = \sup_{y \in B \cap [r_1, 1]} \delta(y) = \delta(\sup(B \cap [a^s, 1]))$ *if* $B \cap [a^s, 1] = \phi$,
(4) $\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) = \delta(r_1)$ *if* $B \cap [a^s, 1] \neq \phi$.

Proof. For (3), since $0 < r_1 \leq y < a^s$,

$$\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) \geq \sup_{y \in B \cap [r_1, 1]} \mathrm{Dim}(F[r_1, y]) = \sup_{y \in B \cap [r_1, 1]} \delta(y).$$

Assume that $\sup_{y \in B \cap [r_1, 1]} \delta(y) \neq s$. Then $\sup(B \cap [r_1, 1]) < a^s$.

$$F[r_1, y] \subset \overline{F}(y) = \overline{E}^{(p)}_{g(y, p)}$$

if $0 < p < a^s$. So $F[r_1, y] \subset \overline{E}^{(y_2)}_{g(y, y_2)}$ where $y_2 = \sup(B \cap [r_1, 1]) < a^s$. It is not difficult to show that $g(y, y_2)$ is an increasing continuous function for y since $0 < y_2 < a^s$(cf. [1]). Since $g(y, y_2) \leq g(y_2, y_2) = \delta(y_2) = \sup_{y \in B \cap [r_1, 1]} \delta(y)$, $\mathrm{Dim}(\cup_{y \in B} F[r_1, y]) \leq \delta(y_2) = \delta(\sup(B \cap [a^s, 1]))$ by the proposition 2.3 of [5]. (1),(2),(4) follow from similar arguments. $\square$

Remark 3. We easily get $\mathrm{Dim}(\cup_{x\in A}F[x,r_2])$ and $\mathrm{Dim}(\cup_{y\in B}F[r_1,y])$ if A and B are countable sets since packing dimension is countable stable. But it is not easy to compute

$$\mathrm{Dim}(\cup_{x\in A}F[x,r_2])$$

or

$$\mathrm{Dim}(\cup_{y\in B}F[r_1,y])$$

if A and B are uncountable sets. In these cases, we apply our Theorems above to the computation of their packing dimensions. If $r_1 \leq a^s$ or $r_2 \geq a^s$, then we apply the above Theorem to the computation of its packing dimension. Precisely, if $r_1 \leq a^s$, then we easily find the counterpart r_2 such that $(r_1, r_2) \in \Delta$. Similarly if $r_2 \geq a^s$, then we easily find the counterpart r_1 such that $(r_1, r_2) \in \Delta$. If not, we apply Theorems 8 and 9 to the computation of their packing dimensions.

References

[1] I. S. Baek, Relation between spectral classes of a self-similar Cantor sets, J. Math. Anal. Appl. **292**(1)(2004), 294-302.

[2] I. S. Baek, L. Olsen and N. Snigireva, Divergence points of self-similar measures and packing dimension, Adv. Math. **214**(1) (2007), 267-287.

[3] I. S. Baek, Characteristic multifractal in a self-similar Cantor set, to appear in J. Chungcheong Math. Soc..

[4] I. S. Baek, Dimensions of the subsets in the spectral classes of a self-similar Cantor set, Journal of Applied Mathematics and Informatics **26** (3-4)(2008), 733-738

[5] K. J. Falconer, Techniques in fractal geometry, John Wiley and Sons(1997).

[6] H. H. Lee and I. S. Baek, Dimensions of a Cantor type set and its distribution sets, Kyungpook Math. Journal **32**(2) (1992), 149-152.

[7] L. Olsen and S. Winter, Normal and non-normal points of self-similar sets and divergence points of self-similar measures, J. London Math. Soc. **67**(2) (2003), 103-122.

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