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FOURIER-YEH-FEYNMAN TRANSFORM AND CONVOLUTION ON YEH-WIENER SPACE

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Abstract. We define Fourier-Yeh-Feynman transform and convolution product on the Yeh-Wiener space, and establish the existence of Fourier-Yeh-Feynman transform and convolution product for functionals in a Banach algebra $\mathcal{S}(Q)$. Also we obtain Parseval's relation for those functionals.

1. Introduction

In 1976 [2], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [9], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ that extended the results in [2]. In [7], Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they obtained various results on the Fourier-Feynman transform and the convolution product [7,8]. In [11], Park, Skoug and Storvick investigated various relationships among the first variation, the convolution product, and the Fourier-Feynman transform for functionals on classical Wiener space that belong to the Banach algebra $\mathcal{S}$.

Recently Ahn, Chang, Kim and Yoo [1,5] extended the above relationships among the Fourier-Feynman transform, the convolution product and the first variation for functionals in $\mathcal{S}$ on classical Wiener space to those for functionals in the Fresnel class $\mathcal{F}(B)$ on abstract Wiener space. Moreover they [6] obtained the above results for functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing $\mathcal{F}(B)$.

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For a detailed survey of previous work on Fourier-Feynman transform and related topics, see the expository paper [12] by Skoug and Storvick.

On the other hand, Yeh [13] extended Wiener space to Yeh-Wiener space, that is, a space of functions of two variables. Much varied work on the Yeh-Wiener space has been done. For example, the first author [10] studied integral transforms of square integrable functionals on Yeh-Wiener space.

The purpose of this paper is to extend the results for functionals on Wiener space or on abstract Wiener space to those for functionals on Yeh-Wiener space. In Section 2, we introduce Yeh-Wiener space and Yeh-Feynman integral, and then define Fourier-Yeh-Feynman transform and convolution of functionals on Yeh-Wiener space. In Section 3, we establish the existence of Fourier-Yeh-Feynman transform and convolution product for functionals in a Banach algebra $\mathcal{S}(Q)$. Also we obtain Parseval's relation for those functionals.

2. Definitions and preliminaries

Let $Q = [0, S] \times [0, T]$ and let $C(Q)$ denote Yeh-Wiener space; that is, the space of all real-valued continuous functions $x(s, t)$ on Q with $x(s, 0) = x(0, t) = 0$ for all $0 \le s \le S$ and $0 \le t \le T$. Yeh [13] defined a Gaussian measure m_Y on $C(Q)$ (later modified in [14]) such that as a stochastic process $\{x(s, t) : (s, t) \in Q\}$ has mean $E[x(s, t)] = 0$ and covariance $E[x(s, t)x(u, v)] = \min\{s, u\} \min\{t, v\}$.

Let $\mathcal{M}$ denote the class of all Yeh-Wiener measurable subsets of $C(Q)$ and we denote the Yeh-Wiener integral of a Yeh-Wiener integrable functional F by

$$\int_{C(Q)} F(x)\, dm_Y(x). \tag{2.1}$$

A subset E of $C(Q)$ is said to be scale-invariant measurable provided ρE is Yeh-Wiener measurable for every $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_Y(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). Given two complex-valued functions F and G on $C(Q)$, we say that $F = G$ s-a.e. and write $F \approx G$ if $F(\rho x) = G(\rho x)$ for m_Y almost every $x \in C(Q)$ for all $\rho > 0$.

Let $\mathbb{C}_+$ and $\mathbb{C}_+^{\sim}$ denote the set of complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively.

Definition 2.1. Let F be a complex valued measurable functional on $C(Q)$ such that

$$J_F(\lambda) = \int_{C(Q)} F(\lambda^{-1/2}x)\, dm_Y(x) \tag{2.2}$$

exists as a finite number for all real $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic Yeh-Wiener integral of F over $C(Q)$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C(Q)}^{\text{anw}_\lambda} F(x)\, dm_Y(x) = J_F^*(\lambda). \tag{2.3}$$

If the following limit exists for nonzero real q, then we call it the analytic Yeh-Feynman integral of F over $C(Q)$ with parameter q and we write

$$\int_{C(Q)}^{\text{anf}_q} F(x)\, dm_Y(x) = \lim_{\lambda \to -iq} \int_{C(Q)}^{\text{anw}_\lambda} F(x)\, dm_Y(x) \tag{2.4}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Now we introduce the definitions of analytic Fourier-Yeh-Feynman transform and convolution product for functionals defined on $C(Q)$. Let $1 \le p < \infty$ and let q be a nonzero real number throughout this paper.

Definition 2.2. For $\lambda \in \mathbb{C}_+$ and $y \in C(Q)$, let

$$T_\lambda(F)(y) = \int_{C(Q)}^{\text{anw}_\lambda} F(x+y)\, dm_Y(x). \tag{2.5}$$

For $1 < p < \infty$, we define the L_p analytic Fourier-Yeh-Feynman transform $T_q^{(p)}(F)$ of F on $C(Q)$ by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = \operatorname*{l.\,i.\,m.}_{\lambda \to -iq} T_\lambda(F)(y), \tag{2.6}$$

whenever this limit exists; that is, for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C(Q)} |T_\lambda(F)(\rho x) - T_q^{(p)}(F)(\rho x)|^{p'}\, dm_Y(x) = 0 \tag{2.7}$$

where $1/p + 1/p' = 1$. We define the L_1 analytic Fourier-Yeh-Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y), \tag{2.8}$$

for s-a.e. $y \in C(Q)$, whenever this limit exists.

By the definition of the analytic Yeh-Feynman integral and the L_1 analytic Fourier-Yeh-Feynman transform, it is easy to see that for a nonzero real number q,

$$T_q^{(1)}(F)(y) = \int_{C(Q)}^{\mathrm{anf}_q} F(x+y)\, dm_Y(x) \tag{2.9}$$

and

$$T_q^{(1)}(F)(0) = \int_{C(Q)}^{\mathrm{anf}_q} F(x)\, dm_Y(x). \tag{2.10}$$

In our next example we evaluate Fourier-Yeh-Feynman transforms of some functionals.

Example 2.3. Let α be of bounded variation on Q in the sense of Hardy and Krause. Let

$$F(x) = \int_Q x(s,t)\, d\alpha(s,t), \quad G(x) = \int_Q [x(s,t)]^2\, d\alpha(s,t).$$

We evaluate Fourier-Yeh-Feynman transforms of F and G. For $\lambda > 0$, using Fubini theorem we have

$$T_\lambda(F)(y) = \int_Q \int_{C(Q)} [\lambda^{-1/2} x(s,t) + y(s,t)]\, dm_Y(x)\, d\alpha(s,t).$$

Since $\int_{C(Q)} x(s,t)\, dm_Y(x) = 0$, we have

$$T_\lambda(F)(y) = \int_Q y(s,t)\, d\alpha(s,t)$$

and so

$$T_q^{(p)}(F)(y) = \int_Q y(s,t)\, d\alpha(s,t).$$

Similarly we have for $\lambda > 0$

$$T_\lambda(G)(y) = \int_Q \int_{C(Q)} [\lambda^{-1/2} x(s,t) + y(s,t)]^2\, dm_Y(x)\, d\alpha(s,t).$$

Since $\int_{C(Q)}[x(s,t)]^2\,dm_Y(x) = st/2$, we have

$$T_\lambda(G)(y) = \int_Q \Big(\frac{1}{2\lambda}st + [y(s,t)]^2\Big)\,d\alpha(s,t).$$

Letting $\lambda \to -iq$ we obtain

$$T_q^{(p)}(G)(y) = \frac{i}{2q}\int_Q st\,d\alpha(s,t) + \int_Q [y(s,t)]^2\,d\alpha(s,t).$$

DEFINITION 2.4. Let F and G be functionals on $C(Q)$. For $\lambda \in \mathbb{C}_+$, we define their convolution product (if it exists) by

$$(F*G)_\lambda(y) = \int_{C(Q)}^{\mathrm{anw}_\lambda} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,dm_Y(x). \tag{2.11}$$

Moreover if $\lambda = -iq$ for nonzero real q, the convolution product is defined by

$$(F*G)_q(y) = \int_{C(Q)}^{\mathrm{anf}_q} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,dm_Y(x). \tag{2.12}$$

It is easy to see that commutative law and distributive law hold for the convolution product.

Next we describe the class of functionals that we work with in this paper. The Banach algebra $\mathcal{S}(Q)$ consists of functionals expressible in the form

$$F(x) = \int_{L_2(Q)} \exp\{i\langle\alpha, x\rangle\}\,df(\alpha) \tag{2.13}$$

for s-a.e. x in $C(Q)$, where f is an element of $M(L_2(Q))$, the space of complex valued countably additive Borel measures on $L_2(Q)$ and $\langle\alpha, x\rangle$ denotes the Paley-Wiener-Zigmund stochastic integral $\int_Q \alpha(s,t)\,dx(s,t)$.

The Banach algebra $\mathcal{S}(Q)$ is the Yeh-Wiener space version of the Banach algebra $\mathcal{S}$ on classical Wiener space introduced by Cameron and Storvick in [3].

It is known that

$$\int_{C(Q)} \exp\{i\langle\alpha, x\rangle\}\,dm_Y(x) = \exp\Big\{-\frac{1}{2}\|\alpha\|^2\Big\}, \tag{2.14}$$

where $\|\alpha\|^2 = \int_Q (\alpha(s,t))^2\, ds\, dt$. Moreover if $F \in \mathcal{S}(Q)$ is given by (2.13), then F is analytic Yeh-Feynman integrable and

$$\int_{C(Q)}^{\mathrm{anf}_q} F(x)\, dm_Y(x) = \int_{L_2(Q)} \exp\Big\{-\frac{i}{2q}\|\alpha\|^2\Big\}\, df(\alpha). \tag{2.15}$$

3. Fourier-Yeh-Feynman transform and convolution

We begin this section by proving the existence of L_p analytic Fourier-Yeh-Feynman transform for functionals in $\mathcal{S}(Q)$.

THEOREM 3.1. *Let $F \in \mathcal{S}(Q)$ be given by (2.13). Then for all p with $1 \le p < \infty$ and for all nonzero real q, the L_p analytic Fourier-Yeh-Feynman transform $T_q^{(p)}(F)$ exists, belongs to $\mathcal{S}(Q)$, and is given by the formula*

$$T_q^{(p)}(F)(y) = \int_{L_2(Q)} \exp\{i\langle\alpha, y\rangle\}\, df_t(\alpha) \tag{3.1}$$

for s-a.e. y in $C(Q)$, where f_t is the measure defined by

$$f_t(E) = \int_E \exp\Big\{-\frac{i}{2q}\|\alpha\|^2\Big\}\, df(\alpha) \tag{3.2}$$

for $E \in \mathcal{B}(L_2(Q))$.

Proof. Using the Fubini theorem and integration formula (2.14) we obtain

$$\begin{aligned} T_\lambda(F)(y) &= \int_{C(Q)} F(\lambda^{-1/2}x + y)\, dm_Y(x) \\ &= \int_{L_2(Q)} \int_{C(Q)} \exp\{i\langle\alpha, \lambda^{-1/2}x + y\rangle\}\, dm_Y(x)\, df(\alpha) \\ &= \int_{L_2(Q)} \exp\Big\{i\langle\alpha, y\rangle - \frac{1}{2\lambda}\|\alpha\|^2\Big\}\, df(\alpha) \end{aligned}$$

for all $\lambda > 0$ and s-a.e. y in $C(Q)$. Let $\lambda \in \mathbb{C}_+^\sim$ and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}_+^\sim$ which converges to λ. Then $|\exp\{i\langle\alpha, y\rangle - \frac{1}{2\lambda_n}\|\alpha\|^2\}| \le 1$ for all $n = 1, 2, \ldots$ and so by the dominated convergence theorem, the right hand side of the above equation is a bounded continuous function of

$\lambda \in \mathbb{C}_+^\sim$. Let Δ be a closed contour in $\mathbb{C}_+$. Then by the Fubini theorem and the Cauchy theorem,

$$\int_\Delta \int_{L_2(Q)} \exp\Big\{ i\langle \alpha, y\rangle - \frac{1}{2\lambda}\|\alpha\|^2 \Big\}\, df(\alpha)\, d\lambda = 0$$

and so by the Morera theorem, $\int_{L_2(Q)} \exp\{i\langle\alpha,y\rangle - \frac{1}{2\lambda}\|\alpha\|^2\}\, df(\alpha)$ is an analytic function of $\lambda \in \mathbb{C}_+$. Hence for $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C(Q)$,

$$T_\lambda(F)(y) = \int_{L_2(Q)} \exp\Big\{ i\langle \alpha, y\rangle - \frac{1}{2\lambda}\|\alpha\|^2 \Big\}\, df(\alpha).$$

In case $p = 1$, by the dominated convergence theorem,

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) = \int_{L_2(Q)} \exp\Big\{ i\langle \alpha, y\rangle - \frac{i}{2q}\|\alpha\|^2 \Big\}\, df(\alpha)$$

for s-a.e. $y \in C(Q)$. If $1 < p < \infty$, again by the dominated convergence theorem,

$$\int_{C(Q)} \Big| \int_{L_2(Q)} \exp\Big\{ i\langle \alpha, \rho y\rangle - \frac{i}{2q}\|\alpha\|^2 \Big\}\, df(\alpha) - T_\lambda(F)(\rho y) \Big|^{p'} dm_Y(y) \to 0$$

as $\lambda \to -iq$ for each $\rho > 0$. Hence $T_q^{(p)}(F)(y)$ exists and is given by

$$T_q^{(p)}(F)(y) = \int_{L_2(Q)} \exp\Big\{ i\langle \alpha, y\rangle - \frac{i}{2q}\|\alpha\|^2 \Big\}\, df(\alpha)$$

for all desired values of p and q. Finally it is easy to see that the last equation can be expressed as (3.1) and (3.2). □

As we have seen in (2.10), the L_1 analytic Fourier-Yeh-Feynman transform of F evaluated at 0 is equal to the analytic Yeh-Wiener integral of F. If we restrict our attention to the functional $F \in \mathcal{S}(Q)$, then from (2.15) and Theorem 3.1, we have

$$T_q^{(p)}(F)(0) = \int_{C(Q)}^{\mathrm{anf}_q} F(x)\, dm_Y(x) \tag{3.3}$$

for all $1 \le p < \infty$ and nonzero real q.

We adopt the convention $\frac{1}{\pm\infty} = 0$. Thus if $q = \pm\infty$, then we mean $T_q^{(p)}(F)$ to be F itself. Now the following corollary is immediate from Theorem 3.1.

Corollary 3.2. *Let $F \in \mathcal{S}(Q)$ be given by* (2.13). *Let $1 \le p < \infty$ and let q_1, q_2 be nonzero extended real numbers. Then*

$$T_{q_1}^{(p)}(T_{q_2}^{(p)}(F)) \approx T_q^{(p)}(F) \tag{3.4}$$

where q is an extended real number such that $1/q_1 + 1/q_2 = 1/q$. If $q_1 = -q_2$ in (3.4), *then we obtain the following inverse transform theorem for $F \in \mathcal{S}(Q)$*

$$T_{-q}^{(p)}(T_q^{(p)}(F)) \approx F. \tag{3.5}$$

Moreover if n is a natural number, then

$$\overbrace{T_q^{(p)}(\cdots(T_q^{(p)}}^{n\ times}(F))) \approx T_{q/n}^{(p)}(F). \tag{3.6}$$

In [4], Cameron and Storvick presented a new translation theorem for the analytic Feynman integral on a classical Wiener space. In our next theorem we will give a simple proof of a Yeh-Wiener space version of the translation theorem.

Theorem 3.3. *Let $F \in \mathcal{S}(Q)$ be given by* (2.13) *and let $w \in C(Q)$. Then for every nonzero real q, both members of the following equation exist and satisfy*

$$\begin{aligned} &\int_{C(Q)}^{\mathrm{anf}_q} F(x+w)\, dm_Y(x) \\ &= \exp\Big\{\frac{iq}{2}\|w\|^2\Big\} \int_{C(Q)}^{\mathrm{anf}_q} F(x) \exp\{-iq\langle w, x\rangle\}\, dm_Y(x). \end{aligned} \tag{3.7}$$

Proof. Let $G(x) = F(x)\exp\{-iq\langle w, x\rangle\}$. Using (2.13) we can rewrite $G(x)$ as follows:

$$G(x) = \int_{L_2(Q)} \exp\{i\langle \alpha - qw, x\rangle\}\, df(\alpha) = \int_{L_2(Q)} \exp\{i\langle \beta, x\rangle\}\, d\tilde{f}(\beta)$$

where $\tilde{f}$ is a measure in $M(L_2(Q))$ such that $\tilde{f}(E) = f(E+qw)$ for $E \in \mathcal{B}(L_2(Q))$. Thus by Theorem 3.1, we have

$$\begin{aligned} T_q^{(1)}(G)(0) &= \int_{L_2(Q)} \exp\Big\{-\frac{i}{2q}\|\alpha - qw\|^2\Big\}\, df(\alpha) \\ &= \exp\Big\{-\frac{iq}{2}\|w\|^2\Big\} \int_{L_2(Q)} \exp\Big\{i\langle\alpha, w\rangle - \frac{i}{2q}\|\alpha\|^2\Big\}\, df(\alpha) \\ &= \exp\Big\{-\frac{iq}{2}\|w\|^2\Big\} T_q^{(1)}(F)(w). \end{aligned}$$

By (2.9) and (2.10), the proof is completed. □

In the next theorem we will show the existence of the convolution product.

Theorem 3.4. *Let $F, G \in \mathcal{S}(Q)$ with corresponding finite Borel measures f and g in $M(L_2(Q))$, respectively. Then for all nonzero real q, the convolution product $(F * G)_q$ exists, belongs to $\mathcal{S}(Q)$ and is given by the formula*

$$(F * G)_q(y) = \int_{L_2(Q)} \exp\{i\langle\gamma, y\rangle\}\, dh_c(\gamma) \tag{3.8}$$

for s-a.e. y in $C(Q)$, where $h_c = h \circ \phi^{-1}$ and $\phi : L_2(Q)^2 \to L_2(Q)$ is a function defined by $\phi(\alpha, \beta) = \frac{1}{\sqrt{2}}(\alpha + \beta)$ and h is the measure defined by

$$h(E) = \int_E \exp\Big\{-\frac{i}{4q}\|\alpha - \beta\|^2\Big\}\, df(\alpha)\, dg(\beta) \tag{3.9}$$

for $E \in \mathcal{B}(L_2(Q)^2)$.

Proof. For all $\lambda > 0$ and s-a.e. y in $C(Q)$, using the Fubini theorem, we have

$$\begin{aligned} (F * G)_\lambda(y) &= \int_{C(Q)} F\Big(\frac{y + \lambda^{-1/2}x}{\sqrt{2}}\Big) G\Big(\frac{y - \lambda^{-1/2}x}{\sqrt{2}}\Big)\, dm_Y(x) \\ &= \int_{L_2(Q)^2} \int_{C(Q)} \exp\Big\{\frac{i}{\sqrt{2}}\langle\alpha + \beta, y\rangle + \frac{i}{\sqrt{2\lambda}}\langle\alpha - \beta, x\rangle\Big\} \\ &\qquad dm_Y(x)\, df(\alpha)\, dg(\beta). \end{aligned}$$

By integration formula (2.14), we have

$$(F * G)_\lambda(y) = \int_{L_2(Q)^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle\alpha + \beta, y\rangle - \frac{1}{4\lambda}\|\alpha - \beta\|^2\Big\}\, df(\alpha)\, dg(\beta).$$

But by the same method as in the proof of Theorem 3.1, we can show that the last expression in the above equation is analytic in $\lambda \in \mathbb{C}_+$ and is bounded continuous in $\lambda \in \mathbb{C}_+^\sim$. Hence $(F * G)_q$ exists and is given by

$$(F * G)_q(y) = \int_{L_2(Q)^2} \exp\Big\{ \frac{i}{\sqrt{2}} \langle \alpha + \beta, y \rangle - \frac{i}{4q} \|\alpha - \beta\|^2 \Big\} \, df(\alpha) \, dg(\beta).$$

Now it is easy to see that the last equation can be expressed as (3.8) and (3.9). □

COROLLARY 3.5. *Let F and G be given as in Theorem 3.4. Let q_1 and q_2 be nonzero extended real numbers. Let $1 \le p < \infty$. Then*

$$(3.10) \qquad (T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \int_{L_2(Q)} \exp\{i\langle \gamma, y \rangle\} \, dh_{tc}(\gamma)$$

for s-a.e. y in $C(Q)$, where $h_{tc} = h_t \circ \phi^{-1}$ and ϕ is the function as in Theorem 3.4 and h_t is the measure defined by

$$(3.11) \quad h_t(E) = \int_E \exp\Big\{ -\frac{i}{2q_1} \|\alpha\|^2 - \frac{i}{2q_2} \|\beta\|^2 - \frac{i}{4q} \|\alpha - \beta\|^2 \Big\} \, df(\alpha) \, dg(\beta)$$

for $E \in \mathcal{B}(L_2(Q)^2)$.

Proof. By Theorem 3.1 we know that $T_{q_1}^{(p)}(F)$ and $T_{q_2}^{(p)}(G)$ exist and belong to $\mathcal{S}(Q)$. Applying Theorem 3.4 to the corresponding expression (3.1) for $T_{q_1}^{(p)}(F)$ and $T_{q_2}^{(p)}(G)$, we obtain

$$(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \int_{L_2(Q)} \exp\{i\langle \gamma, y \rangle\} \, dh_{tc}(\gamma)$$

for s-a.e. y in $C(Q)$, where $h_{tc} = h_t \circ \phi^{-1}$ and ϕ is the function as in Theorem 3.4 and h_t is the measure defined by

$$h_t(E) = \int_E \exp\Big\{ -\frac{i}{4q} \|\alpha - \beta\|^2 \Big\} \, df_t(\alpha) \, dg_t(\beta).$$

Now using the corresponding expression (3.2) for f_t and g_t, we know that h_t above can be expressed as (3.11) and this completes the proof. □

From now on we establish relationships between Fourier-Yeh-Feynman transform and convolution product for functionals in $\mathcal{S}(Q)$.

THEOREM 3.6. *Let $F, G \in \mathcal{S}(Q)$ be given as in Theorem 3.4. Let $1 \le p < \infty$ and let q_1, q_2 be nonzero extended real numbers. Then for all nonzero real q,*

$$T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = T_{q_1'}^{(p)}(F)\Big(\frac{y}{\sqrt{2}}\Big)\, T_{q_2'}^{(p)}(G)\Big(\frac{y}{\sqrt{2}}\Big) \tag{3.12}$$

for s-a.e. y in $C(Q)$, where q_j' is a nonzero extended real number such that $1/q + 1/q_j = 1/q_j'$ for $j = 1, 2$. Also both sides of the above expression are given by the formula

$$\int_{L_2(Q)^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle \alpha+\beta, y\rangle - \frac{i}{2q_1'}\|\alpha\|^2 - \frac{i}{2q_2'}\|\beta\|^2\Big\}\, df(\alpha)\, dg(\beta). \tag{3.13}$$

Proof. Note that $(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y)$ is expressed as (3.10). Applying Theorem 3.1 to the expression in (3.10), we know that

$$T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \int_{L_2(Q)} \exp\{i\langle \gamma, y\rangle\}\, dh_{tct}(\gamma),$$

where

$$h_{tct}(E) = \int_E \exp\Big\{-\frac{i}{2q}\|\gamma\|^2\Big\}\, dh_{tc}(\gamma)$$

and h_{tc} is given as in Corollary 3.5. Now a simple calculation together with (3.11) shows that the left hand side of (3.12) is expressed as (3.13). On the other hand, by Theorem 3.1, the right hand side of (3.12) also is expressed as (3.13) and this completes the proof. □

In our next theorem we establish an interesting Parseval's relation for Fourier-Yeh-Feynman transform on $\mathcal{S}(Q)$.

THEOREM 3.7. *Let F, G, p, q_1 and q_2 be given as in Theorem 3.6. Then for all nonzero real q, the following Parseval's relation holds.*

$$\begin{aligned} &T_{-q}^{(p)}[T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q](0) \\ =&T_q^{(p)}\Big(T_{q_1}^{(p)}(F)\Big(\frac{\cdot}{\sqrt{2}}\Big)\, T_{q_2}^{(p)}(G)\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0). \end{aligned} \tag{3.14}$$

Also both sides of the above expression are given by the formula

$$\int_{L_2(Q)^2} \exp\Big\{-\frac{i}{2q_1}\|\alpha\|^2 - \frac{i}{2q_2}\|\beta\|^2 - \frac{i}{4q}\|\alpha-\beta\|^2\Big\}\, df(\alpha)\, dg(\beta). \tag{3.15}$$

Proof. By the inverse transform theorem (3.5) we have

$$T_{-q}^{(p)}[T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q](0) = (T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(0).$$

But the right hand side of the above equation is expressed as (3.15) by Corollary 3.5. On the other hand, applying Theorem 3.1 to the expression

$$\begin{aligned} &T_{q_1}^{(p)}(F)\Big(\frac{y}{\sqrt{2}}\Big)\, T_{q_2}^{(p)}(G)\Big(-\frac{y}{\sqrt{2}}\Big) \\ &= \int_{L_2(Q)^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle \alpha - \beta, y\rangle - \frac{i}{2q_1}\|\alpha\|^2 - \frac{i}{2q_2}\|\beta\|^2\Big\}\, df(\alpha)\, dg(\beta) \end{aligned}$$

we know that the right hand side of (3.14) is also expressed as (3.15) which completes the proof. □

Using (3.3) we can express (3.14) alternatively as the following corollary.

Corollary 3.8. *Let F, G, p, q_1 and q_2 be given as in Theorem 3.6. Then for all nonzero real q, the following Parseval's relation holds.*

$$\begin{aligned} &\int_{C(Q)}^{\mathrm{anf}_{-q}} T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(x)\, dm_Y(x) \\ &\qquad = \int_{C(Q)}^{\mathrm{anf}_q} T_{q_1}^{(p)}(F)\Big(\frac{x}{\sqrt{2}}\Big)\, T_{q_2}^{(p)}(G)\Big(-\frac{x}{\sqrt{2}}\Big)\, dm_Y(x). \end{aligned} \tag{3.16}$$

In particular if $q_1 = q_2 = \infty$, then

$$\begin{aligned} &\int_{C(Q)}^{\mathrm{anf}_{-q}} T_q^{(p)}(F)\Big(\frac{x}{\sqrt{2}}\Big)\, T_q^{(p)}(G)\Big(\frac{x}{\sqrt{2}}\Big)\, dm_Y(x) \\ &\qquad = \int_{C(Q)}^{\mathrm{anf}_q} F\Big(\frac{x}{\sqrt{2}}\Big)\, G\Big(-\frac{x}{\sqrt{2}}\Big)\, dm_Y(x). \end{aligned} \tag{3.17}$$

Our final corollary follows immediately from (3.17) by choosing $G = F$ for (3.18) and $G = 1$ for (3.19).

Corollary 3.9. *Let F, p and q be given as in Theorem 3.6. Then we have*

$$\int_{C(Q)}^{\mathrm{anf}_{-q}} \Big[T_q^{(p)}(F)\Big(\frac{x}{\sqrt{2}}\Big)\Big]^2 dm_Y(x) = \int_{C(Q)}^{\mathrm{anf}_q} F\Big(\frac{x}{\sqrt{2}}\Big)\, F\Big(-\frac{x}{\sqrt{2}}\Big)\, dm_Y(x) \tag{3.18}$$

and

$$\int_{C(Q)}^{\mathrm{anf}_{-q}} T_q^{(p)}(F)\Big(\frac{x}{\sqrt{2}}\Big)\, dm_Y(x) = \int_{C(Q)}^{\mathrm{anf}_q} F\Big(\frac{x}{\sqrt{2}}\Big)\, dm_Y(x). \tag{3.19}$$

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