Korean J. Math. **16** (2008), No. 3, pp. 323–334

INJECTIVE AND PROJECTIVE PROPERTIES OF REPRESENTATIONS OF QUIVERS WITH n EDGES

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Abstract. We define injective and projective representations of quivers with two vertices with n arrows. In the representation of quivers we denote n edges between two vertices as $\Rightarrow$ and n maps as $f_1 \sim f_n$, and $E \oplus E \oplus \cdots \oplus E$ (n times) as $\oplus_n E$. We show that if E is an injective left R-module, then

$$\oplus_n E \overset{p_1 \sim p_n}{\Longrightarrow} E$$

is an injective representation of $Q = \bullet \Rightarrow \bullet$ where $p_i(a_1, a_2, \cdots, a_n) = a_i,\ i \in \{1, 2, \cdots, n\}$. Dually we show that if $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ is an injective representation of a quiver $Q = \bullet \Rightarrow \bullet$ then M_1 and M_2 are injective left R-modules. We also show that if P is a projective left R-module, then

$$P \overset{i_1 \sim i_n}{\Longrightarrow} \oplus_n P$$

is a projective representation of $Q = \bullet \Rightarrow \bullet$ where i_k is the kth injection. And if $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ is an projective representation of a quiver $Q = \bullet \Rightarrow \bullet$ then M_1 and M_2 are projective left R-modules.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex

Received July 2, 2008. Revised July 16, 2008.
2000 Mathematics Subject Classification: Primary 16E30; Secondary 13C11, 16D80.
Key words and phrases: injective module, injective representation, quiver.
This paper was supported by Dong-A University Research fund, in 2006.

to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2$, V_1 and V_2 are vector spaces and f is a linear map (morphism). Then we can define a morphism of two representations of the same quiver i.e., given a quiver $Q = \bullet \to \bullet$, we can define two representations $V_1 \xrightarrow{f} V_2$ and $W_1 \xrightarrow{g} W_2$.

Now we can define a morphism between these two representations. A morphism of $V_1 \xrightarrow{f} V_2$ to $W_1 \xrightarrow{g} W_2$ is given by a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ W_1 & \xrightarrow[g]{} & W_2 \end{array}$$

with s_1, s_2 linear maps.

In ([3]) a homotopy of quiver was developed and in ([2]) cyclic quiver ring was studies. Recently, the theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]).

Definition 1.1. ([7]) A left R-module E is said to be injective if given any injective linear map $\sigma : M' \to M$ and any linear map $h : M' \to E$, there is a linear map $g : M \to E$ such that $g \circ \sigma = h$. That is

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{\sigma} & M \\ & & {\scriptstyle h}\downarrow & \swarrow{\scriptstyle g} & \\ & & E & & \end{array}$$

can always be completed to a commutative diagram.

Definition 1.2. ([7]) A left R-module P is said to be projective if given any surjective linear map $\sigma : M' \to M$ and any linear map $h : P \to M$, there is a linear map $g : P \to M'$ such that $\sigma \circ g = h$. That is

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle h} & & \\ M' & \xrightarrow{\sigma} & M & \longrightarrow & 0 \end{array}$$

can always be completed to a commutative diagram.

Let $G = G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n$ be a direct product of groups. The projection map $\pi_i : G \to G_i$ where $\pi_i(g_1, g_2, \cdots, g_i, \cdots, g_n) = g_i$ is a homomorphism for each $i = 1, 2, \cdots, n$. This follows immediately from the fact that the binary operation of G coincides in the ith component with the binary operation in G_i. Let $\phi_i : G_i \to G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n$ be given by $\phi_i(g_i) = (e_1, e_2, \cdots, g_i, \cdots, e_n)$ where $g_i \in G_i$ and e_j is the identity of G_j. This is an injection map. Let $F = \{X_i | i \in I\}$ be an indexed family of left R-modules X_i and denote $P = \prod_{i \in I} X_i$ the cartesian product of F. Define an element of P as a function $f : I \to \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for every $i \in I$. Define $(P, +)$ by $(f + g)(i) = f(i) + g(i) \in X_i$ for every $i \in I$, and $0(i) = 0 \in X_i$ and $(-f)(i) = -[f(i)]$. Then easily $(P, +)$ is an abelian group. Define $\mu : R \times P \to P$ by $(rf)(i) = r[f(i)]$ for every $i \in I$. Then easily P is a left R-module. We say P as the direct product of F over R. Consider S the subset of P such that $f(i) = 0$ except only finite $i \in I$. Then easily S is a submodule of P. We say S as direct sum of F and is denoted by $S = \bigoplus_{i \in I} X_i$.

Remark 1. $p_j|_s : \bigoplus_{i \in I} X_i \to X_j$ is called the natural projection of $S = \bigoplus_{i \in I} X_i$.

So we have morphisms

$$X_j \xrightarrow{d_j} \bigoplus_{i \in I} X_i \xrightarrow{i} \prod_{i \in I} X_i \xrightarrow{p_k} X_k$$

$p_k \circ i \circ d_j : X_j \to X_k$ is trivial if $j \neq k$, and is identity if $j = k$.

Remark 2. The natural injection $d_j : X_j \to \bigoplus_{i \in I} X_i$ is a monomorphism and the natural projection $p_j : \prod_{i \in I} X_i \to X_j$ is an epimorphism.

Notation : In the representation of quivers we denote n arrows between two vertices as $\Rightarrow$ and n maps as $f_1 \sim f_n$ and $E \oplus E \oplus \cdots \oplus E$ (n times) as $\oplus_n E$.

2. Injective representation of a quiver $Q = \bullet \Rightarrow \bullet$ with n edges

We define injective representation of a quiver with two vertices and multiple arrows. And consider their various injective representations as left R-modules.

Definition 2.1. A representation $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ of a quiver $Q = \bullet \Rightarrow \bullet$ is called an injective representation if for any representation $N_1 \overset{g_1 \sim g_n}{\Longrightarrow} N_2$ with a subrepresentation

$$S_1 \overset{S_2|g_1|_{S_1} \sim S_2|g_n|_{S_1}}{\Longrightarrow} S_2$$

and morphisms

$$\begin{array}{ccc} S_1 & \Longrightarrow & S_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2 \end{array}$$

there exist $H \in Hom_R(N_1, M_1)$ and $K \in Hom_R(N_2, M_2)$ such that the following diagram

$$\begin{array}{ccc} N_1 & \overset{g_1 \sim g_n}{\Longrightarrow} & N_2 \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle K} \\ M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2 \end{array}$$

commutes and $H|_{S_1} = h$ $K|_{S_2} = k$.
In other words, every diagram of representations

$$\begin{array}{ccccccc} (0 \Longrightarrow 0) & \longrightarrow & (S_1 & \Longrightarrow & S_2) & \longrightarrow & (N_1 \overset{g_1 \sim g_n}{\Longrightarrow} N_2) \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \\ & & (M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2) & & \end{array}$$

can be completed to a commutative diagram as follows :

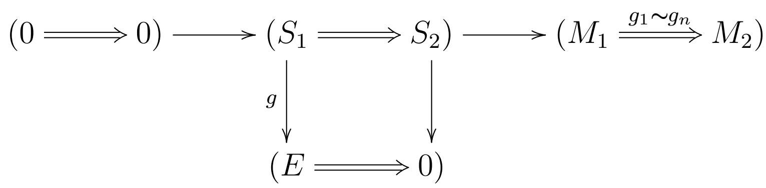
$$\begin{array}{ccccccc} (0 \Longrightarrow 0) & \longrightarrow & (S_1 & \Longrightarrow & S_2) & \longrightarrow & (N_1 \overset{g_1 \sim g_n}{\Longrightarrow} N_2) \\ & & \downarrow & & \downarrow & {\scriptstyle H} \swarrow \quad \swarrow {\scriptstyle K} & \\ & & (M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2) & & \end{array}$$

Theorem 2.2. If E is an injective left R-module, then

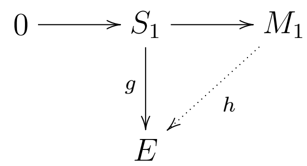
$$E \Longrightarrow 0$$

is an injective representation of $Q = \bullet \Rightarrow \bullet$.

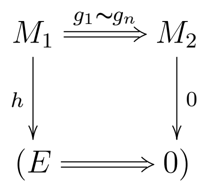
Proof. Let M_1, M_2 be left R-modules, S_1 be a submodule of M_1, S_2 be a submodule of M_2 and $g : S_1 \to E$ be an R-linear map. Consider the following diagram



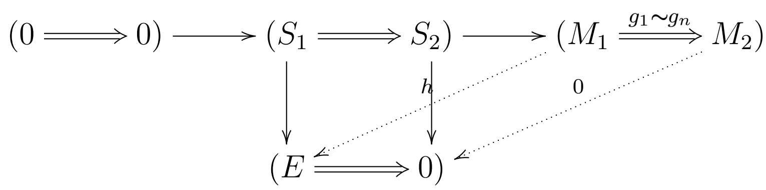
Then since E is an injective left R-module, we can complete the following commutative diagram by h.



Then $0(h(m)) = 0 = 0(g_1(m))$, $0(h(m)) = 0 = 0(g_2(m))$, $\cdots$, $0(h(m)) = 0 = 0(g_n(m))$. Thus, we can complete the following diagram



as a commutative diagram by $0 : M_2 \to 0$.
Therefore, we can complete the diagram



as a commutative diagram. Hence, $E \Longrightarrow 0$ is an injective representation. □

THEOREM 2.3. If E is an injective left R-module, then

$$\oplus_n E \overset{p_1 \sim p_n}{\Longrightarrow} E$$

is an injective representation of $Q = \bullet \Rightarrow \bullet$ where $p_i(a_1, a_2, \cdots, a_n) = a_i$, $i \in \{1, 2, \cdots, n\}$.

Proof. Let M_1, M_2 be a left R-module, S_1 be a submodule of M_1, S_2 be a submodule of M_2 and $g : S_2 \to E$ be an R-linear map. Consider the following diagram

$$\begin{array}{ccccc} (0 \Longrightarrow 0) & \longrightarrow & (S_1 \Longrightarrow S_2) & \longrightarrow & (M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2) \\ & & \downarrow \qquad\quad \downarrow g & & \\ & & (\oplus_n E \overset{p_1 \sim p_n}{\Longrightarrow} E) & & \end{array}$$

Then since E is an injective left R-module, we can consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & S_2 & \longrightarrow & M_2 \\ & & g \downarrow & \swarrow h & \\ & & E & & \end{array}$$

by h. Define $H : M_1 \to \oplus_n E$ by $H(m) = (h(f_1(m)), h(f_2(m)), \cdots, h(f_n(m)))$. Then $p_i(H(m)) = h(f_i(m))$, $i = 1, 2, \cdots, n$ and we can complete the following diagram

$$\begin{array}{ccc} M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2 \\ H \downarrow & & \downarrow h \\ (\oplus_n E & \overset{p_1 \sim p_n}{\Longrightarrow} & E) \end{array}$$

as a commutative diagram. Hence, we can complete the diagram

$$\begin{array}{ccccc} (0 \Longrightarrow 0) & \longrightarrow & (S_1 \Longrightarrow S_2) & \longrightarrow & (M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2) \\ & & \downarrow \qquad\quad \downarrow & \swarrow H \qquad \swarrow h & \\ & & (\oplus_n E \overset{p_1 \sim p_n}{\Longrightarrow} E) & & \end{array}$$

as a commutative diagram. Therefore $\oplus_n E \overset{p_1 \sim p_n}{\Longrightarrow} E$ is an injective representation. □

THEOREM 2.4. If $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ is an injective representation of a quiver $Q = \bullet \Rightarrow \bullet$ then M_1 and M_2 are injective left R-modules.

Proof. First we show that M_2 is an injective left R-module. Let S be a submodule of N and $g : S \to M_2$ be an R-linear map and we consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & N \\ & & {\scriptstyle g}\downarrow & & \\ & & M_2 & & \end{array}$$

Then since $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ is an injective representation, there exist $h : N \to M_2$ which completes the following

$$\begin{array}{ccccc} (0 \Longrightarrow 0) & \longrightarrow & (0 \Longrightarrow S) & \longrightarrow & (0 \Longrightarrow N) \\ & & \downarrow \qquad\quad \downarrow & \swarrow^{h} & \\ & & (M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2) & & \end{array}$$

as a commutative diagram. Thus, $h : N \to M_2$ completes the above diagram as a commutative diagram. Therefore, M_2 is an injective left R-module.

Let $g : S \to M_1$ be an R-linear map and we consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & N \\ & & {\scriptstyle g}\downarrow & & \\ & & M_1 & & \end{array}$$

Consider the following diagram

$$\begin{array}{ccccc} (0 \Longrightarrow 0) & \longrightarrow & (S \overset{i_1 \sim i_n}{\Longrightarrow} \oplus_n S) & \longrightarrow & (N \overset{j_1 \sim j_n}{\Longrightarrow} \oplus_n N) \\ & & {\scriptstyle g}\downarrow \qquad\quad \downarrow{\scriptstyle G} & & \\ & & (M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2) & & \end{array}$$

where $G((s_1, s_2, \cdots, s_n)) = \sum_{k=1}^{n} f_k(g(s_k))$, and $i_k(s)$ is the kth injection.

Then since $M_1 \overset{f_1 \sim f_n}{\Longrightarrow} M_2$ is an injective representation, there exist $h : N \to M_1$ and $\alpha : \oplus_n E \to M_2$ such that the following diagram

$$\begin{array}{ccc} N & \overset{j_1 \sim j_n}{\Longrightarrow} & \oplus_n N \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ (M_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & M_2) \end{array}$$

as a commutative diagram.
Thus, $h : N \to M_1$ completes the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & N \\ & & {\scriptstyle g}\big\downarrow & \swarrow{\scriptstyle h} & \\ & & M_1 & & \end{array}$$

as a commutative diagram. Therefore, M_1 is an injective left R-module. □

3. Projective representation of a quiver $Q = \bullet \Rightarrow \bullet$ with n edges

DEFINITION 3.1. A representation $P_1 \overset{f_1 \sim f_n}{\Longrightarrow} P_2$ of a quiver $Q = \bullet \Rightarrow \bullet$ is called a projective representation if every diagram of representations

$$\begin{array}{ccccccc} & & (P_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & P_2) & & \\ & & \big\downarrow & & \big\downarrow & & \\ (M_1 \overset{g_1 \sim g_n}{\Longrightarrow} M_2) & \longrightarrow & (N_1 & \overset{h_1 \sim h_n}{\Longrightarrow} & N_2) & \longrightarrow & (0 \Longrightarrow 0) \end{array}$$

can be completed to a commutative diagram as follows :

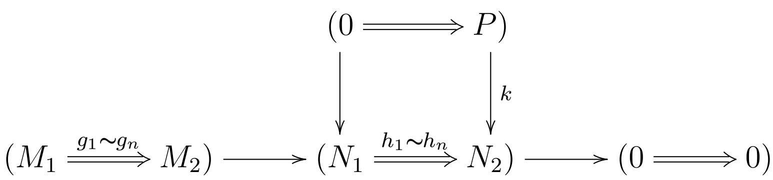
$$\begin{array}{ccccccc} & & (P_1 & \overset{f_1 \sim f_n}{\Longrightarrow} & P_2) & & \\ & \swarrow{\scriptstyle H} \quad \swarrow{\scriptstyle K} & \big\downarrow & & \big\downarrow & & \\ (M_1 \overset{g_1 \sim g_n}{\Longrightarrow} M_2) & \longrightarrow & (N_1 & \overset{h_1 \sim h_n}{\Longrightarrow} & N_2) & \longrightarrow & (0 \Longrightarrow 0) \end{array}$$

Theorem 3.2. If P is a projective left R-module, then

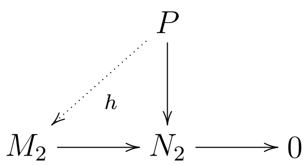
$$0 \Longrightarrow P$$

is a projective representation of $Q = \bullet \Rightarrow \bullet$.

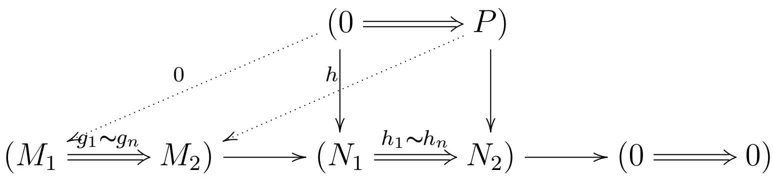
Proof. Let M_1, M_2, N_1, N_2 be left R-modules, and $k : P \to N_2$ be an R-linear map. Consider the following diagram



Then since P is a projective left R-module, we can complete the following commutative diagram by h.



Then $0 : 0 \to M_1$ completes the following diagram



as a commutative diagram. Hence, $0 \Longrightarrow P$ is a projective representation. □

Theorem 3.3. If P is a projective left R-module, then

$$P \overset{i_1 \sim i_n}{\Longrightarrow} \oplus_n P$$

is a projective representation of $Q = \bullet \Rightarrow \bullet$ where i_k is the kth injection.

Proof. Let $M_1, M_2,\ N_1,\ N_2$ be left R-modules and $g : P \to N_1$ be a R-linear map. Consider the following diagram

$$\begin{array}{ccccc} & & (P \overset{i_1\sim i_n}{\Longrightarrow} \oplus_n P) & & \\ & & g\downarrow \qquad\qquad \downarrow & & \\ (M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2) & \longrightarrow & (N_1 \Longrightarrow N_2) & \longrightarrow & (0 \Longrightarrow 0) \end{array}$$

Since P is a projective left R-module we can complete the following diagram by h.

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow g & & \\ M_1 & \longrightarrow & N_1 & \longrightarrow & 0 \end{array}$$

Define $H((a_1, a_2, \cdots, a_n)) = f_1(h(a_1)) + f_2(h(a_2)) + \cdots + f_n(h(a_n))$. Then $f_1(h(a)) = H(i_1(a)),\ f_2(h(a)) = H(i_2(a)), \cdots, f_n(h(a)) = H(i_n(a))$. Thus we can complete the following diagram

$$\begin{array}{ccccc} & & (P \overset{i_1\sim i_n}{\Longrightarrow} \oplus_n P) & & \\ & \overset{h}{\swarrow} \quad \overset{H}{\swarrow} & \downarrow \qquad\qquad \downarrow & & \\ (M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2) & \longrightarrow & (N_1 \Longrightarrow N_2) & \longrightarrow & (0 \Longrightarrow 0) \end{array}$$

as a commutative diagram. Hence, $P \overset{i_1\sim i_n}{\Longrightarrow} \oplus_n P$ is a projective representation. □

Theorem 3.4. If $M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2$ is an projective representation of a quiver $Q = \bullet \Rightarrow \bullet$ then M_1 and M_2 are projective left R-modules.

Proof. First we show that M_1 is a projective left R-module. Let S and N be left R-modules and $g : M_1 \to S$ be an R-linear map and we consider the following diagram

$$\begin{array}{ccccc} & & M_1 & & \\ & & \downarrow g & & \\ N & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

Then since $M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2$ is a projective representation, there exist $h : M_1 \to N$ which completes the following

$$
\begin{array}{ccccc}
 & & (M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2) & & \\
 & \swarrow^{h} \;\; \swarrow & \downarrow \qquad \downarrow & & \\
(N \Longrightarrow 0) & \longrightarrow & (S \Longrightarrow 0) & \longrightarrow & (0 \Longrightarrow 0)
\end{array}
$$

Therefore, M_1 is a projective left R-module. Let $g : M_2 \to S$ be an R-linear map and we consider the following diagram

$$
\begin{array}{ccccc}
 & & M_2 & & \\
 & & \downarrow{\scriptstyle g} & & \\
N & \longrightarrow & S & \longrightarrow & 0
\end{array}
$$

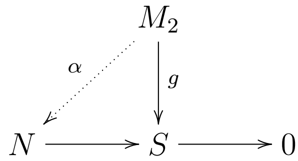
Define $G : M_1 \to \oplus_{i=1}^n S_i$ where $p_k(G(m)) = g(f_k(m))$, for $k = 1, ..., n$ and $G(m) = (g(f_1(m)), g(f_2(m)), \cdots, g(f_n(m)))$, and $p_k((a_1, a_2, \cdots, a_n)) = a_k$, and consider the following diagram

$$
\begin{array}{ccccc}
 & & (M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2) & & \\
 & & {\scriptstyle G}\downarrow \qquad\quad \downarrow{\scriptstyle g} & & \\
(\oplus_{i=1}^n N_i \overset{q_1\sim q_n}{\Longrightarrow} N) & \longrightarrow & (\oplus_n S \overset{p_1\sim p_n}{\Longrightarrow} S) & \longrightarrow & (0 \Longrightarrow 0)
\end{array}
$$

Then since $M_1 \overset{f_1\sim f_n}{\Longrightarrow} M_2$ is a projective representation, there exist $h : M_1 \to \oplus_n N$ and $\alpha : M_2 \to N$ such that the following diagram

$$
\begin{array}{ccc}
M_1 & \overset{f_1\sim f_n}{\Longrightarrow} & M_2 \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\
(\oplus_n N & \overset{i_1\sim i_n}{\Longrightarrow} & N)
\end{array}
$$

as a commutative diagram.Thus, $\alpha : M_2 \to N$ completes the following diagram



Therefore, M_2 is a projective left R-module. □

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