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EXISTENCE OF INFINITELY MANY SOLUTIONS OF THE NONLINEAR HIGHER ORDER ELLIPTIC EQUATION

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Abstract. We prove the existence of infinitely many solutions of the nonlinear higher order elliptic equation with Dirichlet boundary condition $(-\Delta)^m u = q(x,u) \quad$ in Ω, where $m \geq 1$ is an integer and $\Omega \subset R^n$ is a bounded domain with smooth boundary, when q(x,u) satisfies some conditions.

1. Introduction and the main result

In this paper we investigate the multiplicity result for the solutions of the following nonlinear higher order elliptic equations with Dirichlet boundary condition

$$(-\Delta)^m u = q(x,u) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0, \quad \dots \quad \Delta^{m-1} u = 0 \text{ on } \partial\Omega,$$

where $m \geq 1$ is an integer and $\Omega \subset R^n$ is a bounded domain with smooth boundary. We assume that q satisfies the following conditions:
$(q1)$ $q \in C(\Omega \times R)$ is nonnegative,
$(q2)$ There is a constant $C > 0$ such that

$$|q(x,s)| \leq C(1 + |s|), \qquad x \in \Omega, \tag{1.2}$$

in particular, we assume that

$$q(x,s) = \lambda_1^m s + p(x,s), \tag{1.3}$$

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where λ_1 is the smallest eigenvalue of the problem

$$-\Delta\phi = \lambda\phi, \qquad \phi|_{\partial\Omega} = 0, \tag{1.4}$$

and $p(x,s)$ is a fast oscillating function of an arbitrarily small amplitude;
$(q3)$ There are sequences σ_j, τ_k such that $\sigma_j \to \infty$, $\tau_k \to \infty$,

$$\int_\Omega \mathcal{P}(x, \sqrt{2\sigma_j}\phi_1(x))dx \to -\infty, \qquad \int_\Omega \mathcal{P}(x, -\sqrt{2\sigma_j}\phi_1(x))dx \to -\infty, \tag{1.5}$$

and either

$$\int_\Omega \mathcal{P}(x, \sqrt{2\tau_k}\phi_1)dx \geq 0 \qquad \text{or} \qquad \int_\Omega \mathcal{P}(x, -\sqrt{2\tau_k}\phi_1)dx \geq 0, \tag{1.6}$$

where ϕ_1 is the positive eigenfunction of the eigenvalue problem (1.4) corresponding to the eigenvalue λ_1 and $\mathcal{P}(x,s) = \int_0^s p(x,\sigma)d\sigma$.
$(q4)$ the function p is uniformly bounded on $\Omega \times R$.

Several authors were concerned with the multiple solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$-\Delta u = g(u) \qquad \text{in } \Omega, \tag{1.7}$$
$$u = 0, \qquad \text{on } \partial\Omega.$$

Castro and Lazer ([3]) showed that if the interval $(g'(0), g'(\infty)) \cup (g'(\infty), g'(0))$ contains the eigenvalues $\lambda_k, \ldots, \lambda_j$ and $g'(t) < \lambda_{j+1}$ for all $t \in R$, then (1.7) has at least three solutions. The proofs in [4] are based on global Lyapunov-Schmidt arguments applied to variational problems. Castro and Cossio ([5]) proved that problem (1.2) has at least five solutions if g is a differentiable function such that $g(0) = 0$, $g'(0) < \lambda_1$, $g'(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \geq 2$, and $g'(t) \leq \gamma < \lambda_{k+1}$. They proved this by using Lyapunov-Schmidt reduction arguments, the mountain pass lemma, and characterizations of the local degree of critical points. Chang ([6]) also approached the same problems using Morse theory. For other results in the study of this type problems we refer [2], [7], [9].

Our main result is the following:

Theorem 1.1. *Assume that q satisfies $(q1)$-$(q4)$. Then problem (1.1) has infinitely many solutions.*

The proof of Theorem 1.1 is organized as follows: In section 2 we introduce a Banach space H spanned by eigenfunctions and a corresponding functional of (1.1) whose critical points correspond to the weak solutions of problem (1.1), and study the Lagrange multipliers involved in such critical point equations. In Section 3 we prove Theorem 1.1.

2. The functional on the Banach space

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. We define a complete normed inner product subspace H of $L^2(\Omega)$ as follows: Let $\lambda_k, k = 1, 2, \dots$ denote the eigenvalues and $\phi_k, k = 1, 2, \dots$ the corresponding orthogonal eigenfunctions in $L^2(\Omega)$ with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots, \lambda_i \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. For any integer m the eigenvalue problem

$$(-\Delta)^m u = \mu u \qquad \text{in } \Omega, \qquad u = 0, \Delta u = 0, \dots \Delta^{m-1} u = 0 \qquad \text{on } \partial\Omega,$$

has infinitely many eigenvalues

$$\mu_k = \lambda_k^m, \qquad k \geq 1,$$

and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthogonal base for $L^2(\Omega)$. Let us denote an element u in $L^2(\Omega)$ as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) \mid \sum \lambda_k^m h_k^2 < \infty, \qquad m: \text{ integer } \}.$$

Now, we define an inner product $(\,,\,)_H$ by

$$(u, v)_H = ((-\Delta)^m u, v)_{L^2(\Omega)}$$

and a norm $\|\cdot\|$ in H by

$$\|u\|^2 = (u, u)_H = ((-\Delta)^m u, u)_{L^2(\Omega)}.$$

Then H is a complete normed inner product space with a norm $\|\cdot\|$. Since $\lambda_k^m \to +\infty$ and c is fixed, we have the following simple properties.

PROPOSITION 2.1. *(i)* $(-\Delta)^m u \in H$ *implies* $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ *for some* $C > 0$.
Moreover $\|u\| \geq C_r\|u\|^r_L(\Omega)$ *for* $r \geq 2$, $C_r > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ *if and only if* $\|u\| = 0$.

Proof. (i) Let $(-\Delta)^m u = \sum \lambda_k^m h_k \phi_k \in H$. Then

$$\sum \lambda_k^m \lambda_k^{2m} h_k^2 < \infty.$$

Since

$$\infty > \sum \lambda_k^m \lambda_k^{2m} h_k^2 \geq C \sum \lambda_k^m h_k^2$$

for some $C > 0$, it follows that

$$\sum \lambda_k^m h_k^2 < \infty.$$

Thus $u \in H$.
To prove (ii) we compute

$$\begin{aligned} \|u\| &= (u, u)_H \\ &= ((-\Delta)^m u, u)_{L^2(\Omega)} = \sum \int_\Omega [\lambda_k^m h_k^2 \phi_k^2] dx \\ &\geq C \sum \int_\Omega h_k^2 \phi_k^2 dx = C\|u\|_{L^2(\Omega)} \end{aligned}$$

for some $C > 0$. Next, we will prove the second statement. Let

$$\|u\|_{L^r(\Omega)} = \left(\int_\Omega |u|^r \right)^{\frac{1}{r}}, \qquad r \geq 1, \qquad u = \sum h_k \phi_k.$$

By a theorem of Riesz [5, p.525] we have

$$\|u\|_{L^r(\Omega)} \leq C'(\sum_k |h_k|^{r'})^{\frac{1}{r'}}, \quad r \geq 2, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Since for every $\varepsilon > 0$

$$\sum_k \frac{1}{|\lambda_k^m|^{1+\varepsilon}} < \infty,$$

it follows that for every $r \in [2, +\infty)$ there is $C'' > 0$ such that

$$\|u\|_{L^r(\Omega)} \geq C''\|u\|.$$

This proves the second statement of (ii).
To prove (iii) we have:

$$\|u\| = 0 \Longleftrightarrow \int_\Omega [\sum \lambda_k^m h_k^2 \phi_k^2] dx = 0 \Longleftrightarrow h_k = 0 \Longleftrightarrow \|u\|_{L^2(\Omega)}.$$

□

We define the functional f in H

$$f(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^{2m} + \frac{1}{2}\lambda^m |u|^2 + \mathcal{P}(x,u)\right] dx,$$

where $\mathcal{P}(x,s) = \int_0^s p(x,\sigma)d\sigma$. Then the critical points of f coincide with the solutions of equation (1.1). Now we have the following proposition.

PROPOSITION 2.2. *The functional $f(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative f'*

$$(f'(u),v)_H = \int_\Omega [(-\Delta)^m u + \lambda_1^m u + p(x,u)]\, v dx$$

for all $v \in H$.

Proof. Let $u \in H$. To prove the continuity of $f(u)$, we consider

$$\begin{aligned} f(u+v) - f(u) \;&=\; \int_\Omega [u \cdot ((-\Delta)^m v) + \frac{1}{2} v \cdot ((-\Delta)^m v) \\ &\quad + \lambda_1^m uv + \frac{1}{2}\lambda_1^m v^2 + \mathcal{P}(x,u+v) - \mathcal{P}(x,u)] dx. \end{aligned}$$

Let $u = \sum h_k \phi_k$, $v = \sum \tilde{h_k} \phi_k$. By Hölder inequality we have

$$\left|\int_\Omega u \cdot ((-\Delta)^m v) dx\right| = \left|\int_\Omega \sum \lambda_k^m h_k \tilde{h_k} \phi_k^2 dx\right| \le \|u\| \cdot \|v\|,$$

$$\left|\int_\Omega \frac{1}{2} v \cdot ((-\Delta)^m v)\right| \le \frac{1}{2}\|v\|^2.$$

By the Mean Value Theorem we get

$$\mathcal{P}(x, \xi + \eta) - \mathcal{P}(x,\xi) = p(x, \xi + \theta\eta)\eta,$$

where $\theta \in [0,1]$. Therefore by (q2) we have

$$\begin{aligned}
\int_\Omega |\mathcal{P}(x,u+v)-\mathcal{P}(x,u)|\,dx &= \int_\Omega |p(x,u+\theta v)|\cdot|v|dx \\
&\le C\int_\Omega (1+|u|+|v|)|v|dx \\
&\le C\int_\Omega (1+|u|+|v|)|v|dx \\
&\le C\|v\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C\|v\|^2_{L^2(\Omega)} \\
&\le C_1\|v\| + C_2\|u\|\cdot\|v\| + C_3\|v\|^2 \\
&= (C_1 + C_2\|u\| + C_3\|v\|)\|v\|.
\end{aligned}$$

for some constants C, C_1, C_2, C_3. With the above results, we see that $f(u)$ is continuous at u. To prove that $f(u)$ is Fréchet differentiable at $u \in H$, it is enough to compute the following :

$$\begin{aligned}
&\left| f(u+v) - f(u) - \int_\Omega \left[(-\Delta)^m u + \lambda_1^m u + p(x,u)\right] v dx \right| \\
&= \left| \begin{array}{l} \frac{1}{2}\int_\Omega \left[v\cdot((-\Delta)^m v) + \lambda_1^m v^2\right] dx \\ \quad + \int_\Omega \left[\mathcal{P}(x,u+v) - \mathcal{P}(x,u) - p(x,u)v\right] dx \end{array} \right| \\
&\le \frac{1}{2}\int_\Omega \left|v\cdot((-\Delta)^m v) + \lambda_1^m v^2\right| dx \\
&+ \int_\Omega |\mathcal{P}(x,u+v) - \mathcal{P}(x,u) - p(x,u)v|\,dx \\
&\le \frac{1}{2}\int_\Omega |v\cdot((-\Delta)^m v)| + \frac{1}{2}\int_\Omega |\lambda_1^m v|^2 dx \\
&+ \int_\Omega |\mathcal{P}(x,u+v) - \mathcal{P}(x,u) - p(x,u)v|\,dx \\
&\le \frac{1}{2}\|v\|^2 + C_4\|v\|^2 + \int_\Omega |\mathcal{P}(x,u+v) - \mathcal{P}(x,u) - p(x,u)v|\,dx
\end{aligned}$$

for $C_4 > 0$. On the other hand, by the Mean Value Theorem, we have

$$\int_\Omega |\mathcal{P}(x,u+v)-\mathcal{P}(x,u)-p(x,u)v|dx = \int_\Omega |p(x,u+\theta v)v - -p(x,u)v|dx.$$

Define

$$\Omega_1 \equiv \{x \in \overline{\Omega} : |u(x)| \geq \beta\}$$

$$\Omega_2 \equiv \{x \in \overline{\Omega} : |v(x)| \geq \gamma\}$$

$$\Omega_3 = \{x \in \overline{\Omega} : |u(x)| < \beta \text{ and } |v(x)| < \gamma\}$$

with β and γ free for the moment. By the Mean Value Theorem, (q2), and the Hölder inequality we have

$$\int_{\Omega_1} |\mathcal{P}(x, u+v) - \mathcal{P}(x,u)|dx = \int_{\Omega_1} p(x, u+\theta v)v dx$$

$$\leq \int_{\Omega_1} C(1 + |u+\theta v|)|v|dx$$

$$\leq \int_{\Omega_1} C(1 + |u| + |v|)|v|dx$$

$$\leq C\int_{\Omega_1} |v|dx + C\int_{\Omega_1} |u||v|dx + C\int_{\Omega_1} |v|^2 dx$$

$$\leq C|\Omega_1|^{\frac{n+2}{2n}} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)} + C|\Omega_1|^{\frac{1}{\sigma}}(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)})\|v\|_{L^{\frac{2n}{n-2}}(\Omega)},$$

where $\frac{1}{\sigma} + \frac{1}{2} + \frac{n-2}{2n} = 1$. Observe that $\frac{1}{2} + \frac{n-2}{2n} < 1$ and hence there exists a $\sigma \geq 1$ satisfying $\frac{1}{\sigma} + \frac{1}{2} + \frac{n-2}{2n} = 1$. Combining Propostion 2.1 (ii) and the above inequality, we have

$$\int_{\Omega_1} |\mathcal{P}(x, u+v) - \mathcal{P}(x,u)|dx \leq C_2\|v\| \left[|\Omega|^{\frac{n+2}{2n}} + |\Omega_1|^{\frac{1}{\sigma}}(\|u\| + \|v\|)\right].$$

Similarly

$$\int_{\Omega_1} |p(x,u)v|dx \leq C_3\|v\| \left[|\Omega_1|^{\frac{n+2}{2n}} + |\Omega_1|^{\frac{1}{\sigma}}\|u\|\right].$$

By Proposition 2.1 (ii) and the Hölder inequality,

$$\|u\| \geq C_4\|u\|_{L^2(\Omega)} \geq C_4\|u\|_{L^2(\Omega_1)} \geq C_4\beta|\Omega_1|^{\frac{1}{2}}.$$

Therefore

$$|\Omega_1|^{\frac{1}{\sigma}} \leq \left(\frac{\|u\|}{C_4\beta}\right)^{\frac{2}{\sigma}} \equiv M_1,$$

$$|\Omega_1|^{\frac{n+2}{2n}} \leq \left(\frac{\|u\|}{C_4\beta}\right)^{\frac{n+2}{n}} \equiv M_2,$$

where M_1, $M_2 \to 0$ as $\beta \to \infty$. Thus we have

$$\int_{\Omega_1} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \leq C_5\|v\|\left[M_2 + M_1(\|u\| + \|v\|)\right].$$

We may assume $\|v\| \leq 1$. Further choose β so large that $C_5[M_2 + M_1(\|u\| + 1)] \leq \frac{\varepsilon}{3}$. Hence

$$\int_{\Omega_1} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \leq \frac{\varepsilon}{3}\|v\|.$$

Similar estimate show that

$$\begin{aligned}
&\int_{\Omega_2} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \\
&\leq C_6 \int_{\Omega_2} (1 + |u| + |v|)\, |v| dx \\
&\leq C_7 \left[\int_{\Omega_2} (1 + |u| + |v|)^2 dx\right]^{\frac{1}{2}} \|v\|_{L^2(\Omega)} \\
&\leq C_8(1 + \|u\| + \|v\|) \left(\int_{\Omega_2} |v|^2 \left(\frac{|v|}{\gamma}\right)^{\frac{2n}{n-2}-2} dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore we have

$$\int_{\Omega_2} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \leq C_9\gamma^{\frac{2-\frac{2n}{n-2}}{2}}(1+\|u\|+\|v\|)\|v\|^{\frac{n}{n-2}}.$$

Next since $P \in C^1(\overline{\Omega}\times R, R)$, given any $\hat{\epsilon}$, $\hat{\beta} > 0$ there exists a $\hat{\gamma} = \hat{\gamma}(\hat{\varepsilon}, \hat{\beta})$ such that

$$|\mathcal{P}(x, \xi + h) - \mathcal{P}(x, \xi) - p(x, \xi)h| \leq \hat{\epsilon}|h|$$

whenever $x \in \overline{\Omega}$, $|\xi| \leq \hat{\beta}$, and $|h| \leq \hat{\gamma}$. In particular if $\hat{\beta} = \beta$ and $\gamma \leq \hat{\gamma}$,

$$\int_{\Omega_3} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \leq \hat{\varepsilon} \int_{\Omega_3} |v| dx \leq C_{10}\hat{\varepsilon}\|v\|.$$

Choose $\hat{\varepsilon}$ so that $3C_{10}\hat{\epsilon} \leq \epsilon$. This determines $\hat{\gamma}$. Choose $\gamma = \hat{\gamma}$.

$$\begin{aligned}
&\int_{\Omega} |\mathcal{P}(x, u+v) - \mathcal{P}(x, v) - p(x, u)v|\, dx \\
&\leq \frac{2\varepsilon}{3}\|v\| + C_9\gamma^{\frac{2-\frac{2n}{n-2}}{2}}(1 + \|u\| + \|v\|)\|v\|^{\frac{n}{n-2}}.
\end{aligned}$$

Finally choose δ so small that

$$C_9\gamma^{1-\frac{n}{n-2}}(2+\|u\|)\delta^{\frac{n}{n-2}-1}\leq\frac{\varepsilon}{3}$$

Thus we prove that $f(u)$ is Fréchet differentiable in H. □

Proposition 2.3. *Let*

$$g(u)=\int_\Omega\left[\frac{1}{2}\lambda_1^m u^2(x)+\mathcal{P}(x,u(x))\right]dx.$$

Then g' is continuous with respect to weak convergence and

$$(g'(u),v)_H=\int_\Omega\left[\lambda_1^m uv+p(x,u)v\right]dx \tag{2.2}$$

for all $v\in H$. This implies that g itself is weakly continuous.

To prove Proposition 2.3, we need the following lemma.

Lemma 2.1. *If p satisfies (q1)-(q2), the map $u(x)\to p(x,u(x))$ belongs to* $C(L^2(\Omega),L^2(\Omega))$.

Proof. If $u\in L^2(\Omega)$, then we have

$$\int_\Omega|p(x,u(x))|^2dx\leq C\int_\Omega(1+|u|)^2dx\leq C\int_\Omega(1+|u|^2)dx,$$

which shows that $p: L^2(\Omega)\to L^2(\Omega)$. To prove the continuity of this map, observe that it is continuous at v if and only if $f(x,z)=p(x,z(x)+v(x))-p(x,v(x))$ is continuous at $z=0$. We show that $f(x,u)$ is continuous at $u=0$. Let $\varepsilon>0$ be given. We claim there is a $\delta>0$ such that $\|u\|_{L^2(\Omega)}\leq\delta$ implies $\|f(\cdot,u)\|_{L^2(\Omega)}\leq\varepsilon$. By (q1) and $f(x,0)=0$, given any $\hat{\varepsilon}>0$, there is a $\hat{\delta}>0$ such that $|f(x,\xi)|\leq\hat{\varepsilon}$ if $x\in\overline{\Omega}$ and $|\xi|\leq\hat{\delta}$. Let $u\in L^2(\Omega)$ with $\|u\|_{L^2(\Omega)}\leq\delta$, δ being free from now, and set

$$\Omega_1\equiv\{x\in\overline{\Omega}:|u(x)|\leq\hat{\delta}\}.$$

Therefore

$$\int_{\Omega_1}|f(x,u(x))|^2dx\leq\hat{\varepsilon}^2|\Omega_1|\leq\hat{\varepsilon}^2|\Omega|,$$

where $|\Omega_1|$ denotes the measure of Ω_1. Choose $\hat{\varepsilon}^2$ so that $\hat{\varepsilon}^2|\Omega|<\left(\frac{\varepsilon}{2}\right)^2$. This determines $\hat{\varepsilon}$. Let $\Omega_2=\overline{\Omega}\setminus\Omega_1$. Then

$$\int_{\Omega_2}|f(x,u(x)|^2dx\leq C_1\left(|\Omega_2|+\delta^2\right).$$

Moreover

$$\delta^2 \geq \int_{\Omega_2} |u|^2 dx \geq \hat{\delta}^2 |\Omega_2|$$

or $|\Omega_2| \leq \left(\delta\hat{\delta}^{-1}\right)^2$. Thus we have

$$\int_{\Omega_2} |f(x,u(x)|^2 dx \leq C_1(1+\hat{\delta}^{-2})\delta^2.$$

Choose δ so that $C_1(1+\hat{\delta}^{-2})\delta^2 \leq \left(\frac{\varepsilon}{2}\right)^2$. Thus

$$\int_{\Omega} |f(x,u(x)|^2 dx = \int_{\Omega_1} |f(x,u(x)|^2 dx + \int_{\Omega_2} |f(x,u(x)|^2 dx \leq \left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2,$$

which implies

$$\|f(\cdot,u)\|_{L^2(\Omega)} \leq \varepsilon \quad \text{if} \quad \|u\|_{L^2(\Omega)} \leq \delta$$

and the proof is complete. □

Proof of Proposition 2.3

Let $u_m \to u$ in H. Then $u_m \to u$ in $L^2(\Omega)$. By Hölder inequality we have

$$\begin{aligned}
&\|g'(u_m) - g'(u)\| \\
&= \sup_{\|v\|\leq 1} \left| \int_{\Omega} [\lambda_1^m(u_m - u) + (p(x,u_m(x)) - p(x,u(x)))]\, v(x)dx \right| \\
&\leq \sup_{\|v\|\leq 1} \left[\int_{\Omega} |\lambda_1^m||u_m - u||v(x)|dx + \int_{\Omega} |p(x,u_m(x)) - p(x,u(x))||v(x)|dx\right] \\
&\leq \lambda_1^m \|u_m - u\|_{L^2(\Omega)} + \|p(x,u_m) - p(x,u)\|_{L^2(\Omega)}.
\end{aligned}$$

By Lemmma 2.1, the right-hand side of the above inequality tends to 0 as $m \to \infty$ and hence g' is continuous. Finally to prove that g is weakly continuous, let u_m converge weakly to u in H. Then by Proposition 2.1 (ii), u_m converges to u in $L^2(\Omega)$. Consequently Lemma 2.1 implies $g(u_m) \to g(u)$.

For every $t \geq 0$, we define

$$S_t = \{u \in H : \|u\|^2 = 2t\}. \tag{2.4}$$

THEOREM 2.1. *Let*

$$\gamma(t) = \sup_{u \in S_t} g(u). \tag{2.5}$$

Then $\gamma(t)$ *is a continuous, nondecreasing function in* $[0,\infty)$. *For every* $t>0$, $\gamma(t)$ *has left and right hand derivatives* $\gamma'_{\pm}$ *satisfying*

$$0 < {\gamma_-}'(t) \ \leq \ {\gamma_+}'(t), \quad t>0, \tag{2.6}$$

If $\gamma'_+(t)\neq 0$, *then there is* $u\in\sum_t=\{u\in S_t : g(u)=\gamma(t)\}$ *such that*

$$g'(u) = {\gamma_+}'(t)u. \tag{2.7}$$

If $\gamma'_-(t)\neq 0$, *then there is a* $u\in\sum_t=\{u\in S_t : g(u)=\gamma(t)\}$ *such that*

$$g'(u) = {\gamma_-}'(t)u. \tag{2.8}$$

The proof is found in [7,8].

3. Proof of theorem 1.1

From the confinuity of γ and (2.6) we have

LEMMA 3.1. *If* $0<a<c<b$ *and* $\gamma(a)\leq a$, $\gamma(b)\leq b$, $\gamma(c)\geq c$, *then there exists a point* $d\in[a,b]$ *such that* $\gamma'(d)$ *exists and equals to 1.*

COROLLARY 1. *If there are sequences* θ_j, τ_k *such that* $\theta_j\to\infty$, $\tau_k\to\infty$, $\gamma(\theta_j)\leq\theta_j$ *and* $\gamma(\tau_k)\geq\tau_k$, *then there are infinitely many solutions of*

$$u = g'(u) \tag{3.1}$$

Proof of Theorem 1.1

In order to apply Corollary 3.1, we note that by Proposition 2.2, $(g'(u),v)_H$ is weakly continuous. We shall show that under hypothesis $(q2)-(q3)$, there are sequences $\{\sigma_j\}$, $\{\tau_k\}$ satisfying the hypotheses of Corollary 3.1. This will produce an infinite number of solutions of

$$(u,v)_H = (g'(u),v)_H, \tag{3.2}$$

which, given smooth Ω, translates into the solutions of (1.1). Suppose that the first inequality of (1.5) holds. Let

$$\phi_t = \sqrt{2t}\,\phi_1. \tag{3.3}$$

Then |
$phi_t\|^2 = 2t = \lambda_1^m\|\phi_t\|^2_{L^2(\Omega)}$. Thus

$$\gamma(t) \geq g(\phi_t) = t + \int_\Omega \mathcal{P}(x,\phi_t(x))dx. \tag{3.4}$$

Thus by the first inequality in (1.5),

$$\gamma(\tau_k) \geq \tau_k, \quad k \geq 1. \tag{3.5}$$

If the second inequality in (1.5) holds, the above argument can be repeated with

$$\phi_t = -\sqrt{2t}\phi_1.$$

Let $\epsilon > 0$ be given. For arbitary $t > 0$ there is a $u_t \in S_t$ such that $\gamma(u_t) \leq g(u_t) + \epsilon$. We can represent this function as

$$u_t = \sqrt{2t}\,(\pm \cos\theta_t \cdot \phi_1 + \sin\theta_t \cdot w_t) \tag{3.6}$$

for some θ_t and w_t with $w_t \perp \phi_1$, $\|w_t\| = 1$. Representation (3.6) holds with $\cos\theta_t \geq 0$ and an appropriate choice of the sign $\pm$. Thus we have

$$\gamma(t) \leq t(\cos^2\theta_t + \frac{\lambda_1^m}{\lambda}\sin^2\theta_t) + \int_\Omega \mathcal{P}(x, u_t(x))dx + \epsilon \tag{3.7}$$

with $\lambda > \lambda_1^m$. Let

$$\phi_t = \pm\sqrt{2t}\,\phi_1$$

with the same choice of sign as in (3.6). From (3.7), by (3.4) and with v_t being some convex combination of u_t and ϕ_t, we have

$$\begin{aligned}
&t(1 - \frac{\lambda_1^m}{\lambda})\sin^2\theta_t - \epsilon \leq \int_\Omega [\mathcal{P}(x, u_t) - \mathcal{P}(x, \phi_t)]\,dx \\
&= \int_\Omega p(x, v_t)(u_t - \phi_t)dx \leq C\sqrt{t}\int_\Omega [(1 - \cos\theta_t)|\phi_1| + |\sin\theta_t||w_t|]\,dx \\
&\leq C\sqrt{t}|\sin\theta_t|.
\end{aligned}$$

Hence

$$\sin^2\theta_t \leq \frac{C}{t}. \tag{3.8}$$

In particular, we have $\theta_t \to 0$ as $t \to \infty$. We see from (3.7), (3.8) that

$$\int_\Omega [\mathcal{P}(x, u_t) - \mathcal{P}(x, \phi_t)] \leq C. \tag{3.9}$$

Consequently, by (3.7),

$$\begin{aligned}
\gamma(t) - t &\leq \int_\Omega \mathcal{P}(x, u_t)dx - t(1 - \frac{\lambda_1^m}{\lambda})\sin^2\theta_t + \varepsilon \\
&\leq \int_\Omega [\mathcal{P}(x, u_t) - \mathcal{P}(x, \phi_t)]\,dx + \int_\Omega \mathcal{P}(x, \phi_t)dx + \varepsilon.
\end{aligned}$$

Therefore, by (3.9) and since ε is arbitrary,

$$\gamma(t) - t \leq C + \int_\Omega \mathcal{P}(x, \phi_t(x))dx.$$

We see that (1.4), even though the sign of ϕ_t is not determined, implies $\gamma(\sigma_k) - \sigma_k \to -\infty$ as $k \to \infty$. The theorem follows from Corollary 3.1.

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