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HODGE NUMBERS OF CALABI–YAU MANIFOLDS BY SMOOTHING OF NORMAL CROSSING VARIETIES

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Abstract. We give a formula for the Hodge numbers of Calabi–Yau 3-folds, constructed by smoothing and calculate the Hodge numbers of Calabi–Yau 3-folds of Y. Kawamata and Y. Namikawa.

1. Introduction

A *Calabi–Yau manifold* is a compact Kähler manifold with trivial canonical class such that the intermediate cohomology groups of its structure sheaf are trivial ($h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim(X)$). Simple examples are a smooth quintic hypersurface in $\mathbb{P}^4$ and a complete intersection of two cubics in $\mathbb{P}^5$. Y. Kawamata and Y. Namikawa suggested a different way of construction, that is, by smoothing normal crossing varieties ([3], see Theorem 3.1 in this note).

A *normal crossing variety* is a reduced complex analytic space which is locally isomorphic to a normal crossing divisor on a smooth variety and it is called simple if its components are all smooth. From now on, a normal crossing variety will mean a simple one unless stated otherwise. By smoothing normal crossing varieties, Y. Kawamata and Y. Namikawa constructed 26 classes of examples of Calabi–Yau 3-folds up to the topological Euler numbers([3]). However they did not provide the Hodge numbers.

The purpose of this short note is to give a formula for the Hodge numbers (Corollary 3.3) and calculate the Hodge numbers of Calabi–Yau 3-folds of Y. Kawamata and Y. Namikawa (Table 3, 185 pairs). It turns

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out that many of examples have the different Hodge numbers although their Euler number are identical – so they were undistinguishable by the Euler number calculation of Kawamata and Namikawa.

2. Notations and preliminary results

Let $\pi : X \to \overline{\Delta}$ be a proper map from a Kähler $(n+1)$-fold X with boundary onto a closed disk $\overline{\Delta} = \{t \in \mathbb{C} \big| \|t\| \leq 1\}$ such that the fibers $X_t = \pi^{-1}(t)$ are connected n-folds for every $t \neq 0$ (generic) and the central fiber

$$X_0 = \pi^{-1}(0) = \bigcup_{\alpha=1}^{r} Y_\alpha$$

is a normal crossing of n-folds. We denote the generic fiber by X_t. The condition, $t \neq 0$, is assumed in this notation. We call such a map π (or simply the total space X) a semi-stable degeneration of X_t and X_0 the central fiber.

We will need the following result (Theorem 3.2.1 in [6]), obtained by analyzing the Clemens-Schmid exact sequence ([1]).

Theorem 2.1. *Suppose that* $h^{2,0}(X_t) = 0$. *Then*

$$h^{1,1}(X_t) = h^2(X_0) - r + 1,$$

where r *is the number of components of* X_0.

Note that we have $h^{2,0}(X_t) = 0$ for a Calabi–Yau manifold X_t of dimension higher than two. There is a well-known formula for Euler numbers. So we can determine all the Hodge numbers of X_t by the above theorem when $\dim X_t = 3$.

3. Hodge numbers of Calabi–Yau 3-folds by Y. Kawamata and Y. Namikawa

Now we state the theorem of Y. Kawamata and Y. Namikawa ([3]), which is a generalization of a result of R. Friedman ([2]).

Theorem 3.1 (Y. Kawamata, Y. Namikawa). *Let* $X_0 = \bigcup_i Y_i$ *be (not necessarily simple) normal crossing of dimension* $n \geq 3$ *such that*

1. *Its dualizing sheaf is trivial:* $\omega_{X_0} = \mathcal{O}_{X_0}$.
2. $H^{n-2}(Y_i, \mathcal{O}_{Y_i}) = 0$ *for any* i *and* $H^{n-1}(X_0, \mathcal{O}_{X_0}) = 0$.
3. *It is Kähler.*
4. *It has a logarithmic structure.*

Then X_0 is smoothable to an n-fold with the smooth total space (semistable degeneration).

The 'logarithmic structure' is also called as d-semistability. It will be explained shortly for a simpler case that is necessary in this note. From now on, we restrict ourselves to the three dimension.

The case in which the central fiber has only two components occurs very often. That is, $X_0 = Y_1 \cup Y_2$. Let $D = Y_1 \cap Y_2$. Then the conditions in the above theorem correspond to the followings:

1. $D \in |-K_{Y_i}|$ for $i = 1, 2$.
2. $H^1(\mathcal{O}_{Y_i}) = H^1(\mathcal{O}_D) = 0$. Note that this condition, together with (1), implies that D is a $K3$ surface.
3. There are ample divisors H_1, H_2 on Y_1, Y_2 respectively such that $H_1|_D$ is linearly equivalent to $H_2|_D$.
4. $N_{D/Y_1} \otimes N_{D/Y_2} = \mathcal{O}_D$, where N_{D/Y_i} is the normal bundle to D in Y_i. This condition is called d-semistability.

By the above theorem, X_0 is smoothable to a 3-fold, X_t with $K_{X_t} = 0$ and $h^1(\mathcal{O}_{X_t}) = h^2(\mathcal{O}_{X_t}) = 0$. Accordingly X_t is a Calabi–Yau 3-fold.

Let us calculate the Hodge numbers of X_t:

Theorem 3.2. *Let X_t be the smoothing of X_0 as the above. Then*

1. $h^{1,1}(X_t) = h^2(Y_1) + h^2(Y_2) - k - 1$, *where*
$$k = \mathrm{rk}(\mathrm{im}(H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \to H^2(D, \mathbb{Z}))).$$
2. $h^{1,2}(X_t) = 21 + h^{1,2}(Y_1) + h^{1,2}(Y_2) - k$.

Proof.

1. From Theorem 2.1,
$$\begin{aligned} h^{1,1}(X_t) &= h^2(X_t) \\ &= h^2(X_0) - 2 + 1 \\ &= (h^2(Y_1) + h^2(Y_2) - k) - 1 \\ &= h^2(Y_1) + h^2(Y_2) - k - 1. \end{aligned}$$
2. $e(X_t) = e(Y_1) + e(Y_2) - 2e(Y_1 \cap Y_2) = e(Y_1) + e(Y_2) - 48$. Then the result easily comes from
$$e(X_t) = 2(h^{1,1}(X_t) - h^{1,2}(X_t)),$$
$$e(Y_i) = 2(h^{1,1}(Y_i) - h^{1,2}(Y_i) + 1).$$

□

We go over the examples of Calabi–Yau 3-folds which are introduced in [3], p. 408. They are constructed from two copies of $\mathbb{P}^3$.

Let D be a smooth quartic $K3$ surface in $\mathbb{P}^3$. Let $C = C_1 + \cdots + C_s$ and $C' = C'_1 + \cdots + C'_t$ be reduced divisors of D which are composed of smooth curves. We make Y_1 (resp., Y_2) by blowing up successively along with the proper transforms of $C_1, \cdots, C_s$ (resp. $C'_1, \cdots, C'_t$) in this order. Let D_i be the proper transform in Y_i of D. Note that D_i is isomorphic to D. We make a normal crossing variety $X_0 = Y_1 \cup Y_2$ by identifying D_1 in Y_1 with D_2 in Y_2. According to Theorem 4.2 in [3], there is a semistable degeneration of a Calabi–Yau 3-fold which has X_0 as its central fiber if $C + C' \in |\mathcal{O}_D(8)|$ and X_0 is Kähler. In [3], they gave the topological Euler numbers.

Let us specify D as

$$D = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^3,$$

which is the Fermat quartic $K3$ surface. We can define various divisors on it.

1. For a primitive 8-th root of unity ξ, define the divisor $\Gamma_{i,j,k}$ of D as
$$\Gamma_{i,j,k} = \{(x_0, \ldots, x_3) \in D | x_i = \xi^k x_j\}$$
Let $L = \{\Gamma_{0,2,1}, \Gamma_{0,2,2}, \Gamma_{0,2,3}, \Gamma_{1,2,4}\}$. It can be easily shown that each of these $\Gamma_{i,j,k}$'s in L consists of 4 lines which meet at a single point.
2. For the sixteen lines on D, coming from those $\Gamma_{i,j,k}$'s in L, we can assign a divisor $F \in |\mathcal{O}_D(1)|$ as
$$F = D \cap H,$$
where H is a generic hyperplane which contains the line. Then F is composed of the line and a smooth cubic curve. Let N be the set of such divisors. This cubic curves are newly introduced in this note to construct more examples of Calabi–Yau 3-folds.

Now we take

$$C = E_1 + \cdots + E_s + \Gamma_1 + \cdots + \Gamma_a + F_1 + \cdots + F_u$$

and

$$C' = E'_1 + \cdots + E'_t + \Gamma'_1 + \cdots + \Gamma'_b + F'_1 + \cdots + F'_v$$

where E_i and E'_j for $1 \le i \le s$, $1 \le j \le t$ are members of $|\mathcal{O}_D(e_i)|$ and $|\mathcal{O}_D(e'_i)|$, Γ_i and Γ'_j are members of L, F_i and F'_j are members of N

respectively such that

$$a + b + \sum_i e_i + \sum_j e'_j + u + v = 8.$$

Then X_0 is smoothable and the Hodge numbers are given by the following corollary, which is a simple application of Theorem 3.2.

COROLLARY 3.3. *Let X_t be the smoothing of X_0, then the Hodge numbers of X_t are:*

$$h^{1,1} = \beta + 4\gamma + 2\delta,$$
$$h^{1,2} = 20 + 2\alpha + \beta + 4\gamma + 2\delta,$$

where $\alpha = \sum_i e_i^2 + \sum_j {e'}_j^2$, $\beta = s + t$, $\gamma = a + b$ *and* $\delta = u + v$.

The following (Table 3) is the exhaustive list of Hodge numbers of Calabi–Yau 3-folds constructible in this way. For example, if one take $a = b = 0$, $s = 1, t = 0, e_1 = 8$, and $u = v = 0$,

$$h^{1,1} = 1 + 0 + 0 = 1,$$
$$h^{1,2} = 20 + 2 \cdot 64 + 1 + 0 + 0 = 149,$$
$$e = 2(h^{1,1} - h^{1,2}) = -296.$$

Note that there are many examples with the same Euler number that have different Hodge numbers in the table.

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$h^{1,1}$	$h^{2,1}$	e												
1	149	−296	2	122	−240	2	102	−200	2	90	−176	2	86	−168
3	120	−234	3	99	−192	3	83	−160	3	75	−144	3	71	−136
3	67	−128	4	80	−152	4	73	−138	4	68	−128	4	64	−120
4	60	−112	4	56	−104	5	119	−228	5	95	−180	5	66	−122
5	65	−120	5	58	−106	5	57	−104	5	53	−96	6	76	−140
6	72	−132	6	64	−116	6	63	−114	6	60	−108	6	55	−98
6	54	−96	6	51	−90	6	50	−88	6	36	−60	7	94	−174
7	74	−134	7	65	−116	7	61	−108	7	61	−108	7	57	−100
7	53	−92	7	52	−90	7	49	−84	7	48	−82	7	47	−80
8	75	−134	8	63	−110	8	62	−108	8	59	−102	8	59	−102
8	54	−92	8	51	−86	8	50	−84	8	50	−84	8	46	−76
8	45	−74	8	44	−72	9	93	−168	9	73	−128	9	60	−102
9	57	−96	9	52	−86	9	51	−84	9	48	−78	9	48	−78
9	47	−76	9	44	−70	9	43	−68	9	42	−66	10	74	−128
10	62	−104	10	58	−96	10	50	−80	10	49	−78	10	46	−72
10	45	−70	10	44	−68	10	42	−64	10	41	−62	10	40	−60
11	72	−122	11	59	−96	11	59	−96	11	56	−90	11	51	−80
11	47	−72	11	47	−72	11	44	−66	11	43	−64	11	42	−62
11	39	−56	11	38	−54	12	57	−90	12	49	−74	12	48	−72
12	45	−66	12	44	−64	12	41	−58	12	40	−56	12	39	−54
12	37	−50	12	36	−48	13	71	−116	13	55	−84	13	46	−66
13	43	−60	13	42	−58	13	41	−56	13	38	−50	13	37	−48
13	35	−44	13	34	−42	14	56	−84	14	48	−68	14	44	−60
14	40	−52	14	39	−50	14	38	−48	14	36	−44	14	35	−42
14	32	−36	15	54	−78	15	45	−60	15	42	−54	15	41	−52
15	37	−44	15	36	−42	15	34	−38	15	33	−36	15	30	−30
16	43	−54	16	39	−46	16	38	−44	16	35	−38	16	34	−36
16	31	−30	16	28	−24	17	53	−72	17	41	−48	17	36	−38
17	35	−36	17	33	−32	17	32	−30	17	29	−24	18	42	−48
18	38	−40	18	34	−32	18	33	−30	18	30	−24	18	27	−18
19	40	−42	19	35	−32	19	32	−26	19	31	−24	19	28	−18
20	33	−26	20	32	−24	20	29	−18	20	26	−12	21	39	−36
21	31	−20	21	30	−18	21	27	−12	22	32	−20	22	28	−12
22	25	−6	23	30	−14	23	29	−12	23	26	−6	24	27	−6
24	24	0	25	29	−8	25	25	0	26	23	6	27	24	6
28	28	0	28	22	12	29	23	12	30	21	18	32	20	24

TABLE 1. Hodge numbers of Calabi–Yau 3-folds

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