JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 4, December 2008

SOME CRITERIA OF FIBER PRODUCTS

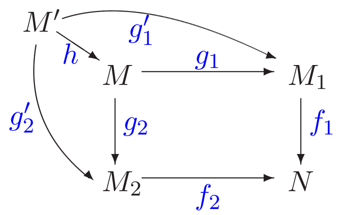
Sang-Cho Chung*

Abstract. The purpose of this paper is to show that we give some criteria of fiber products.

1. Introduction

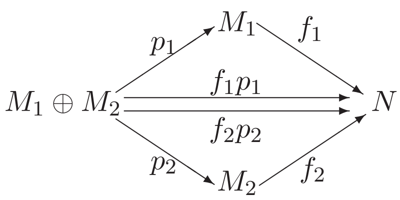
In this paper all rings are commutative with identity. The purpose of this paper is to show that we give some criteria of fiber product. Fiber product is one of the basic tools constructing Projective modules. Mandal[1] gave a criterion of fiber products (cf, Lemma 2.1 (2)). we have an extended criteror of fiber products in Theorem 2.2. We introduce the definition of the fiber product.

Definition 1.1. ([1], Definition 2.1.1.) Let R be a commutative ring with identity and let M, M_1, M_2 and N be R-modules. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be homomorphisms of R-modules. The *fiber product* of M_1 and M_2 over N is a triple (M, g_1, g_2), where $g_1 : M \to M_1$ and $g_2 : M \to M_2$ are R-linear maps such that $f_1 g_1 = f_2 g_2$ and the triple is universal in the sense that given any other triple (M', g_1', g_2') of this kind with $f_1 g_1' = f_2 g_2'$ there is a unique homomorphism $h : M' \to M$ such that $g_1 h = g_1'$ and $g_2 h = g_2'$.



Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be maps and let $p_1 : M_1 \oplus M_2 \to M_1$ and $p_2 : M_1 \oplus M_2 \to M_2$ be natural projections.

Received October 16, 2008; Accepted November 20, 2008.
2000 Mathematics Subject Classification: Primary 13C99.
Key words and phrases: Fiber Product.



Then we have that

$$Im(f_1p_1 - f_2p_2) = \{f_1(m_1) - f_2(m_2) : (m_1, m_2) \in M_1 \oplus M_2\}$$

where $(f_1p_1 - f_2p_2)(m_1, m_2) = (f_1p_1)(m_1, m_2) - (f_2p_2)(m_1, m_2)$ for all $(m_1, m_2) \in M_1 \oplus M_2$ and

$$Ker(f_1p_1 - f_2p_2) = \{(m_1, m_2) \in M_1 \oplus M_2 : f_1(m_1) = f_2(m_2)\}.$$

2. Main Theorem

In the following Lemma 2.1(2), Mandal shows a criteria of fiber products.

LEMMA 2.1. ([1] Proposition 2.1.1 and Proposition 2.1.2) *Let R be a commutative ring with identity and M_1, M_2, N be R-modules. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be R-homomorphisms. Let*

$$M = \{(m_1, m_2) \in M_1 \oplus M_2 : f_1(m_1) = f_2(m_2)\}.$$

Then we have the following.

(1) $(M, p_1|_M, p_2|_M)$ *is a fiber product of M_1 and M_2 over N.*

(2) *The diagram*

$$\begin{array}{ccc} M & \xrightarrow[g_1]{} & M_1 \\ \Big\downarrow{\scriptstyle g_2} & & {\scriptstyle f_1}\Big\downarrow \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a fiber product diagram of M_1 and M_2 over N if and only if for each pair of elements $m_1 \in M_1$ and $m_2 \in M_2$ with $f_1(m_1) = f_2(m_2)$ there is a unique element $m \in M$ with $g_1(m) = m_1$ and $g_2(m) = m_2$.

(3) *If (M', g_1, g_2) is a fiber product of M_1 and M_2 over N, then we have*

  (i) *there is a unique isomorphism $h : M' \to M$ such that $(p_1|_M)h = g_1$ and $(p_2|_M)h = g_2$.*

  (ii) $Ker(g_1) \cap Ker(g_2) = \{0\}$.

Proof. (1), (2) and (3)(i) follow from ([1], Prop. 2.1.1 and Prop. 2.1.2).

(3)(ii) Let $m \in Ker(g_1) \cap Ker(g_2)$. Then we have

$$0 = g_1(m) = p_1|_M(h(m)) \quad \text{and} \quad 0 = g_2(m) = p_2|_M(h(m)).$$

Therefore

$$h(m) \in Ker(p_1|_M) \cap Ker(p_2|_M).$$

Since $Ker(p_1|_M) \cap Ker(p_2|_M) = \{(0,0)\}$, we have

$$h(m) = (0,0)$$

and then $m = 0$. □

In the next Theorem 2.2, we have more extended criteria of fiber products than Mandal's.

THEOREM 2.2. *Let M, M_1, M_2, N be R-modules. Let $g_i : M \to M_i$ for $i = 1,2$ and $f_i : M_i \to N$ for $i = 1,2$ be R-homomorphisms. Let*

$$p_i : M_1 \oplus M_2 \to M_i \text{ for } i = 1,2$$

be natural projections. Consider the map

$$(g_1, g_2) : M \to M_1 \oplus M_2$$

where $(g_1, g_2)(m) = (g_1(m), g_2(m))$ for all $m \in M$ and

$$(f_1p_1 - f_2p_2) : M_1 \oplus M_2 \to N$$

where $(f_1p_1 - f_2p_2)(m_1, m_2) = (f_1p_1)(m_1, m_2) - (f_2p_2)(m_1, m_2)$ for all $(m_1, m_2) \in M_1 \oplus M_2$. Then the following are equivalent.

(1) *The following triple (M, g_1, g_2)*

$$\begin{array}{ccc} M & \xrightarrow[g_1]{} & M_1 \\ \Big\downarrow{\scriptstyle g_2} & & \Big\downarrow{\scriptstyle f_1} \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a fiber product diagram of M_1 and M_2 over N.

(2) *The following sequence*

$$0 \longrightarrow M \xrightarrow{(g_1,g_2)} M_1 \oplus M_2 \xrightarrow{f_1p_1 - f_2p_2} Im(f_1p_1 - f_2p_2) \longrightarrow 0$$

is exact.

(3) *For the diagram in (1), we have*
   (i) *(g_1, g_2) is injective.*
   (ii) *$\{(m_1, m_2) \in M_1 \oplus M_2 : f_1(m_1) = f_2(m_2)\} \subset Im(g_1, g_2)$.*

Proof. (1) $\Rightarrow$ (2) Consider the sequence

$$0 \longrightarrow M \xrightarrow{(g_1,g_2)} M_1 \oplus M_2 \xrightarrow{f_1p_1-f_2p_2} Im(f_1p_1 - f_2p_2) \longrightarrow 0. \cdots (*)$$

Clearly $f_1p_1 - f_2p_2$ is surjective.

We show that (g_1, g_2) is injective. Let $m \in Ker(g_1, g_2)$. Then $(0, 0) = (g_1, g_2)(m) = (g_1(m), g_2(m))$. That is

$$g_1(m) = 0 \ \text{ and } \ g_2(m) = 0.$$

Hence $m \in Ker(g_1) \cap Ker(g_2)$. By Lemma 2.1(3), we have

$$m = 0.$$

$$\therefore (g_1, g_2) \text{ is injective.}$$

For all $m \in M$, we have

$$\begin{aligned}(f_1p_1 - f_2p_2)((g_1, g_2)(m)) &= (f_1p_1 - f_2p_2)(g_1(m), g_2(m)) \\ &= f_1(g_1(m)) - f_2(g_2(m)) = 0,\end{aligned}$$

since $f_1g_1 = f_2g_2$. Hence we have $Im(g_1, g_2) \subset Ker(f_1p_1 - f_2p_2)$.

Let $(m_1, m_2) \in Ker(f_1p_1 - f_2p_2)$. Then we have

$$0 = (f_1p_1 - f_2p_2)(m_1, m_2) = f_1(m_1) - f_2(m_2).$$

Therefore

$$f_1(m_1) = f_2(m_2).$$

By Lemma 2.1, there is an element $m \in M$ such that

$$g_1(m) = m_1 \text{ and } g_2(m) = m_2.$$

Hence

$$(m_1, m_2) = (g_1(m), g_2(m)) = (g_1, g_2)(m).$$

We have $Ker(f_1p_1 - f_2p_2) \subset Im(g_1, g_2)$.

$$\therefore Ker(f_1p_1 - f_2p_2) = Im(g_1, g_2).$$

Thus the sequence $(*)$ is exact.

(2) $\Rightarrow$ (1) Suppose that the sequence

$$0 \longrightarrow M \xrightarrow{(g_1,g_2)} M_1 \oplus M_2 \xrightarrow{f_1p_1-f_2p_2} Im(f_1p_1 - f_2p_2) \longrightarrow 0$$

is exact.

For all $m \in M$, we have

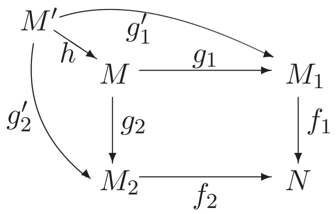
$$\begin{aligned}0 = (f_1p_1 - f_2p_2)(g_1, g_2)(m) &= (f_1p_1 - f_2p_2)(g_1(m), g_2(m)) \\ &= f_1(g_1(m)) - f_2(g_2(m)).\end{aligned}$$

Then we have

$$f_1(g_1(m)) = f_2(g_2(m)).$$

$$\therefore f_1g_1 = f_2g_2.$$

Next we show that the triple (M, g_1, g_2) is universal. Let (M', g_1', g_2') be any triple with $f_1g_1' = f_2g_2'$.



For each $m' \in M'$, we have

$$f_1(g_1'(m')) = f_2(g_2'(m')).$$

Hence we have $(g_1'(m'), g_2'(m')) \in Ker(f_1p_1 - f_2p_2) = Im(g_1, g_2)$. There is an element $m \in M$ such that

$$(g_1, g_2)(m) = (g_1'(m'), g_2'(m')).$$

That is

$$g_1(m) = g_1'(m') \text{ and } g_2(m) = g_2'(m').$$

Define a map

$$h : M' \to M$$

by $h(m') = m$, where $m' \in M'$ and $m \in M$ such that

$$(g_1'(m'), g_2'(m')) = (g_1(m), g_2(m)).$$

Since (g_1, g_2) is injective, h is a well-defined R-homomorphism. Furthermore, for all $m' \in M'$.

$$g_i(h(m')) = g_i(m) = g_i'(m') \text{ for } i = 1, 2.$$

Thus we have

$$g_1h = g_1', \quad g_2h = g_2'.$$

Assume that there is an R-homomorphism $h' : M' \to M$ such that

$$g_1h' = g_1', \quad g_2h' = g_2'.$$

Then we have, for all $m' \in M'$

$$g_i(h(m')) = g_i'(m') = g_i(h'(m')) \text{ for } i = 1, 2.$$

Therefore

$$h(m') - h'(m') \in Ker(g_1) \cap Ker(g_2).$$

On the other hand, since (g_1, g_2) is injective, we have $Ker(g_1) \cap Ker(g_2) = \{0\}$ and then

$$h(m') - h'(m') = 0.$$

Thus we have

$$h = h'.$$

Therefore (M, g_1, g_2) is a fiber product of M_1 and M_2 over N.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (2) Since $Ker(f_1 p_1 - f_2 p_2) = \{(m_1, m_2) \in M_1 \oplus M_2 : f_1(m_1) = f_2(m_2)\}$, it suffices to show that

$$Im(g_1, g_2) \subset Ker(f_1 p_1 - f_2 p_2).$$

But it follows that the diagram is commutative. □

COROLLARY 2.3. ([1], Example 2.1.3) *With the same notation as in Theorem 2.2, let S be a multiplicative closed subset of R. If*

$$\begin{array}{ccc} M & \longrightarrow & M_1 \\ \downarrow & & \downarrow f_1 \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

is a fiber product diagram of R-modules, then

$$\begin{array}{ccc} M_S & \longrightarrow & M_{1S} \\ \downarrow & & \downarrow f_{1S} \\ M_{2S} & \xrightarrow{f_{2S}} & N_S \end{array}$$

is a fiber product diagram of R_S-modules.

Proof. By Theorem 2.2, we have the exact sequence

$$0 \longrightarrow M \longrightarrow M_1 \oplus M_2 \longrightarrow Im(f_1 p_1 - f_2 p_2) \longrightarrow 0.$$

Then since $(M_1 \oplus M_2)_S \cong M_{1S} \oplus M_{2S}$, we have the following exact sequence

$$0 \longrightarrow M_S \longrightarrow M_{1S} \oplus M_{2S} \longrightarrow Im(f_{1S} p_{1S} - f_{2S} p_{2S}) \longrightarrow 0.$$

Hence the conclusion follows from Theorem 2.2. □

DEFINITION 2.4. Let R, S, T, U be commutative rings with identity. Let $g_1 : R \to S$, $g_2 : R \to T$, $f_1 : S \to U$ and $f_2 : T \to U$ be ring homomorphisms. The *fiber product* of commutative rings

$$\begin{array}{ccc} R & \xrightarrow[g_1]{} & S \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ T & \xrightarrow{f_2} & U \end{array}$$

is defined as in Definition 1.1.

Next we define the fiber product for commutative rings.

THEOREM 2.5. *Let R, S, T, U be commutative rings with identity. Let $g_1 : R \to S$, $g_2 : R \to T$, $f_1 : S \to U$ and $f_2 : T \to U$ be ring maps. Then the following*

$$\begin{array}{ccc} R & \xrightarrow[g_1]{} & S \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ T & \xrightarrow{f_2} & U \end{array}$$

is a fiber product diagram of rings if and only if, for the ring homomorphism

$$(g_1, g_2) : R \to S \oplus T$$

defined by $(g_1, g_2)(r) = (g_1(r), g_2(r))$ where $r \in R$ and the ring homomorphism

$$f_1p_1 - f_2p_2 : S \oplus T \to Im(f_1p_1 - f_2p_2)$$

defined by $(f_1p_1 - f_2p_2)(s, t) = f_1(s) - f_2(t)$ where $(s, t) \in S \oplus T$, we have

(i) *(g_1, g_2) is injective.*
(ii) *$Im(g_1, g_2) = Ker(f_1p_1 - f_2p_2)$ and therefore $R \cong Ker(f_1p_1 - f_2p_2)$.*

PROOF: The proof is similar to Theorem 2.2. □

COROLLARY 2.6. *We have the following.*

(1) *(See [1] Example 2.1.1) Let I, J be ideals of R. Then*

$$\begin{array}{ccc} R/I \cap J & \xrightarrow{g_1} & R/I \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ R/J & \xrightarrow{f_2} & R/I + J \end{array}$$

is a fiber product diagram of rings.

(2) *(See [1] Example 2.1.2) Let s, t be elements of R such that $R_s + R_t = R$. Then*

$$\begin{array}{ccc} M & \xrightarrow[g_s]{} & M_s \\ \Big\downarrow{\scriptstyle g_t} & & \Big\downarrow{\scriptstyle f_s} \\ M_t & \xrightarrow{f_t} & M_{st} \end{array}$$

is a fiber product diagram of R-modules.

Proof. (1) Consider a ring homomorphism

$$(g_1, g_2) : R/I \cap J \to R/I \oplus R/J$$

and a map

$$f_1p_1 - f_2p_2 : R/I \oplus R/J \to Im(f_1p_1 - f_2p_2).$$

Then clearly (g_1, g_2) is injective.

For all $r + I \cap J \in R/I \cap J$,

$$\begin{aligned} & (f_1p_1 - f_2p_2)(g_1, g_2)(r + I \cap J) \\ = \;& (f_1p_1 - f_2p_2)(g_1(r), g_2(r)) \\ = \;& (f_1p_1 - f_2p_2)(r + I, r + J) = f_1(r + I) - f_2(r + J) \\ = \;& r + (I + J) - r + (I + J) \\ = \;& 0 + (I + J) \end{aligned}$$

Thus $Im(g_1, g_2) \subset Ker(f_1p_1 - f_2p_2)$.

Let $(r + I, s + J) \in Ker(f_1p_1 - f_2p_2)$. Then

$$r - s \in I + J.$$

There is $i \in I, j \in J$ such that

$$r + i = s + j.$$

Let $u = r + i \in R$. Then we have

$$\begin{aligned} (g_1, g_2)(u + I \cap J) &= (u + I, u + J) = (r + i + I, r + i + J) \\ &= (r + i + I, s + j + J) = (r + I, s + J). \end{aligned}$$

Therefore $Ker(f_1p_1 - f_2p_2) \subset Im(g_1, g_2)$ and then

$$Ker(f_1p_1 - f_2p_2) = Im(g_1, g_2).$$

Hence the conclusion follows from Theorem 2.5.

(2) Consider the following sequence

$$0 \longrightarrow M \xrightarrow{(g_s, g_t)} M_s \oplus M_t \xrightarrow{f_sp_s - f_tp_t} Im(f_sp_s - f_tp_t) \longrightarrow 0.$$

Clearly $f_sp_s - f_tp_t$ is surjective.

Let $m \in Ker(g_s, g_t)$. Then we have

$$(0,0) = (g_s, g_t)(m) = (m/1, m/1) \in M_s \oplus M_t.$$

Hence there is non-zero integer u, v such that

$$s^u m = 0 \quad \text{and} \quad t^v m = 0.$$

Since $Rs^u + Rt^v = R$, there is elements $a, b \in R$ such that

$$as^u + bt^v = 1.$$

Then

$$m = (as^u + bt^v)m = as^u m + bt^v m = 0.$$

Therefore (g_s, g_t) is injective.

Next for all $m \in M$,

$$(f_s p_s - f_t p_t)(g_s, g_t)(m) = (f_s p_s - f_t p_t)(g_s(m), g_t(m))$$
$$= (f_s p_s - f_t p_t)(m/1, m/1) = m/1 - m/1 = 0.$$

Thus $Im(g_s, g_t) \subset Ker(f_s p_s - f_t p_t)$.

Let $(m/s^u, m'/t^v) \in Ker(f_s p_s - f_t p_t)$. Then we have

$$\begin{aligned} 0 &= (f_s p_s - f_t p_t)(m/s^u, m'/t^v) \\ &= f_s(m/s^u) - f_t(m'/t^v) \\ &= t^u m/(st)^u - s^v m'/(st)^v \\ &= [(st)^v t^u m - (st)^u s^v m']/(st)^{u+v}. \end{aligned}$$

Hence there is a non-zero integer i such that

$$(st)^i((st)^v t^u m - (st)^u s^v m') = 0.$$

$$\therefore s^{i+v} t^{i+u} t^v m = s^{i+v} t^{i+u} s^u m'.$$

Since $Rs^{i+u+v} + Rt^{i+u+v} = R$, there is elements $x, y \in R$ such that

$$xs^{i+u+v} + yt^{i+u+v} = 1.$$

$$\begin{aligned} s^{i+v} m &= s^{i+v} m(xs^{i+u+v} + yt^{i+u+v}) \\ &= s^{2i+u+2v} xm + s^{i+v} t^{i+u} t^v ym \\ &= s^{2i+u+2v} xm + s^{i+v} t^{i+u} s^u ym' \\ &= s^{i+v} s^u (s^{i+v} xm + t^{i+u} ym'). \end{aligned}$$

Hence we have

$$(s^{i+v} xm + t^{i+u} ym')/1 = m/s^u \text{ in } M_s.$$

On the other hand,

$$\begin{aligned} t^{i+u}m' &= t^{i+u}m'(xs^{i+u+v}+yt^{i+u+v}) \\ &= t^{i+u}s^{i+v}s^{u}xm' + t^{2i+2u+v}ym' \\ &= t^{i+u}s^{i+v}t^{v}xm + t^{2i+2u+v}ym' \\ &= t^{i+u}t^{v}(s^{i+v}xm + t^{i+u}ym'). \end{aligned}$$

Hence we have

$$(s^{i+v}xm + t^{i+u}ym')/1 = m'/t^{v} \text{ in } M_t.$$

Let $n = s^{i+v}xm + t^{i+u}ym' \in M$. Then we have

$$\begin{aligned} (g_s, g_t)(n) &= (g_s, g_t)(s^{i+v}xm + t^{i+u}ym') \\ &= (g_s(s^{i+v}xm + t^{i+u}ym'), g_t(s^{i+v}xm + t^{i+u}ym')) \\ &= (s^{i+v}xm + t^{i+u}ym'/1, s^{i+v}xm + t^{i+u}ym'/1) \\ &= (m/s^{u}, m/t^{v}). \end{aligned}$$

Thus $Ker(f_s p_s - f_t p_t) \subset Im(g_s, g_t)$ and then

$$Ker(f_s p_s - f_t p_t) = Im(g_s, g_t).$$

Hence the conclusion follows from Theorem 2.2.

□

References

[1] Satya Mandal, *Projective Modules and Complete Intersections*, Lecture Notes in Mathematics 1672, Springer, 1997.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: scchung@cnu.ac.kr