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r-HOMOMORPHISMS IN TRANSFORMATION GROUPS

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Abstract. In this paper, it will be given a necessary and sufficient condition for a function to be an r-homomorphism in connection with the subgroups of the automorphism group of a universal minimal set.

1. Introduction

In [6], R. Ellis proved that given any abstract group T there exists a minimal set (M, T) with compact phase space M such that any minimal set (X, T) with compact X is a homomorphic image of (M, T). This (M, T) is called the universal minimal set associated with a group T, and he also showed that this universal minimal set is unique up to an isomorphism. J. Auslander [2] studied minimal sets and their homomorphisms by means of certain subgroups of automorphism group of the universal minimal set (M, T), and he obtained several properties about homomorphisms of distal minimal sets and regular minimal sets. The class of regular minimal sets is shown to coincide with the minimal right ideals of the enveloping semigroups of transformation groups.

The proximal relation $P(X, T)$ in a transformation group (X, T) with compact Hausdorff X has been introduced by R. Ellis and W. H. Gottschalk [7]. The notions of proximal relation and regular minimal sets has been strengthened and extended by consideration of homomorphisms by Shoenfeld [8]. In [9], the author introduced the r-homomorphism between two transformation groups and investigated some properties in relation to the regionally proximal relations and relative regionally regular relations.

In this paper, we study some certain subgroups of the automorphism group of a universal minimal set in connection with the regular relations

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of transformation groups and it will be given a necessary and sufficient condition for a function to be an r-homomorphism.

2. Preliminaries

Throughout this paper, (X, T) will denote a transformation group with compact Hausdorff phase space X. A closed non-empty subset A of X is called a *minimal set* if for every $x \in A$ the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called *an almost periodic point.* If X is itself minimal, we say that it is a *minimal transformation group or a minimal set.*

Let (X, T) and (Y, T) be transformation groups. A function $\pi : X \to Y$ is called a *homomorphism* if π is continuous and $\pi(xt) = \pi(x)t$ $(x \in X, t \in T)$. If Y is minimal, π is always onto. Onto homomorphism is called an *epimorphism.* A homomorphism π from (X, T) into itself is called an *endomorphism* of (X, T) and bijective endomorphism is called an *automorphism* of X. We denote the group of automotphisms of X by $A(X)$.

We define $E(X)$, the *enveloping semigroup* of (X, T), to be the closure of T in X^X, providing X^X with its product topology. The *minimal right ideal* I is the non-empty subset of $E(X)$ with $IE(X) \subset I$, which contains no proper non-empty subset of the same property.

Theorem 2.1. *[3, 4] Let $E(X)$ be the enveloping semigroup of (X, T). Then*

(1) The maps $\theta_x : E(X) \to X$ defined by $\theta_x(p) = xp$ are homomorphisms with range $\overline{xT}$ for $x \in X$.

(2) Given an epimorphism $\pi : X \to Y$, there exists a unique epimorphism $\theta : (E(X), T) \to (E(Y), T)$ such that $\pi\theta_x = \theta_{\pi(x)}\theta$ for $x \in X$.

(3) If (X, T) coincides with (Y, T), then θ is the identity map.

Definition 2.2. [6] Let T be an arbitrary topological group. A minimal transformation group (M, T) is said to be *universal* if every minimal transformation group with acting group T is a homomorphic image of (M, T).

For any group T, a universal minimal set exists and is unique up to isomorphism.

The compact Hausdorff space X carries a natural uniformaity $\mathcal{U}[X]$ whose indices are all the neighborhoods of the diagonal in $X \times X$.

Definition 2.3. [7] Let (X,T) be a transformation group. Two points x and x' of X are called *proximal* provided that for each index $\alpha \in \mathcal{U}[X]$, there exists a $t \in T$ such that $(xt, x't) \in \alpha$. The set of all proximal pairs of points is called the *proximal relation* and is denoted by $P(X,T)$.

Definition 2.4. [11] Let (X,T) be a transformation group. Two points x and x' are said to be *regular* if $h(x)$ and x' are proximal for some automorphism h of X. that is, $(h(x), x') \in P(X,T)$ for some $h \in A(X)$. The set of all regular pairs of points in X is called the *regular relation* and is denoted by $R(X,T)$.

Definition 2.5. [2] Minimal transformation group (X,T) is called *regular minimal* if (x,y) is an almost periodic point of $(X \times X, T)$, then there is an endomorphism h of (X,T) such that $h(x) = y$.

3. r-Homomorphisms

Let $\pi : (X,T) \to (Y,T)$ be a homomorphism. It is well-known that if x and x' are proximal then $\pi(x)$ and $\pi(x')$ are also proximal. Furthermore, in a pointwise almost periodic transformation group, x and x' are proximal if and only if $\pi(x)$ and $\pi(x')$ are proximal. It is noted that the homomorphic images of any two regular points are not regular, in general. But, it will be shown that the r-homomorphic images of any two regular points are also regular.

Definition 3.1. [9] Let (X,T) and (Y,T) be transformation groups. An epimorphism $\pi : X \to Y$ is called an *r-homomorphism* if a given $h \in A(X)$, there exists a $k \in A(Y)$ such that $\pi h = k\pi$.

r-Homomorphisms are closely related to certain subgroups of the automorphism group of a universal minimal set.

Let (M,T) be a universal minimal transformation group, which will be fixed from now on, and the group of automorphisms of (M,T) is denoted by $A(M)$. Given a minimal transformation group (X,T) and a homomorphism $\gamma : M \to X$, $G(X,\gamma) = \{\alpha \in A(M) \mid \gamma\alpha = \gamma\}$ is defined in ([1]).

The following definition is a generalized notion of $G(X,\gamma)$.

Definition 3.2. [9] Let (M,T) be a universal minimal transformation group and let (X,T) be a minimal transformation group. Given a homomorphism $\gamma : M \to X$, we define a subgroup of $A(M)$ as follows.

$$S(X,\gamma) = \{\alpha \in A(M) \mid h\gamma\alpha = \gamma \text{ for some } h \in A(X)\}$$

Similarly, given a homomorphism $\pi : X \to Y$, $S(Y, \pi\gamma)$ is defined obviously. that is,

$$S(Y, \pi\gamma) = \{ \alpha \in A(M) \mid k\pi\gamma\alpha = \pi\gamma \text{ for some } k \in A(Y) \}$$

LEMMA 3.3. *[1] Let (X,T) be a minimal transformation group and let $\gamma : M \to X$ and $\delta : M \to X$ be homomorphisms. Then there exists an $\alpha \in A(M)$ such that $\delta = \gamma\alpha$.*

LEMMA 3.4. *[12] Let (X,T) be a minimal transformation group and let $x, x' \in X$ with (x, x') an almost periodic point. Then $(x, x') \in R(X,T)$ if and only if $h(x) = x'$ for some automorphism h of (X,T).*

THEOREM 3.5. *Let (X,T) and (Y,T) be minimal and let $\pi : X \to Y$, $\gamma : M \to X$ be homomorphsims. If (X,T) is regular minimal, then $S(X,\gamma) = A(M)$. Similarly, if (Y,T) is regular minimal, then $S(Y,\pi\gamma) = A(M)$.*

Proof. Suppose that (X,T) is regular minimal. We need only to show that $A(M) \subset S(X,\gamma)$, because $S(X,\gamma)$ is a subgroup of $A(M)$. Let $\alpha \in A(M)$, $m \in M$. Put $x_1 = \gamma\alpha(m), x_2 = \gamma(m)$. Since X is regular minimal and (x_1, x_2) is an almost periodic point of $(X \times X, T)$, there exists an $h \in A(X)$ such that $h(x_1) = x_2$, by Lemma 3.4. Therefore, $h\gamma\alpha(m) = \gamma(m)$. This shows that $h\gamma\alpha = \gamma$ and hence $\alpha \in S(X,\gamma)$. Therefore $A(M) \subset S(X,\gamma)$. The rest of the theorem is proved similarly. □

Let $\gamma : M \to X$ be an $1 : 1$ homomorphism and let $h : X \to X$ be an automorphism. Then, for $h\gamma : M \to X$ and $\gamma : M \to X$ there exists an $\alpha \in A(M)$ such that $h\gamma\alpha = \gamma$ by Lemma 3.3. Note that α is uniquely determined. In fact, suppose that $h\gamma\alpha = \gamma$ and $h\gamma\beta = \gamma$. Then $h\gamma\alpha = h\gamma\beta$. Since h and γ are $1 : 1$ homomorphisms, we obtain $\alpha = \beta$. Therefore, we can establish a homomorphism ϕ from the group $A(X)$ of automorphisms of (X,T) to the group $A(M)$ of automorphisms of (M,T).

THEOREM 3.6. *Let (X,T) be minimal and let $\gamma : M \to X$ be an 1 : 1 homomorphism. Let $\phi : A(X) \to A(M)$ be a function defined by $\phi(h) = \alpha^{-1}$, where α^{-1} is the inverse of the unique automorphism α of M satisfying $h\gamma\alpha = \gamma$. Then ϕ is a group homomorphism and the image of ϕ is $S(X,\gamma)$.*

Proof. Let $\phi : A(X) \to A(M)$ be a function defined by $\phi(h) = \alpha^{-1}$ with $h\gamma\alpha = \gamma$. Let h and g be in $A(X)$ and let α and β be in $A(M)$ satisfying $h\gamma\alpha = \gamma$, $g\gamma\beta = \gamma$. Then , since $(hg)\gamma(\beta\alpha) = h(g\gamma\beta)\alpha = h\gamma\alpha = \gamma$, hg is associated with $\beta\alpha$. Therefore,

$$\phi(hg) = (\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1} = \phi(h)\phi(g),$$

which shows that ϕ is a group homomorphism. Next, we show that $\phi(A(X)) = S(X, \gamma)$. Let $h \in A(X)$. Then $\phi(h) = \alpha^{-1}$ with $h\gamma\alpha = \gamma$. $\alpha \in S(X, \gamma)$ implies $\alpha^{-1} \in S(X, \gamma)$, because $S(X, \gamma)$ is a subgroup of $A(M)$. Therefore, $\phi(h) \in S(X, \gamma)$. that is, $\phi(A(X)) \subset S(X, \gamma)$. Conversely, let $\alpha \in S(X, \gamma)$. Then $h\gamma\alpha = \gamma$ for some $h \in A(X)$. We also have $h^{-1}\gamma\alpha^{-1} = \gamma$. Since $h^{-1} \in A(X)$, $\phi(h^{-1}) = \alpha$. Therefore, $\alpha \in \phi(A(X))$. that is, $S(X, \gamma) \subset \phi(A(X))$. □

THEOREM 3.7. *Let (X, T) and (Y, T) be minimal transformation groups, let $\pi : X \to Y$ be a homomorphism, and let $\widetilde{\pi} : X \times X \to Y \times Y$ be the induced homomorphism $\widetilde{\pi}(x, x') = (\pi(x), \pi(x'))$ by π. The following holds.*

(1) If $S(X, \gamma) \subset S(Y, \pi\gamma)$, then $\widetilde{\pi}R(X, T) \subset R(Y, T)$

(2) If $S(X, \gamma) = S(Y, \pi\gamma)$, then $\widetilde{\pi}R(X, T) = R(Y, T)$

Proof. (1) Let $(x_1, x_2) \in R(X, T)$. Then there exists an $h \in A(X)$ such that $h(x_1)p = x_2p$ for some $p \in E(X)$. By Lemma 3.3, for given $h\gamma : M \to X$ and $\gamma : M \to X$, there exists an $\alpha \in A(M)$ such that $h\gamma\alpha = \gamma$. Let $m_1 \in M$. Put $\alpha(m_1) = m_2$ and $\gamma(m_2) = x_1p$. Then we get $\gamma(m_1) = h\gamma\alpha(m_1) = h\gamma(m_2) = h(x_1)p = x_2p$. Since $\alpha \in S(X, \gamma)$ and $S(X, \gamma) \subset S(Y, \pi\gamma)$, $k\pi\gamma\alpha = \pi\gamma$ for some $k \in A(Y)$. Since

$$\begin{aligned} k\pi(x_1)\theta(p) &= k\pi(x_1p) = k\pi\gamma(m_2) = k\pi\gamma\alpha(m_1) \\ &= \pi\gamma(m_1) = \pi(x_2p) = \pi(x_2)\theta(p), \end{aligned}$$

we have $(k\pi(x_1), \pi(x_2)) \in P(Y, T)$ and therefore, $(\pi(x_1), \pi(x_2)) \in R(Y, T)$.

(2) It suffices to show that $R(Y, T) \subset \widetilde{\pi}R(X, T)$. Let $(y_1, y_2) \in R(Y, T)$. There exists a $k \in A(Y)$ such that $k(y_1)p = y_2p$ for some $p \in E(Y)$. Since y_2 is an almost periodic point of (Y, T), there is an idenpotent $v^2 = v$ such that $y_2v = y_2$. We also have $u^2 = u \in E(X)$ such that $\theta(u) = v$. By Lemma 3.3, for given $k\pi\gamma : M \to Y$ and $\pi\gamma : M \to Y$ there exists an $\alpha \in A(M)$ such that $k\pi\gamma\alpha = \pi\gamma$. that is, $\alpha \in S(Y, \pi\gamma)$. Since $S(Y, \pi\gamma) \subset S(X, \gamma)$, there is an $h \in A(X)$ such that $h\gamma\alpha = \gamma$ and $k\pi = \pi h$. In fact, take $x_1 \in X$, and $m_1, m_2 \in M$ so that $\pi(x_1) = y_1,\ \gamma(m_2) = x_1,\ \gamma(m_1) = h(x_1),\ \alpha(m_1) = m_2$. Then

$$k\pi(x_1) = k\pi\gamma(m_2) = k\pi\gamma\alpha(m_1) = \pi\gamma(m_1) = \pi h(x_1),$$

which shows that $k\pi = \pi h$. Since $(x_1, h(x_1)u) \in R(X, T)$ and

$$\begin{aligned} \widetilde{\pi}(x_1, h(x_1)u) &= (\pi(x_1), \pi(h(x_1)u)) = (\pi(x_1), \pi h(x_1)\theta(u)) \\ &= (\pi(x_1), \pi h(x_1)v) = (y_1, k\pi(x_1)v) \\ &= (y_1, k(y_1)v) = (y_1, y_2v) = (y_1, y_2), \end{aligned}$$

we have $(y_1, y_2) \in \widetilde{\pi}R(X, T)$. This completes the proof. □

COROLLARY 3.8. *Let (X,T) and (Y,T) be minimal transformation groups, and let $\pi : X \to Y$ be a homomorphism. Suppose that $S(X,\gamma) = S(Y,\pi\gamma)$. If $R(X,T)$ is an equivalence relation, so is $R(Y,T)$.*

Proof. Obviously $R(Y,T)$ is reflexive and symmetric, so we need only to show that $R(Y,T)$ is transitive. Let $(y_1,y_2) \in R(Y,T)$ and $(y_2,y_3) \in R(Y,T)$. Since $S(X,\gamma) = S(Y,\pi\gamma)$, $\widetilde{\pi}R(X,T) = R(Y,T)$ by Theorem 3.7. There exist $(x_1,x_2) \in R(X,T)$ and $(x_2,x_3) \in R(X,T)$ such that

$$\widetilde{\pi}(x_1,x_2) = (y_1,y_2), \qquad \widetilde{\pi}(x_2,x_3) = (y_2,y_3).$$

Since $R(X,T)$ is transitive, we have $(x_1,x_3) \in R(X,T)$ and hence $\widetilde{\pi}(x_1,x_3) = (y_1,y_3) \in R(Y,T)$. Therefore, $R(Y,T)$ is transitive. □

THEOREM 3.9. *Let $\pi_1 : (X,T) \to (Y,T)$, and $\pi_2 : (Y,T) \to (Z,T)$ be r-homomorphisms. Then the composition $\pi_2\pi_1 : (X,T) \to (Z,T)$ is also an r-homomorphism.*

Proof. Let π_1 and π_2 be r-homomorphisms, and let $h \in A(X)$. Since π_1 and π_2 are r-homomorphisms, $\pi_1 h = k\pi_1$ for given k in $A(Y)$, and $\pi_2 k = l\pi_2$ for some l in $A(Z)$. It follows that

$$(\pi_2\pi_1)h = \pi_2(\pi_1 h) = \pi_2(k\pi_1) = (\pi_2 k)\pi_1 = (l\pi_2)\pi_1 = l(\pi_2\pi_1),$$

which shows that $\pi_2\pi_1$ is an r-homomorphism. □

THEOREM 3.10. *Let (X,T) and (Y,T) be minimal transformation groups and let $\pi : X \to Y$ and $\gamma : M \to X$ be homomorphisms. Then the following are equivalent.*

(1) $\pi : X \to Y$ is an r-homomorphism.
(2) For $x \in \pi^{-1}(y)$, $x' \in \pi^{-1}(y')$ with $(x,x') \in R(X,T)$ and (x,x') is an almost periodic point of $(X \times X,T)$, there exists a $k \in A(Y)$ such that $k(y) = y'$ and $\pi h = k\pi$ for some $h \in A(X)$.
(3) $S(X,\gamma) \subset S(Y,\pi\gamma)$.

Proof. (1) ⇒ (2). Let $x \in \pi^{-1}(y), x' \in \pi^{-1}(y')$ with $(x,x') \in R(X,T)$ and (x,x') is an almost periodic point of $(X \times X,T)$. Then $h(x) = x'$ for some $h \in A(X)$ by Lemma 3.4. Since π is an r-homomorphism, for given $h \in A(X)$, there exists a $k \in A(Y)$ such that

$$k\pi = \pi h$$

and

$$k(y) = k(\pi(x)) = \pi h(x) = \pi(x') = y'.$$

(2) ⇒ (3). Let $\alpha \in S(X,\gamma)$. Then $h\gamma\alpha = \gamma$ for some $h \in A(X)$. Let $y \in Y$ and $\pi(x) = y$. There exists a $m_2 \in M$ such that $\gamma(m_2) = x$.

$\alpha \in S(X,\gamma) \subset A(M)$ implies that $\alpha(m_1) = m_2$ for some $m_1 \in M$. Put $\gamma(m_1) = x'$, $\pi\gamma(m_1) = \pi(x') = y'$ Then, since $h\gamma\alpha = \gamma$, we have

$$x' = \gamma(m_1) = h\gamma\alpha(m_1) = h\gamma(m_2) = h(x)$$

Therefore, $x \in \pi^{-1}(y)$, $x' \in \pi^{-1}(y')$ and $(x,x') \in R(X,T)$. Obviously (x,x') is almost periodic. By (2), there exists a $k \in A(Y)$ such that $k(y) = y'$ and $\pi h = k\pi$. It follows that

$$k\pi\gamma\alpha(m_1) = \pi h\gamma\alpha(m_1) = \pi\gamma(m_1)$$

and therefore, $k\pi\gamma\alpha = \pi\gamma$ and hence $\alpha \in S(Y,\pi\gamma)$.

(3) $\Rightarrow$ (1). Let $h \in A(X)$. Then there exists $\alpha \in A(M)$ such that $h\gamma\alpha = \gamma$ by Lemma 3.3. This shows that $\alpha \in S(X,\gamma)$. Since $S(X,\gamma) \subset S(Y,\pi\gamma)$, we have $k\pi\gamma\alpha = \pi\gamma$ for some $k \in A(Y)$. Therefore, $k\pi\gamma\alpha(m) = \pi\gamma(m)$ for all $m \in M$. Since $h\gamma\alpha = \gamma$, we have $k\pi\gamma\alpha(m) = \pi\gamma(m) = \pi h\gamma\alpha(m)$. Since $k\pi$ and πh agree at $\gamma\alpha(m)$, we obtain $k\pi = \pi h$. Therefore, $\pi : X \to Y$ is an r-homomorphism. □

From Theorem 3.7 and Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *Let (X,T) and (Y,T) be minimal transformation groups and let $\pi : X \to Y$ be an r-homomorphism. If x and x' are regular in X, then $\pi(x)$ and $\pi(x')$ are regular in Y.*

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