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DIMENSIONALLY EQUIVALENT SPACES

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Abstract. We compare a coding space which has an ultra metric with the unit interval which has an associated generalized dyadic expansion. The two spaces are not homeomorphic but dimensionally equivalent in the sense that the Hausdorff and packing dimensions of the corresponding distribution sets in the two spaces coincide.

1. Introduction

We([2]) studied the multifractal spectrum of a distribution set in a general coding space(cf.[6]). The dimensions of the distribution sets in a general coding space can be obtained from a bi-Hölder correspondence between a self-similar Cantor set and a general coding space with an ultra metric. In fact, each self-similar Cantor set has its copy of a coding space with an associated ultra metric related to its contraction ratios. The coding space with an associated ultra metric generates a string of coding spaces with ultra metrics related to the powers of the contraction ratios. We easily obtained the multifractal spectrum of a distribution set in the generated coding space with powered ultra metrics using the a bi-Hölder correspondence between a self-similar Cantor set and the general coding space. Recently we([4]) investigated dimensions of distribution sets in the unit interval having the usual metric using the technique([1]) used in a self-similar Cantor set. We showed that the multifractal spectrum formula holds for the unit interval as that in a self-similar Cantor set. The difference between the unit interval and a self-similar Cantor set is that they are not homeomorphic to each other. Precisely, a continuous map from a self-similar Cantor set to the unit interval is not bi-continuous. More precisely, the pre-images of two close

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members in the unit interval are not close any more in the self-similar Cantor set. In this paper, we find that the multifractal spectrum of a distribution set in a general coding space whose ultra metric structure resembles the lengths of generalized dyadic intervals is the same as that of the distribution set in the unit interval having the generalized dyadic expansion. Finally we give a conjecture that the map which gives this dimensional equivalence is a dimension-preserving map.

2. Preliminaries

We denote F a self-similar Cantor set([7]), which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0,1]$ with $a > 0$, $b > 0$ and $1-(a+b) > 0$. Let the fundamental interval $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{0,1\}$ and $1 \le j \le k$. Sometimes we use the notation $F_{a,b}$ instead of F to distinguish it from another self-similar Cantor set having different contraction ratios. We note that $F_{\frac{1}{3},\frac{1}{3}}$ is the classical Cantor ternary set.

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of real numbers. We note that if $x \in F$, then there is $\sigma \in \{0,1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). Without confusion, we identify $x \in F$ with $\sigma \in \{0,1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$.

From now on $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E([7]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E([7]). We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of $x = \sigma$(cf. [1]).
For $q \in [0,1]$, we define lower(upper) distribution set $\underline{F}(q)(\overline{F}(q))$ containing the digit 1 in proportion q by

$$\underline{F}(q) = \{x \in F : \liminf_{k\to\infty} \frac{n_0(x|k)}{k} = q\},$$

$$\overline{F}(q) = \{x \in F : \limsup_{k\to\infty} \frac{n_0(x|k)}{k} = q\}.$$

We write $\underline{F}(q) \cap \overline{F}(q) = F(q)$ and call it a distribution set containing the digit 0 in proportion q.

We recall a coding space $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{x,y}$([2]) such that for $(x,y) \in \{(x,y)|0 < x, y < 1\}$, $\rho_{x,y}(\sigma,\sigma) = 0$ and if $\sigma \ne \tau$ then $\rho_{x,y}(\sigma,\tau) = x^{n_0(x|k)} y^{k-n_0(x|k)}$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$ and $\tau = i_1 i_2 \cdots i_k j_{k+1} \cdots$ where $i_{k+1} \ne j_{k+1}$ for some $k = 0, 1, 2 \cdots$.

Before going into our main results, we need some useful lemma. In this paper, the domain on which a function will be defined will be a subset of $\mathbb{R}$ with the usual metric. The following lemma gives the scaling properties of Hausdorff and packing dimensions of an image of a function satisfying a bi-Hölder condition.

LEMMA 2.1. ([2]) *Let E be a metric space with a metric ρ. Let $f : F \longrightarrow E$ be a function satisfying a bi-Hölder condition*

$$c_1|x-y|^\alpha \leq \rho(f(x), f(y)) \leq c_2|x-y|^\alpha$$

for some constants c_1, c_2 and each $x, y \in F$. Then $\dim(f(F)) = \frac{1}{\alpha}\dim(F)$ and $\mathrm{Dim}(f(F)) = \frac{1}{\alpha}\mathrm{Dim}(F)$.

In this paper, we assume that $0\log 0 = 0$ for convenience.

3. Main results

THEOREM 3.1. *Let $0 < \alpha + \beta < 1$ for positive real numbers α, β. Let*

$$f : F_{\alpha,\beta} \longrightarrow \{0,1\}^{\mathbb{N}}$$

be a function such that $f(x) = \sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ where $\sigma \in \{0,1\}^{\mathbb{N}}$. Consider a positive real number t. In $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric ρ_{α^t,β^t} for each $q \in [0,1]$, we have

$$\begin{aligned}\dim(f(\underline{F}(q))) = \dim(f(\overline{F}(q))) &= \mathrm{Dim}(f(F(q))) \\ &= \frac{q\log q + (1-q)\log(1-q)}{tq\log\alpha + t(1-q)\log\beta}.\end{aligned}$$

Further we have for $q \in [0, \alpha^s]$ where $\alpha^s + \beta^s = 1$, $\mathrm{Dim}(f(\underline{F}(q))) = \frac{s}{t}$ with

$$\mathrm{Dim}(f(\overline{F}(q))) = \frac{q\log q + (1-q)\log(1-q)}{tq\log\alpha + t(1-q)\log\beta}.$$

Similarly we have for $q \in [\alpha^s, 1]$ where $\alpha^s + \beta^s = 1$, $\mathrm{Dim}(f(\overline{F}(q))) = \frac{s}{t}$ with

$$\mathrm{Dim}(f(\underline{F}(q))) = \frac{q\log q + (1-q)\log(1-q)}{tq\log\alpha + t(1-q)\log\beta}.$$

Proof. We have

$$|x-y| \leq \rho_{\alpha,\beta}(f(x), f(y)) \leq [\frac{1}{1-(\alpha+\beta)}]|x-y|$$

for each $x, y \in F_{\alpha,\beta}$. Hence for each positive real number t

$$|x-y|^t \leq \rho_{\alpha^t,\beta^t}(f(x), f(y)) \leq [\frac{1}{1-(\alpha+\beta)}]^t |x-y|^t$$

for each $x, y \in F_{\alpha,\beta}([2])$. We note that $\dim(F_{\alpha,\beta}) = \mathrm{Dim}(F_{\alpha,\beta}) = s$ since $\alpha^s + \beta^s = 1$. From this together with Lemma 2.1 and the results in [1] and the parallel results in [3, 5], it follows. □

We note that if $x \in [0,1]$, then there is a generalized dyadic expansion ([4], cf. [8])) $\sigma \in \{0,1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$. Without confusion, we identify $x \in [0,1]$ with $\sigma \in \{0,1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$. As above, in $[0,1]$, we also define lower(upper) distribution set $\underline{F}(q)(\overline{F}(q))$ containing the digit 0 in proportion q where $q \in [0,1]$.

PROPOSITION 3.2. ([4]) *Let $c + d = 1$ for positive real numbers c, d. Let $\delta(q) = \frac{q\log q + (1-q)\log(1-q)}{q\log c + (1-q)\log d}$ for $q \in [0,1]$. In the unit interval having the generalized dyadic expansion with positive bases c, d with $c + d = 1$, we have*
(1) $\dim(\underline{F}(q)) = \dim(\overline{F}(q)) = \mathrm{Dim}(F(q)) = \delta(q)$ *for* $q \in [0,1]$,
(2) $\mathrm{Dim}(\underline{F}(q))) = 1$ *and* $\mathrm{Dim}(\overline{F}(q)) = \delta(q)$ *for* $q \in [0,c]$,
(3) $\mathrm{Dim}(\overline{F}(q)) = 1$ *and* $\mathrm{Dim}(\underline{F}(q)) = \delta(q)$ *for* $q \in [c,1]$.

As above, in $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{c,d}$, we also define lower(upper) distribution set $\underline{F}(q)(\overline{F}(q))$ containing the digit 0 in proportion q where $q \in [0,1]$.

THEOREM 3.3. *Let $c + d = 1$ for positive real numbers c, d. Let $\delta(q) = \frac{q\log q + (1-q)\log(1-q)}{q\log c + (1-q)\log d}$ for $q \in [0,1]$. In $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{c,d}$, we have*
(1) $\dim(\underline{F}(q)) = \dim(\overline{F}(q)) = \mathrm{Dim}(F(q)) = \delta(q)$ *for* $q \in [0,1]$,
(2) $\mathrm{Dim}(\underline{F}(q))) = 1$ *and* $\mathrm{Dim}(\overline{F}(q)) = \delta(q)$ *for* $q \in [0,c]$,
(3) $\mathrm{Dim}(\overline{F}(q)) = 1$ *and* $\mathrm{Dim}(\underline{F}(q)) = \delta(q)$ *for* $q \in [c,1]$.

Proof. By the theorem 3.6([2]), given positive real numbers c, d such that $c + d = 1$, there exist positive real numbers a, b and $r \in (0,1)$ such that $a + b = 1$ and $(ra)^s = c$ and $(rb)^s = d$ with $(ra)^s + (rb)^s = 1$ for some positive real number s. Putting $\alpha = ra$ and $\beta = rb$ in Theorem 3.1 with

$$f : F_{ra,rb} \longrightarrow \{0,1\}^{\mathbb{N}},$$

we see that in $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{(ra)^s,(rb)^s} = \rho_{c,d}$

$$\dim(f(\underline{F}(q))) = \dim(f(\overline{F}(q))) = \mathrm{Dim}(f(F(q)))$$

$$= \frac{q\log q + (1-q)\log(1-q)}{sq\log ra + s(1-q)\log rb}.$$

Further we have for $q \in [0,(ra)^s] = [0,c]$ where $(ra)^s + (rb)^s = 1$, $\mathrm{Dim}(f(\underline{F}(q))) = \frac{s}{s} = 1$ with

$$\mathrm{Dim}(f(\overline{F}(q))) = \frac{q\log q + (1-q)\log(1-q)}{sq\log ra + s(1-q)\log rb}.$$

Similarly we have for $q \in [(ra)^s, 1] = [c,1]$ where $(ra)^s + (rb)^s = 1$, $\mathrm{Dim}(f(\overline{F}(q))) = \frac{s}{s} = 1$ with

$$\mathrm{Dim}(f(\underline{F}(q))) = \frac{q\log q + (1-q)\log(1-q)}{sq\log ra + s(1-q)\log rb}.$$

We note that $f(\underline{F}(q)) = \underline{F}(q)$ and $f(\overline{F}(q)) = \overline{F}(q)$ for $q \in (0,1)$. Further $f(\underline{F}(q)) = \underline{F}(q)$ and $f(\overline{F}(q)) = \overline{F}(q)$ for $q = 0$ or $q = 1$ essentially in the sense that they differ only for the end points of fundamental intervals which do not affect the values of dimensions since the end points are at most countable. It follows from the above facts with

$$\frac{q\log q + (1-q)\log(1-q)}{sq\log ra + s(1-q)\log rb} = \frac{q\log q + (1-q)\log(1-q)}{q\log(ra)^s + (1-q)\log(rb)^s} = \delta(q).$$

□

Remark 3.4. By comparing the above Theorem with the above Proposition, we easily see that, for $c + d = 1$ where c, d are positive real numbers, the the unit interval $[0,1]$ having the generalized dyadic expansion and the coding space $\{0,1\}^{\mathbb{N}}$ with an ultra metric $\rho_{c,d}$ are dimensionally equivalent in the sense that the Hausdorff and packing dimensions of its distribution sets coincide in each space. From the proof of Theorem 3.1, we find a bi-Lipschitz map

$$f : F_{\alpha,\beta} \longrightarrow \{0,1\}^{\mathbb{N}}$$

from a totally disconnected Cantor set $F_{\alpha,\beta}$ such that $0 < \alpha + \beta < 1$ for positive real numbers α, β onto $\{0,1\}^{\mathbb{N}}$ with an ultra metric $\rho_{\alpha,\beta}$. Further $\{0,1\}^{\mathbb{N}}$ with the ultra metric $\rho_{c,d} = \rho_{\alpha^s,\beta^s}$ where s is a positive real number satisfying $\alpha^s + \beta^s = 1$ with $\alpha^s = c$ and $\beta^s = d$ is the image of $F_{\alpha,\beta}$ under the function f which satisfies a bi-Hölder condition. This means that $\{0,1\}^{\mathbb{N}}$ with the ultra metric $\rho_{c,d}$ is the homeomorphic image of $F_{\alpha,\beta}$. Noting that the unit interval $[0,1]$ is connected and the coding space with an ultra metric $\rho_{c,d}$ such that $c + d = 1$ for positive real numbers c, d is totally disconnected, we see that they are different in topological sense. This means that dimensional equivalence does not give topological invariance. As a result, we see that there is no bi-Lipschitz map

$$g : [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$$

from $[0,1]$ into $\{0,1\}^{\mathbb{N}}$ with a generalized ultra metric $\rho_{c,d}$ with $c+d=1$.

CONJECTURE 3.5. Since the map f such that $f(x)=\sigma$ with $\{x\} = \bigcap_{k=0}^{\infty} I_{\sigma|k}$ where $\sigma \in \{0,1\}^{\mathbb{N}}$ gives dimensional equivalence for distribution sets in each space, we conjecture that f is a dimension-preserving map from $[0,1]$ into $\{0,1\}^{\mathbb{N}}$ with $\rho_{c,d}$ in the sense that $\dim(E) = \dim(f(E))$ and $\mathrm{Dim}(E) = \mathrm{Dim}(f(E))$ for $E \subset [0,1]$. We note that a bi-Lipschitz map is a dimension-preserving map. In this respect, we conjecture that the above map f from $[0,1]$ into $\{0,1\}^{\mathbb{N}}$ with $\rho_{c,d}$ is a dimension-preserving map even though f is not a bi-Lipschitz map.

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