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TAME DIFFEOMORPHISMS WITH C^1-STABLE PROPERTIES

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Abstract. Let f be a diffeomorphisms of a compact C^∞ manifold, and let p be a hyperbolic periodic point of f. In this paper, we prove that if generically, f is tame diffeomorphims then the following conditions are equivalent: (i) f is Ω-stable, (ii)f has the C^1-stable shadowing property (iii) f has the C^1-stable inverse shadowing property.

1. Introduction

Let M be a closed C^∞ manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. In [2], the authors proved that if generically, f is tame diffeomorphisms then the following two conditions are equivalent:

- f is hyperbolic(i.e., structurally stable, or equivalently Axiom A with the transversality condition)
- f is shadowable.

Definition 1.1. [1] Let $f \in$ Diff(M). Then f is *tame diffeomorphisms* if f satisfies the following conditions:

- $\Omega(f) = \bigcup_{i=1}^{n} \Lambda(p_i)$, where $\Lambda(p_i) = H_f^T(p_i)$ for $i = 1, \cdots, n$;
- $\Lambda(p_i) \cap \Lambda(p_j) = \phi$, for $i \neq j$;

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- $\Lambda(p_i)$ are transitive sets, for $i = 1, \cdots, n$, where every p_i are hyperbolic saddle of f.

In this paper, we study that if generically, f is tame diffeomorphisms then the following notions are equvalent; Ω stable, C^1-stable shadowing property and C^1-stable inverse shadowing property.

Let us be more precise. Let X be a compact metric space with metric d, and let $Z(X)$ denote the space of homoemorphisms on X with the C^0-metric d_0. Let $f \in Z(X)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ in X is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. For given $x, y \in X$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=a}^{b} (a < b)$ of f such that $x_a = x$ and $x_b = y$. The set of points $\{x \in X : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. If we denote the set of periodic points of f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wondering set of f. The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* of f. Let $\Lambda \subset X$ be a closed f-invariant set. We say that $f|_\Lambda$ has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of $f(-\infty \leq a < b \leq \infty)$, there is $y \in M$, f-ϵ-shadowing the pseudo-orbit, i.e., $d(f^i(y), x_i) < \epsilon$ for all $a \leq i \leq b-1$. This property does not depend on the metric used and is preserved under topological conjugacy. Note that $f|_\Lambda$ has the shadowing property if and only if $f^n|_\Lambda$ has the shadowing property for $n \in \mathbf{Z} \setminus \{0\}$.

Let $X^{\mathbf{Z}}$ be the compact metric space of all two sided sequences $\xi = \{x_k : k \in \mathbf{Z}\}$ in X, endowed with the product topology. For a constant $\delta > 0$ and $f \in Z(X)$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \to \Phi_f(\delta) \subset X^{\mathbf{Z}}$ satisfying $\varphi(x)_0 = x$, $x \in X$, is said to be a *δ-method* for f. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbf{Z}}$. Say that φ is *continuous δ-method* for f if φ is continuous. The set of all δ-methods[respectively, continuous δ-methods] for f will be denoted by $\mathcal{T}_0(f, \delta)$[respectively, $\mathcal{T}_c(f, \delta)$]. Every $g \in Z(X)$ with $d_0(f, g) < \delta$ induces a continuous δ-method $\varphi_g : X \to X^{\mathbf{Z}}$ for f by defining $\varphi_g(x) = \{g^k(x) : k \in \mathbf{Z}\}$. Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ-methods for f which we introduced by homeomorphisms g on X with $d_0(f, g) < \delta$. Let $\Lambda \subset X$ be a closed f-invariant set. We say that $f|_\Lambda$ has the *inverse* *shadowing property* with respect to a class $\mathcal{T}_\alpha$ ($\alpha = 0, c, h, d$) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f, \delta)$, there is a map $s : \Lambda \to M$ satisfies $d(f^n(x), \varphi(s(x)) < \epsilon$ for all $x \in \Lambda$ and all $n \in \mathbf{Z}$. Note that $f|_\Lambda$ has the inverse shadowing property with respect to $\mathcal{T}_\alpha(\alpha = 0,\ c,\ h)$ if and only if $f^n|_\Lambda$ has the inverse shadowing property with respect to $\mathcal{T}_\alpha$ for $n \in \mathbf{Z} \setminus \{0\}$(see, [7]).

Let M be as before, and let $f \in \mathrm{Diff}(M)$. It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. Then $W^s(p, f)$ is called *stable manifolds* and $W^u(p, f)$ is called *unstable manifold* of p with respect to f.

A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f^T(p)$. It is clear that $H_f^T(p)$ is compact f-invariant sets. Transversal homoclinic classes are the natural candidates to replace hyperbolic basic sets in non-hyperbolic theory.

Let $f \in \mathrm{Diff}(M)$, and let $\Lambda \subset M$ be a closed f-invariant set. We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\bigcap_{n\in\mathbf{Z}} f^n(U) = \Lambda$.

DEFINITION 1.2. We say that an f-invariant set Λ has the *C^1-stable shadowing property[resp. C^1-stable inverse shadowing property]* if Λ is locally maximal in U and there is a C^1-neighborhood of $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g}$ has the shadowing property [resp. inverse shadowing property]. Here $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ and which is called the *continuation* of $\Lambda_f = \Lambda = \bigcap_{n\in\mathbf{Z}} f^n(U)$.

THEOREM 1.3. *Generically, f is tame then the following three conditions are equivalent:*

(a) *f is Ω-stable,*
(b) *f has the C^1-stable shadowing property,*
(c) *f has the C^1-stable inverse shadowing property.*

Let f satisfy Ω-stable. Then it well-known that f satisfies both Axiom A and the no-cycle condition. Thus we know the following facts. Denote by $\mathcal{F}(M)$ the set of $f \in \text{Diff}(M)$ such that there is a C^1-neighborhood $\mathcal{U}(f)$ of f with property that every $p \in P(g)(g \in \mathcal{U}(f))$ is hyperbolic. It is proved by Hayashi [6] that $f \in \mathcal{F}(M)$ if and only if f satisfies both Axiom A and no-cycle condition. Therefore, we are going to make using this result to prove. We prove that the equivalence (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (b).

2. Proof of the Theorem

Let M be as before, and let $f \in \text{Diff}(M)$. The following so-called Franks' lemma will play essential roles in our proofs.

Lemma 2.1. *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \cdots, x_N\}$, a neighborhood U of $\{x_1, x_2, \cdots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \le i \le N$.*

Proof. See [[5], Lemma 1.1]. □

Given an open subset $\mathcal{U}$ of Diff(M), *a subset $\mathcal{G}$ of $\mathcal{U}$ is said to be residual in $\mathcal{U}$ if $\mathcal{G}$ contains a countable intersection of open and dense subsets of $\mathcal{U}$. If P is a property of $f \in \mathcal{U}$, we say that this property is generic in $\mathcal{U}$ if $\{f \in \mathcal{U} : f$ satisfies $P\}$ is residual in $\mathcal{U}$. From [3], we know that generically $H_f^T(p) = C_f(p)$ for any hyperbolic saddle point $p \in M$.*

Lemma 2.2. [3] *There exists a residual subset $\mathcal{G}_1$ of* $\text{Diff}(M)$ *such that if $f \in \mathcal{G}_1$ then $H_f^T(p) = C_f(p)$, where p is a hyperbolic periodic point of f.*

From [1], the author mentioned tame diffeomorphisms which is shortly, if generically, f admits a spectral decomposition then there exists an open neighborhood $\mathcal{U}(f)$ of f such that if $g \in \mathcal{U}(f)$, then g admits a spectral decomposition. And so, in [2] the authors consider the following lemma.

LEMMA 2.3. [2] *There exists a residual subset* $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ *such that* $f \in \mathcal{G}_2$ *then* f *is tame.*

REMARK 2.4. Note that the identity map *id* of the unit interval I and a rotation map ρ of the unit circle S^1 do not have the shadowing property[resp. inverse shadowing property].

LEMMA 2.5. *Let* Λ *be closed* f*-invariant set. Suppose that* Λ *has the* C^1*-stable shadowing property [resp. inverse shadowing property] for* f. *Then there exists a* C^1*-neighborhood* $\mathcal{V}(f) \subset \mathcal{U}(f)$ *of* f *such that for each* $g \in \mathcal{V}(f)$, *every* $q \in \Lambda_g \cap P(g)$ *is hyperbolic, where* $\Lambda_g = \bigcap_{n \in \mathbf{Z}} g^n(U)$ *for a small neighborhood* U *of* Λ.

Proof. See the proof of [[8], Lemma 5.5 and [10], Lemma 3.2]. □

PROPOSITION 2.6. *Let* $f \in \mathcal{G}_1 \cap \mathcal{G}_2$. *Then* f *has the* C^1*-stable shadowing property if and only if* f *is* Ω*-stable.*

Proof. We show that "only if part"; Let $f \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then f is tame and $H_f^T(p) = C_f(p)$, where p is a hyperbolic saddle of f. Since f is tame and $H_f^T(p) = C_f(p)$, we know that $\Omega(f) = \Lambda(p_1) \cup \Lambda(p_2) \cup \cdots \cup \Lambda(p_n)$, where $\Lambda(p_i) = H_f^T(p_i)$ and p_i are hyperbolic saddle for $i = 1, \cdots, n$. And so, we will show that for fixed i, $H_f^T(p_i) = C_f(p_i)$ is hyperbolic. Then for each i, $\Lambda(p_i)$ are hyperbolic and f is Ω-stable. Let $\Lambda(p_i) = \Lambda$ for fixed i and let Λ be a compact f-invariant set which has the C^1-stable shadowing property. Then there are a compact neighborhood U of Λ and a C^1-neighborhood $\mathcal{U}(f)$ of f and such that for any $g \in \mathcal{U}(f), g|_{\Lambda_g}$ has the shadowing property, where $\Lambda_g = \bigcap_{n \in Z} g^n(U)$. Since f is tame, we have $\Lambda_g(p_g) = H_g^T(p_g)$. By [8](see, Theorem 3) Λ is hyperbolic. And so, for each i, $\Lambda(p_i)$ are hyperbolic. This means for all basic sets are hyperbolic. Thus $f \in \mathcal{F}(M)$ and so it is Axiom A and the no-cycle condition. Thus f is Ω-stable.

Next we show that "if" part; We well-known that if f is Ω-stable then it satisfies both Axiom A and no-cycle condition. And so we show that f satisfies both Axiom A and the no-cycle condition. Thus, we may assume that f satisfies both Axiom A and the no-cycle condition. Then $\Omega(f) = \overline{P(f)}$ is hyperbolic. And so $\overline{P(f)}$ is locally maximal; i.e., there exist a compact neighborhood U of $\Omega(f)$ such that $\Omega(f) = \bigcap_{n \in \mathbf{Z}} f^n(U)$. By the stability of locally maximal hyperbolic sets, we can take a compact neighborhood U of $\Omega(f)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $f|_{\Omega(f)}$ and $g|_{\Omega(g)}$ are conjugate, where $\Omega(g) = \bigcap_{n\in\mathbf{Z}} g^n(U)$. Since $\Omega(f)$ is hyperbolic for f, $f|_{\Omega(f)}$ has the C^1-stable shadowing property. Thus $g|_{\Omega(g)}$ has the shadowing property. This completes the proof of Proposition 2.5. □

PROPOSITION 2.7. *Let $f \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then f has the C^1-stable inverse shadowing property if and only if f is Ω-stable.*

Proof. This proof is similarly to Proposition 2.5. □

Therefore by Proposition 2.5 and Proposition 2.6, we can direct to prove that Theorem 1.1. And so we get the following result which obtained by Lemma 2.2 and [8, 10].

COROLLARY 2.8. *Generically, let p be a hyperbolic periodic point of f. Then C^1-stable shadowing property on chain component $C_f(p)$ if and only if C^1-stable inverse shadowing property on chain component $C_f(p)$.*

Proof. First we prove that "only if" part. Suppose $C_f(p)$ is C^1-stable shadowing property. Then there exist a compact neighborhood U of $C_f(p)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ $g|_{\Lambda_g}$ has the shadowing property, where $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$. By [8], $C_f(p) = H_f^T(p)$ and $C_f(p)$ is hyperbolic. And so, $C_f(p)$ has the inverse shadowing property. By [12], $C_f(p)$ is locally maximal. Therefore, by the local stability hyperbolic invariant set theorem, $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ and $f|_{C_f(p)}$ is conjugate $g|_{\Lambda_g}$. Thus Λ_g has the inverse shadowing property.

Next, we prove that "if" part. Let $f \in \mathcal{G}$. Then by Lemma 2.2, $C_f(p) = H_f^T(p)$. Suppose $C_f(p)$ is C^1-stable inverse shadowing property. Then there exist a compact neighborhood U of $C_f(p)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ $g|_{\Lambda_g}$ has the inverse shadowing property, where $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$. By [10], generically, $C_f(p)$ is hyperbolic. And so, $C_f(p)$ has the shadowing property. By [12], $C_f(p)$ is locally maximal. Therefore, by the local stability hyperbolic invariant set theorem, $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ and $f|_{C_f(p)}$ is conjugate $g|_{\Lambda_g}$. Thus Λ_g has the shadowing property. This completes the proof of Corollary 2.8. □

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