JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 4, December 2008

h-STABILITY FOR PERTURBED INTEGRO-DIFFERENTIAL SYSTEMS

Yoon Hoe Goo* and Dae Hee Ry**

Abstract. In this paper, we investigate $h-$stability of nonlinear integro-differential equations.

1. Introduction

Pinto [9, 10] introduced $h-$stability which is an important extension of the notion of exponential asymptotic stability. He introduced the concept of $h-$stability with the intention of obtaining results about stability for a weakly stable systems (at least, weaker than those given by EAS and ULS). Choi and Ryu [3] dealt with $h-$stability of the solutions of ordinary differential systems and Volterra integro-differential systems. In this paper, we investigate $h-$stability of a solution of nonlinear integro-differential equations. The paper is organized as follows. In Section 2, we review definition of $h-$stability and present some related properties needed for our purposes. Finally, in Section 3, we investigate $h-$stability for the nonlinear integro-differential equations.

2. Preliminaries

Let R^n and R^+ be the Euclidean $n-$space and the set of all nonnegative real numbers, respectively. Consider the linear integro-differential equation of Volterra type

$$(2.1) \qquad x' = A(t)x + \int_{t_0}^{t} K(t,s)x(s)\, ds, \quad x(t_0) = x_0$$

Received October 01, 2008; Revised October 29, 2008; Accepted November 20, 2008.

2000 Mathematics Subject Classification: 34D10, 34C11.

Key words and phrases: h-stable, nonlinear integro-differential equation.

and its perturbation,

$$(2.2)\quad y' = A(t)y + \int_{t_0}^{t} K(t,s)y(s)\ ds + f(t,y), \quad y(t_0) = y_0,$$

where $A(t), K(t,s)$ are continuous $n \times n$ matrices on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$, respectively, and $f \in C[R^+ \times R^n, R^n]$. Then the unique solution of (2.2) through $(t_0, y_0) \in R^+ \times R^n$ satisfies the integral equation

$$y(t) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,s)f(s,y(s))\ ds,\ t \geq t_0 \geq 0,$$

where $R(t,s)$ is the solution of the initial value problem

$$(2.3)\quad \frac{\partial}{\partial s}R(t,s) + R(t,s)A(s) + \int_{s}^{t} R(t,u)K(u,s)\ du = 0$$

with $R(t,t) = I$ for $0 \leq s < t < \infty$ [5]. In addition, the unique solution of (2.1) is given by $x(t) = R(t,t_0)x_0$.

We now give the main definition in [9, 10] that we need in the sequel. We consider the nonlinear differential system

$$(2.4)\quad x' = f(t,x), \quad x(t_0) = x_0,$$

where $f \in C[R^+ \times R^n, R^n]$ and $f(t,0) = 0$. Let $x(t) = x(t,t_0,x_0)$ denote the solution of (2.4) through (t_0, x_0) in $R^+ \times R^n$ such that $x(t_0,t_0,x_0) = x_0$.

Definition 2.1. The trivial solution of (2.4) is called (hS) $h-$stable if there exist $c \geq 1$, $\delta > 0$ and a positive bounded continuous function h on R^+ such that

$$|x(t)| \leq c|x_0|h(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

Remark 2.2. If $h(t) = e^{-t}$, then $h-$stability coincides with exponential stability, and if $h(t)$ is constant, then we have uniform Lipschitz stability.

Lemma 2.3 [6]. Let the following condition hold for functions $u(t), v(t) \in C[[t_0,\infty), R^+]$ and $k(t,u) \in C[[t_0,\infty) \times R^n, R^+]$:

$$u(t) - \int_{t_0}^{t} k(s,u(s))\ ds \ \leq\ v(t) - \int_{t_0}^{t} k(s,v(s))\ ds,$$

$t \geq t_0$ and $k(s,u)$ is monotone nondecreasing in u for each fixed $s \geq 0$. If $u(t_0) < v(t_0)$, then $u(t) < v(t),\ t \geq t_0 \geq 0$.

Lemma 2.4 [4]. Let $u(t), a(t), b(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on R^+, for which the inequality

$$u(t) \leq d + \int_{t_0}^{t} a(s)u(s)ds + \int_{t_0}^{t} b(s)\Big(\int_{t_0}^{s} q(\tau)u(\tau)d\tau\Big)ds, \ t \geq t_0 \geq 0,$$

holds, where d is a nonnegative constant, then

$$u(t) \leq d \exp \int_{t_0}^{t} \Big(a(s) + b(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds$$

for $t_0 \leq t \leq L$, where $W(u) = \int_{u_0}^{u} \frac{dz}{z}, \ u > 0, u_0 > 0$, and $W^{-1}(u)$ is the inverse of $W(u)$ and

$$L = \sup\left\{t \geq t_0 : W(d) + \int_{t_0}^{t} \Big(a(s) + b(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds \in \mathrm{Dom}W^{-1}\right\}.$$

Lemma 2.5 [3]. The trivial solution of (2.1) is hS if and only if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h defined on R^+ such that

$$|R(t, t_0)| \leq ch(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$, where $R(t, t_0)$ is the solution of (2.3).

3. Main results

In this section we discuss $h-$stability of the trivial solution of nonlinear integro-differential equations. To begin with, the following theorem is proved.

Theorem 3.1. Let the following conditions hold for the equation (2.2):

(i) $|f(t, x)| \leq k(t, |x|), \ k(t, 0) = 0$ and $k(t, u) \in C[R^+ \times R^n, R^+]$ is monotone nondecreasing with respect to u for each $t \geq 0$,

(ii) the trivial solution of the equation (2.1) is hS with the nonincreasing function h.

If the trivial solution of the scalar differential equation

$$(3.1) \qquad z' = ck(t, z), \ z(t_0) = z_0 = c|y_0|, \ c > 1,$$

is hS, then the trivial solution of (2.2) is also hS.

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of (2.2) through $(t_0, y_0) \in R^+ \times R^n$. Then, by the variation of constants formula, the solution

$y(t, t_0, y_0)$ of (2.2) is given by

$$y = y(t, t_0, y_0) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)f(s, y(s, t_0, y_0))\ ds$$

Thus we obtain from conditions (i),(ii), and Lemma 2.5 that

$$\begin{aligned} |y(t)| &\le ch(t)h(t_0)^{-1}|y_0| + \int_{t_0}^{t} ch(t)h(s)^{-1}k(s, |y(s)|)\ ds \\ &\le c|y_0| + c\int_{t_0}^{t} k(s, |y(s)|)\ ds. \end{aligned}$$

Next, let $z = z(t, t_0, z_0)$ be the solution of (3.1) passing through (t_0, z_0). Then

$$|y(t)| - \int_{t_0}^{t} ck(s, |y(s)|)ds \le z(t) - \int_{t_0}^{t} ck(s, z(s))ds.$$

Therefore applying Lemma 2.3, we obtain that

$$|y(t)| < z(t),\ t \ge t_0.$$

Since the trivial solution of (3.1) is hS, there exist $K_1 \ge 1, \delta_1 > 0$ and a positive bounded continuous function h on R^+ such that $|z(t)| \le K_1 h(t)h(t_0)^{-1}|z_0|$ for $t \ge t_0 \ge 0$ and $|z_0| < \delta_1$. Thus set $\delta = \delta_1/c$ and $K = cK_1$. If $|y_0| < \delta$, then we have that $|y(t)| \le Kh(t)h(t_0)^{-1}|y_0|$ for all $t \ge t_0$, which completes the proof of the theorem. □

Next, we investigate the property of hS for the functional integro-differential equation

$$(3.2)\quad y' = A(t)y + \int_{t_0}^{t} K(t, s)y(s)\ ds + g(t, y, Ty),\ y(t_0) = y_0$$

where $g \in C[R^+ \times R^n \times R^n, R^n]$ and $T : C[R^+, R^n] \to C[R^+, R^n]$ is a continuous operator.

Theorem 3.2. Suppose that the trivial solution of (2.1) is hS with the nonincreasing function h and the solutions of the scalar differential equation

$$(3.3)\qquad z' = ck(t, z, Tz),\ z(t_0) = z_0 = c|y_0|$$

exist on R^+ with $c > 1$. For the perturbed system (3.2), suppose that $|g(t, y, Ty)| \le k(t, |y|, |Ty|)$, where $k(t, u, v) \in C[R^+ \times R^+ \times R^+, R^+]$ is nondecreasing in u, v for each $t \in R^+$ with $k(t, 0, 0) = 0$.

If the trivial solution of (3.3) is hS, then the trivial solution of (3.2) is also hS.

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of (3.2) with an initial value (t_0, y_0), $t_0 \geq 0$. Then the solution is of the form

$$y = y(t, t_0, y_0) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)g(s, y(s), Ty(s))\ ds.$$

Let $z = z(t, t_0, z_0)$ be the solution of (3.3) passing through (t_0, z_0). Following the similar argument as in the proof of Theorem 3.1, we obtain

$$|y(t)| \leq c|y_0| + c\int_{t_0}^{t} k(s, |y(s)|, |Ty(s)|)\ ds.$$

Therefore by Lemma 2.3, we obtain that

$$|y(t)| < z(t),\ t \geq t_0.$$

Since the trivial solution of (3.3) is hS,

$$\begin{aligned} |y(t)| < z(t) \quad &\leq \quad c_1|z_0|h(t)h(t_0)^{-1} \\ &= \quad M|y_0|h(t)h(t_0)^{-1}, \end{aligned}$$

where $M = c_1 c > 1$. This completes the proof. □

Theorem 3.3. Suppose that the trivial solution of (2.1) is hS with increasing function h, and that $g(t, y, z)$ satisfies the inequality

$$|g(s, y, z)| \leq r(s)(|y(s)| + |z(s)|),$$

where $r \in C(R^+)$ and $\int_{t_0}^{\infty} r(s)\ ds < \infty$. Further suppose that the operator T satisfies the inequality

$$|Ty(t)| \leq \int_{t_0}^{t} q(s)|y(s)|\ ds,$$

where $q \in C(R^+)$, $\int_{t_0}^{\infty} q(s)\ ds < \infty$, and

$$M(t) = \exp\left[c\int_{t_0}^{t} r(s)(1 + \int_{t_0}^{s} q(\tau)d\tau)ds\right]$$

Then the trivial solution of (3.2) is hS.

Proof. Note that the solution of (3.2) is given by

$$y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)g(s, y(s), Ty(s))\ ds,\ t \geq t_0 \geq 0,$$

where $R(t,s)$ is the solution of the initial value problem

$$\frac{\partial}{\partial s}R(t,s)+R(t,s)A(s)+\int_s^t R(t,u)K(u,s)du=0$$

with $R(t,t)=I$ for $0\le s<t<\infty$. Since the solution $x=0$ of (2.1) is hS, it follows from Lemma 2.5 that

$$\begin{aligned}|y(t)| &= |R(t,t_0)|\,|y_0|+\int_{t_0}^t |R(t,s)||g(s,y(s),Ty(s)|ds,\\ &\le ch(t)h(t_0)^{-1}|y_0|+\int_{t_0}^t cr(s)h(t)h(s)^{-1}|y(s,t_0,y_0)|\ ds\\ &\quad +\int_{t_0}^t cr(s)h(t)h(s)^{-1}\left(\int_{t_0}^s q(u)|y(u,t_0,y_0)|du\right)\ ds.\end{aligned}$$

Set $u(t)=|y(t)|h(t)^{-1}$. Then by Lemma 2.4 we obtain

$$\begin{aligned}|y(t)| &\le c|y_0|h(t)h(t_0)^{-1}\exp\left[c\int_{t_0}^t r(s)\Big(1+\int_{t_0}^s q(\tau)d\tau\Big)\ ds\right]\\ &\le M|y_0|h(t)h(t_0)^{-1},\quad M=cM(\infty)\ge 1.\end{aligned}$$

Hence the trivial solution of (3.2) is hS and so the proof is complete. □

Acknowledgement: The authors would like to thank the referee for his/her valuable suggestions that helped to improve the original manuscript.

References

[1] F. Brauer, Perturbations of nonlinear systems of differential equations II, J. Math. Anal. Appl. **17** (1967), 418-434.

[2] S. K. Choi, Stabilities in differential systems, Comm. Korean Math. Soc. **9** (1994), 579-591.

[3] S. K. Choi and H. S. Ryu, $h-$stability in differential systems, Bull. Inst. Math. Acad. Sinica **21** (1993), 245-262.

[4] S. K. Choi and N. J. Koo, $h-$stability for nonlinear perturbed systems, Ann. of Diff. Eqs. **11** (1995), 1-9.

[5] S. Hu, V. Lakshmikantham and M. R. M. Rao, Nonlinear variation of parameters formula for integro-differential equations of Volterra type, J. Math. Anal. Appl. **129** (1998), 223-230.

[6] V. Lakshmikantham and M. R. M. Rao, Theory of Integro-Differential Equations, Gordon and Breach Science Publishers, 1995.

[7] L. J. Lin, The stability and growth for the perturbed nonlinear systems of differential equations, Bull. Inst. Math. Acad. Sinica **8** (1980), 535-551.

[8] B. G. Pachpatte, Stability and asymptotic behavior of perturbed nonlinear system, J. Differential equations, **16** (1974), 14-25.
[9] M. Pinto, Perturbations of asymptotically stable differential systems, Analysis **4** (1984), 161-175.
[10] M. Pinto, Stability of nonlinear differential systems, Applicable Analysis **43** (1992), 1-20.
[11] T. Taniguchi, Stability theorems of perturbed linear ordinary differential equations, J. Math. Anal. Appl. **149** (1990), 583-598.

*
Department of Mathematics
Hanseo University
Seosan Chungnam 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr

**
Department of Computer Science
Chungwoon University
Hongseong Chungnam 351-701, Republic of Korea
E-mail: rdh@chungwoon.ac.kr